Note di Matematica Vol. X, Suppl. n. 1, 251-272 (1990)

GENERALIZED FOURIER EXPANSIONS FOR ZERO-SOLUTIONS OF SURJECTIVE CONVOLUTION OPERATORS ON $\mathcal{D}'(\mathbb{R})$ AND $\mathcal{D}'_\omega(\mathbb{R})$

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Dedicated to the memory of Professor Gottfried Köthe

It is well-known that each distribution μ with compact support can be convolved with an arbitrary distribution and that this defines a convolution operator S_μ acting on $\mathcal{D}'(\mathbb{R})$. The surjectivity of S_μ was characterized by Ehrenpreis [5]. Extending this result, we characterize in the present article the surjectivity of convolution operators on the space $\mathcal{D}'_\omega(\mathbb{R})$ of all ω-ultradistributions of Beurling type on $\mathbb{R}$. This is done in two steps. In the first one we show that $ker\ S_\mu$ has an absolute basis whenever S_μ admits a fundamental solution $\nu \in \mathcal{D}'_\omega(\mathbb{R})$. The expansion of an element in $ker\ S_\mu$ with respect to this basis can be regarded as a generalization of the Fourier expansion of periodic ultradistributions. In the second step we use this sequence space representation together with results of Palamodov [15] and Vogt [17], [18] on the projective limit functor to obtain the desired characterization. It turns out that S_μ is surjective if and only if S_μ admits a fundamental solution. Hence the elements of $ker\ S_\mu$ admit a generalized Fourier expansion for each surjective convolution operator S_μ on $\mathcal{D}'_\omega(\mathbb{R})$. Note that this differs from the behavior of convolution operators on the space $\mathcal{E}_{\{\omega\}}(\mathbb{R})$ of ω-ultradifferentiable functions of Roumieu-type, as Braun, Meise and Vogt [4] have shown. Note also that the results of the present article apply to convolution operators on $\mathcal{D}'(\mathbb{R})$, too.

1. PRELIMINARIES

In this preliminary section we introduce most of the notation which will be used in the sequel.

Definition 1.1. *A continuous increasing function* $\omega : [0,\infty[\to [0,\infty[$ *is called a weight function if it satisfies*

(α) *there exists* $K \in \mathbb{N}$ *with* $\omega(2t) \leq K(1+\omega(t))$ *for all* $t \geq 0$

(β) $\int_0^\infty \frac{\omega(t)}{1+t^2}\,dt < \infty$

(γ) $\lim_{t\to\infty} \frac{\log t}{\omega(t)} = 0$

(δ) $\varphi : t \mapsto \omega(e^t)$ *is convex.*

The Young conjugate $\varphi^* : [0,\infty[\to \mathbb{R}$ *of* φ *is defined by*

$$\varphi^*(y) := \sup\{xy - \varphi(x) \mid x \geq 0\}.$$

By abuse of notation we shall write subsequently $\omega(z)$ instead of $\omega(|z|)$ for $z \in \mathbb{C}$.

Definition 1.2. *Let ω be a weight function, and let Ω be an open subset of $\mathbf{R}$. Then we define*

$$\mathcal{E}_\omega(\Omega) := \Big\{ f \in C^\infty(\Omega) \,|\, \text{for each compact subset } K \text{ of } \Omega \text{ and each } m \in \mathbf{N} :$$
$$p_{K,m}(f) := \sup_{x\in K} \sup_{j\in \mathbf{N}_0} |f^{(j)}(x)| \, exp\left(-m\varphi^*\left(\frac{j}{m}\right)\right) < \infty \Big\}$$

and we endow $\mathcal{E}_\omega(\mathbf{R})$ with the Fréchet space topology which is induced by the semi-norms $p_{K,m}$, $K \subset\subset \Omega$, $m \in \mathbf{N}$. The elements of $\mathcal{E}_\omega(\mathbf{R})$ are called ω-ultradifferentiable functions of Beurling type.

For $k \in \mathbf{N}$ we set

$$\mathcal{D}_\omega[-k,k] := \{ f \in \mathcal{E}_\omega(\mathbf{R}) \,|\, Supp(f) \subset [-k,k] \},$$

endowed with the induced topology. Finally, we define

$$\mathcal{D}_\omega(\mathbf{R}) := \operatorname*{ind}_{k\to} \mathcal{D}_\omega[-k,k].$$

The elements of $\mathcal{D}_\omega(\mathbf{R})'$ are called ω-ultradistribution of Beurling type. $\mathcal{D}_\omega(\mathbf{R})'$ will be endowed with the strong topology.

Remark 1.3. (a) For further details concerning the spaces $\mathcal{D}_\omega(\mathbf{R})$ and $\mathcal{E}_\omega(\mathbf{R})$ we refer to Braun, Meise and Taylor [3]. In particular it is shown there that $\mathcal{D}_\omega(\mathbf{R})$ is an infinite dimensional, complete, nuclear (LF)-space for each weight function ω.

(b) By [3], 3.4, the spaces $\mathcal{E}_\omega(\mathbf{R})$ and $\mathcal{D}_\omega(\mathbf{R})$ do not change if we replace ω by σ : $t \mapsto \max(\omega(t) - \omega(1), 0)$. Therefore we can and shall assume in the sequel that φ^* is non-negative.

(c) The function $\omega : t \mapsto log(1+t)$ is not a weight function since 1.1 (γ) is not satisfied. Nevertheless it can be subsumed under the present theory, provided that one uses the right interpretation.

Example 1.4. The following functions $\omega : [0,\infty[\to [0,\infty[$ are examples of weight functions

(1) $\omega(t) := t^\alpha, 0 < \alpha < 1$
(2) $\omega(t) := (log(1+t))^\beta, \beta > 1$
(3) $\omega(t) := t(log(e+t))^{-\beta}, \beta > 1$.

Note that for $\omega(t) = t^\alpha, 0 < \alpha < 1$, the space $\mathcal{E}_\omega(\mathbf{R})$ is the Gevrey class $\mathcal{E}^{(d)}(\mathbf{R})$ for $d := \frac{1}{\alpha}$.

The Fourier-Laplace transform on $\mathcal{D}_\omega(\mathbf{R})$ 1.5. Let $A(\mathbb{C})$ denote the algebra of all entire functions on $\mathbb{C}$. For a weight function ω and $k, m \in \mathbf{N}$ we define the Banach space

$$A(\omega, k, m) = \left\{ f \in A(\mathbb{C}) \mid \| f \|_{k,m} := \sup_{z \in \mathbb{C}} |f(z)| \, exp(-k|Im\ z| + m\omega(z)) < \infty \right\}$$

and the (LF)-space

$$A_\omega := \{ f \in A(\mathbb{C}) \mid \text{there exists } k \in \mathbf{N} \text{ so that for all } m \in \mathbf{N} : \| f \|_{k,m} < \infty \}$$
$$= \underset{k\to}{ind}\ \underset{\leftarrow m}{proj}\ A(\omega, k, m).$$

By [3], 3.5(1), the Fourier-Laplace transform

$$\mathcal{F} : \mathcal{D}_\omega(\mathbf{R}) \to A_\omega,$$

$$\mathcal{F}(f)[z] := \widehat{f}(z) := \int_{\mathbf{R}} f(t) e^{-itz} dt$$

is a linear topological isomorphism.

Convolution operators on $\mathcal{D}_\omega(\mathbf{R})'$ 1.6. Let ω be a weight function. For $\mu \in \mathcal{E}_\omega(\mathbf{R})'$ and $f \in \mathcal{E}_\omega(\mathbf{R})$ we define the convolution $\mu * f : \mathbf{R} \to \mathbb{C}$ by

$$\mu * f(x) := \langle \mu_y, f(x - y) \rangle.$$

By [3], 6.3, $\mu * f$ is in $\mathcal{E}_\omega(\mathbf{R})$. Moreover, by the same reference

$$S_\mu : \mathcal{D}_\omega(\mathbf{R})' \to \mathcal{D}_\omega(\mathbf{R})',$$

defined by

$$S_\mu(\nu) := \mu * \nu : \psi \mapsto \langle \nu, \check{\mu} * \psi \rangle, \quad \psi \in \mathcal{D}_\omega(\mathbf{R}),$$

where $\langle \check{\mu}, f \rangle = \langle \mu_x, f(-x) \rangle$, is a continuous linear map. S_μ is called the convolution operator on $\mathcal{D}_\omega(\mathbf{R})'$ which is induced by $\mu \in \mathcal{E}_\omega(\mathbf{R})'$. Note that by [3], 6.2, we have $S_\mu(f) = \mu * f$ for all $f \in \mathcal{E}_\omega(\mathbf{R})$.

Since $\mathcal{D}_\omega(\mathbf{R})$ is reflexive by [3], 5.6, the adjoint S_μ^t of S_μ can be regarded as a continuous linear operator on $\mathcal{D}_\omega(\mathbf{R})$. It is easy to check that on A_ω, defined in 1.5, we have

$$\mathcal{F} \circ S_\mu^t \circ \mathcal{F}^{-1} = M_{\widetilde{\mu}}, \tag{1}$$

where $M_{\widetilde{\mu}} : A_\omega \to A_\omega$ denotes the operator of multiplication by the function $\widetilde{\mu}$ which is defined by

$$\widetilde{\mu}(z) := \langle \mu_x, e^{ixz} \rangle, \quad z \in \mathbb{C}.$$

By $\widehat{\mu}$ we denote the function $\widehat{\mu} : z \mapsto \widetilde{\mu}(-z)$. Then $\widetilde{\mu}$ and $\widehat{\mu}$ are entire functions on $\mathbb{C}$ and there exist $m \in \mathbb{N}$ and $C > 0$ so that

$$(2) \qquad |\widetilde{\mu}(z)| \leq C\, exp(m|Im\, z| + m\omega(z)) \quad \text{for all } z \in \mathbb{C}.$$

Next note that by [3], 5.6, $\mathcal{D}_\omega(\mathbf{R})'$ is a complete nuclear space. Hence $ker\, S_\mu$ has these properties, too. By Schwartz [16], p. 43, this implies that $(ker\, S_\mu)'$, the strong dual of $ker\, S_\mu$, is ultrabornological. By the reflexivity of $\mathcal{D}_\omega(\mathbf{R})$ we can identify $(\mathcal{D}_\omega(\mathbf{R})')'$ with $\mathcal{D}_\omega(\mathbf{R})$. Then the restriction map

$$\rho : \mathcal{D}_\omega(\mathbf{R}) = (\mathcal{D}_\omega(\mathbf{R})')' \to (ker\, S_\mu)'$$

is continuous and surjective by the theorem of Hahn-Banach. Hence the open mapping theorem implies

$$(3) \qquad (ker\, S_\mu)' \cong \mathcal{D}_\omega(\mathbf{R})/ker\, \rho \cong \mathcal{D}_\omega(\mathbf{R})/(ker\, S_\mu)^\perp.$$

Since $\mathcal{D}_\omega(\mathbf{R})$ is reflexive, $(ker\, S_\mu)^\perp$ equals the closure $\overline{im S_\mu^t}$. Hence (1) and (3) imply

$$(4) \qquad (ker\, S_\mu)' \cong A_\omega / \overline{\widetilde{\mu} A_\omega},$$

where the isomorphism is induced by the map $\Phi := \rho \circ \mathcal{F}^{-1}$. Note that the theorem of Hahn-Banach implies

$$(5) \qquad \left\{ \begin{array}{l} \text{A subset } G \text{ of } (ker\, S_\mu)' \text{ is equicontinuous if and only if there exist } k \in \mathbf{N} \\ \text{and a bounded set } M \text{ in } \underset{\leftarrow m}{proj}\, A(\omega, k, m) \subset A_\omega \text{ so that } G = \Phi(M). \end{array} \right.$$

We want to characterize those $\mu \in \mathcal{E}_\omega(\mathbf{R})'$ for which the convolution operator S_μ is surjective on $\mathcal{D}_\omega(\mathbf{R})'$. A necessary condition for the surjectivity of S_μ is obviously that the equation

$$S_\mu(\nu) = \mu * \nu = \delta$$

has a solution $\nu \in \mathcal{D}_\omega(\mathbf{R})'$, i.e. that S_μ admits a fundamental solution ν. This property of S_μ was characterized already by Braun, Meise and Vogt [4], 2.7. From there and from the diameter estimates obtained in the proof of Meise, Taylor and Vogt [13], 2.3, we know:

Proposition 1.7. *Let ω be a weight function and let $\mu \in \mathcal{E}_\omega(\mathbf{R})'$ be given. Then S_μ admits a fundamental solution $\nu \in \mathcal{D}_\omega(\mathbf{R})'$ if and only if $\widetilde{\mu}$ is slowly decreasing in the following sense: there exist positive numbers ε, C and D such that on each component S of the set*

$$S(\widetilde{\mu},\varepsilon,C) := \{z \in \mathbb{C} \mid |\widetilde{\mu}(z)| < \varepsilon\, exp(-C|Im\ z| - C\omega(z))\}$$

we have

$$\sup_{z\in S}(|Im\ z| + \omega(z)) \leq D\left(1 + \inf_{z\in S}(|Im\ z| + \omega(z))\right).$$

If $\widetilde{\mu}$ is slowly decreasing then one can choose ε, C and D in such a way that

$$(*) \qquad \sup_{z\in S}\omega(z) \leq D\left(1 + \inf_{z\in S}\omega(z)\right)$$

holds for each component S of the set $S(\widetilde{\mu},\varepsilon,C)$.

In the next section we will show that $ker\ S_\mu$ has an absolute basis, whenever S_μ admits a fundamental solution. In doing this we shall use certain sequence spaces which we introduce now.

Definition 1.8. *Let α and β be sequences in $[0,\infty[$ with $\lim_{j\to\infty}\beta_j = \infty$ and let $\mathbf{E} = (E_j, \|\cdot\|_j)_{j\in\mathbf{N}}$ be a sequence of finite dimensional normed spaces. For $k,m \in \mathbf{N}$ we introduce the Banach spaces*

$$\lambda(k,m,\mathbf{E}) := \left\{x \in \prod_{j\in\mathbf{N}} E_j \,\middle|\, \|x\|_{k,m} := \sum_{j=1}^{\infty} \|x_j\|_j\, exp(k\alpha_j - m\beta_j) < \infty\right\}$$

$$K(k,m,\mathbf{E}) := \left\{x \in \prod_{j\in\mathbf{N}} E_j \,\middle|\, |||x|||_{k,m} := \sup_{j\in\mathbf{N}} \|x_j\|_j\, exp\left(-k\alpha_j + m\beta_j\right) < \infty\right\}$$

and we define

$$\lambda(\alpha,\beta,\mathbf{E}) := \underset{\leftarrow k}{proj}\ \underset{m\rightarrow}{ind}\ \lambda(k,m,\mathbf{E})$$

$$K(\alpha,\beta,\mathbf{E}) := \underset{m\rightarrow}{ind}\ \underset{\leftarrow k}{proj}\ K(k,m,\mathbf{E})$$

If $\mathbf{E} = (\mathbb{C}, |\cdot|)_{j\in\mathbf{N}}$ then we write $\lambda(\alpha,\beta)$, $\lambda(k,m)$ etc. instead.

The proof of Meise [9], 1.6, also applies to the present sequence spaces and gives:

Proposition 1.9. *For α, β and $\mathbf{E}$ as in 1.8, the following holds:*

(1) $\lambda(\alpha, \beta, \mathbf{E})$ is a complete Schwartz space

(2) $\lambda(\alpha, \beta, \mathbf{E})'$, the strong dual of $\lambda(\alpha, \beta, \mathbf{E})$, can be identified with $K(\alpha, \beta, \mathbf{E}')$, where $\mathbf{E}' = (E_j', \|\cdot\|_j')_{j\in\mathbf{N}}$.

(3) With respect to the duality in (2), a set $M \subset K(\alpha, \beta, \mathbf{E}')$ is equicontinuous if and only if there exists $k \in \mathbf{N}$ so that

$$\sup_{y\in M}\sup_{j\in\mathbf{N}} \| y_j \|_j' \, exp(-k\alpha_j + m\beta_j) < \infty \text{ for all } m \in \mathbf{N}.$$

Also, the proof of Lemma 1.7 in Meise [9] gives:

Lemma 1.10. *For α, β and $\mathbf{E}$ as in 1.8 assume $1 \leq n_j := dim\, E_j$ for all $j \in \mathbf{N}$. Then the condition*

$$(*) \qquad \text{there exists } l \in \mathbf{N} \text{ such that } \sup_{j\in\mathbf{N}} n_j \, exp(-l(\alpha_j + \beta_j)) < \infty$$

implies

$$\lambda(\alpha, \beta, \mathbf{E}) \cong \lambda(\gamma, \delta) \text{ and } K(\alpha, \beta, \mathbf{E}) \cong K(\gamma, \delta),$$

where the sequence γ (resp. δ) is obtained from α (resp. β) by repeating α_j (resp. β_j) n_j times.

2. GENERALIZED FOURIER EXPANSION

In this section let ω always denote a fixed weight function. We will show that for each convolution operator S_μ on $\mathcal{D}_\omega(\mathbf{R})'$ which admits a fundamental solution, $ker\, S_\mu$ admits an absolute basis of exponential solutions. With respect to this basis $ker\, S_\mu$ is isomorphic to a suitable sequence space $\lambda(\alpha, \beta)$, whenever $ker\, S_\mu$ is infinite dimensional. The idea of proof for this is the same as in Meise [10], however, some modifications are needed. In particular we use a result of Braun and Meise [2] to overcome the difficulty that the function $z \mapsto |Im\, z| - m\omega(z)$ is not subharmonic in general. The proof of the main result is prepared by several lemmas.

Lemma 2.1. *Assume that $f \in C(\mathbf{R})$ satisfies $Supp(f) \subset [-A, A]$ for some $A > 0$ and*

$$\int_{\mathbf{R}} |\widehat{f}(t)| \, exp(K\omega(t))\,dt < \infty,$$

where K is the constant appearing in 1.1 (α). Then for each $\varepsilon > 0$ there exists $g \in \mathcal{D}_\omega(\mathbf{R})$ with $Supp(g) \subset [-A, A]$ such that

$$\int_{\mathbf{R}} |\widehat{f}(t) - \widehat{g}(t)| \, exp(\omega(t))\,dt \leq \varepsilon.$$

Proof. Let $\varepsilon > 0$ be given. By the hypothesis on f we can choose $C > 0$ so that

$$\int_{2|t|\geq C} |\widehat{f}(t)| \, exp(K\omega(t)) \mathrm{d}t \leq \frac{\varepsilon e^{-K}}{6}.$$

For $q > 1$, let f_q denote the function $x \mapsto f(qx)$. Since f_q converges to f uniformly on $\mathbf{R}$ as q tends to 1, we can find $1 < p < 2$ so that

$$\int_{\mathbf{R}} |f(x) - f_p(x)| \mathrm{d}x \leq \frac{\varepsilon e^{-\omega(C)}}{6C}.$$

Next note that by Braun, Meise and Taylor [3], 2.6, for each $0 < \delta < 1$ there exists $h_\delta \in \mathcal{D}_\omega(\mathbf{R})$ satisfying

$$h_\delta \geq 0, \quad Supp(h_\delta) \subset [-\delta, \delta] \text{ and } \int_{\mathbf{R}} h_\delta(x) \mathrm{d}x = 1.$$

Since $\lim_{\delta\downarrow 0} \widehat{h}_\delta(x) = 1$ for all $x \in \mathbf{R}$, we can choose $0 < \eta < A(1 - p^{-1})$ so that

$$\int_{\mathbf{R}} |\widehat{f}(t) \left(1 - \widehat{h}_\eta(t)\right)| \, exp(\omega(t)) \mathrm{d}t \leq \frac{\varepsilon}{3}.$$

Now define $g := f_p * h_\eta$ and note that g is in $\mathcal{D}_\omega[-A, A]$. Moreover, the following estimate holds:

$$\int_{\mathbf{R}} | \widehat{f}(t) - \widehat{g}(t)| \, exp(\omega(t)) \mathrm{d}t =$$

$$= \int_{\mathbf{R}} | \widehat{f}(t) - \widehat{f}(t)\widehat{h}_\eta(t) + \widehat{f}(t)\widehat{h}_\eta(t) - \widehat{f}_p(t)\widehat{h}_\eta(t)| \, exp(\omega(t)) \mathrm{d}t \leq$$

$$\leq \int_{\mathbf{R}} | \widehat{f}(t)(1 - \widehat{h}_\eta(t))| \, exp(\omega(t)) \mathrm{d}t + \int_{\mathbf{R}} | \widehat{f}(t) - \widehat{f}_p(t)| \, exp(\omega(t)) \mathrm{d}t \leq$$

$$\leq \frac{\varepsilon}{3} + \int_{|t|\leq C} | \widehat{f}(t) - \widehat{f}_p(t)| \, exp(\omega(t)) \mathrm{d}t + \int_{|t|\geq C} | \widehat{f}(t)| \, exp(\omega(t)) \mathrm{d}t +$$

$$+ \int_{|t|\geq C} | \widehat{f}_p(t)| \, exp(\omega(t)) \mathrm{d}t \leq$$

$$\leq \frac{\varepsilon}{3} + 2Ce^{\omega(C)} \sup_{t\in\mathbf{R}} | (\widehat{f} - \widehat{f}_p)(t)| + \int_{|t|\geq C} | \widehat{f}(t)| \, exp(\omega(t)) \mathrm{d}t +$$

$$+ \int_{p|t|\geq C} | \widehat{f}(t)| \, exp(\omega(pt)) \mathrm{d}t \leq$$

$$\leq \frac{\varepsilon}{3} + 2Ce^{\omega(C)} \int_{\mathbf{R}} |f(x) - f_p(x)| \mathrm{d}x + 2e^K \int_{2|t|\geq C} |\widehat{f}(t)| \, exp(K\omega(t)) \mathrm{d}t \leq$$

$$\leq \varepsilon.$$

Definition 2.2. *For $A, B > 0$ define the Hilbert space*

$$L_{A,B} := \left\{ f \in L^2_{loc}(\mathbb{C}) \mid \| f \|^2_{A,B} := \int_{\mathbb{C}} \{|f(z)| \, exp(-A|Im\, z| + B\omega(z))\}^2 \, d\lambda(z) < \infty \right\}$$

where λ denotes the Lebesgue measure on $\mathbb{C} = \mathbb{R}^2$.

Moreover, we define the Fréchet space

$$L_A := \underset{\leftarrow m}{proj}\, L_{Am}.$$

Lemma 2.3. *For each $A > 0$ and each bounded set M in L_A there exists a bounded set Q in L_{2A} such that for each $u \in M$ there exists $v \in Q$ with $\overline{\partial} v = u$ in the distributional sense.*

Proof. For $A, B > 0$ and $K \geq 1$ as in $1.1(\alpha)$ we let

$$Y_{2A,B} := \left\{ f \in L_{2A,B} / \overline{\partial} f \in L_{A,2KB} \right\}$$

and we endow $Y_{2A,B}$ with the graph norm

$$|f|_{2A,B} := \| f \|_{2A,B} + \| \overline{\partial} f \|_{A,2KB}, \qquad f \in Y_{2A,B}.$$

Then $Y_{2A,B}$ is a Banach space and we claim that

(1) $$\overline{\partial} : Y_{2A,B} \to L_{A,2KB} \text{ is surjective for all } A, B > 0.$$

To prove this, we recall from Braun and Meise [2], Prop. 5, that there exist a subharmonic function $u : \mathbb{C} \to \mathbb{R}$ and $C > 0$ so that for all $z \in \mathbb{C}$ we have

$$-C - K\frac{2B}{A}\omega(Re z) \leq u(z) \leq |Im\, z| - \frac{2B}{A}\omega(z).$$

Now define $v : z \mapsto Au(z) + A|Im\, z|$, $z \in \mathbb{C}$, and note that

(2) $$A|Im\, z| - 2KB\omega(z) - AC \leq v(z) \leq 2A|Im\, z| - 2B\omega(z)$$

holds for all $z \in \mathbb{C}$. Since ω satisfies $1.1(\gamma)$, we can choose $D > 0$ so that

(3) $$log(1 + |z|^2) \leq B\omega(z) + D \quad \text{for all } z \in \mathbb{C}.$$

Next let $f \in L_{A,2KB}$ be given. Then (2) implies

$$(4) \qquad \int [|f(z)| exp(-v(z))]^2 d\lambda(z) \leq exp(2AC) \, \| f \|^2_{A,2KB} \, .$$

Since v is subharmonic on $\mathbb{C}$, Hörmander [6], 4.4.2, implies the existence of $g \in L^2_{loc}(\mathbb{C})$ so that $\overline{\partial} g = f$ in the distributional sense and

$$\int_{\mathbb{C}} \left[|g(z)| \frac{exp(-v(z))}{1+|z|^2} \right]^2 d\lambda(z) \leq \int_{\mathbb{C}} [|f(z)| exp(-v(z))]^2 d\lambda(z).$$

Because of (2), (3) and (4), this implies

$$\| g \|^2_{2A,B} \leq exp(2D + 2AC) \, \| f \|^2_{A,2KB} < \infty.$$

Consequently, g is in $L_{2A,B}$ and the proof of (1) is complete.
For $A > 0$ and $n \in \mathbf{N}$ we now define

$$W_{2A,n} := \left\{ f \in A(\mathbb{C}) \mid \| f \|_{2A,n} < \infty \right\} = \left\{ f \in L_{2A,n} | \overline{\partial} f = 0 \right\}.$$

Then $W_{2A,n+1} \subset W_{2A,n}$, so that we can consider the projective spectrum $(W_{2A,n}, \iota^n_{n+1})_{n \in \mathbf{N}}$ with inclusion maps (see 3.1 for the notation). It is easy to see that this spectrum is equivalent to the spectrum $(A(\omega, 2A, n), j^n_{n+1})_{n \in \mathbf{N}}$, again with inclusion maps. From Braun, Meise and Taylor [3], 3.3, it follows that this projective spectrum is equivalent to the spectrum $(\mathcal{D}_{2A,n}, \kappa^n_{n+1})$, where

$$\mathcal{D}_{2A,n} := \{ f \in C(\mathbf{R}) \mid Supp(f) \subset [-2A, 2A] \text{ and }$$
$$\| f \|_n := \int_{\mathbf{R}} |\widehat{f}(t)| exp(n\omega(t)) dt < \infty \}.$$

and where κ^n_{n+1} denotes the corresponding inclusion map. From these equivalences and Lemma 2.1 we get:

(5) For each $j \in \mathbf{N}$ there exists $k \in \mathbf{N}, k > j$, so that $\underset{\leftarrow n}{proj}\, W_{2A,n}$ is dense in $W_{2A,k}$ with respect to the topology induced by $W_{2A,j}$.

Moreover, $\underset{\leftarrow n}{proj}\, W_{2A,n}$ is isomorphic to $\underset{\leftarrow n}{proj}\, \mathcal{D}_{2A,n} = \mathcal{D}_\omega[-2A, 2A]$ and hence a nuclear Fréchet space by [3], 3.6. In particular, we have

$$(6) \qquad \underset{\leftarrow n}{proj}\, W_{2A,n} \text{ is quasinormable.}$$

Now note that by (1) we have for each $n \in \mathbf{N}$ the exact sequence of Banach spaces

$$(7) \qquad 0 \to W_{2A,n} \hookrightarrow Y_{2A,n} \xrightarrow{\overline{\partial}} L_{A,2Kn} \to 0.$$

From this and (5) it follows by Komatsu [7], 1.3, that

$$0 \to \underset{\leftarrow n}{proj}\, W_{2A,n} \hookrightarrow \underset{\leftarrow n}{proj}\, Y_{2A,n} \xrightarrow{\overline{\partial}} L_A \to 0$$

is an exact sequence of Fréchet spaces. Therefore, (6) and Merzon [14], Thm. 2, implies that for each bounded set M in L_A there exists a bounded set Q in $\underset{\leftarrow n}{proj}\, Y_{2A,n}$ with $\overline{\partial}(Q) = M$. Since $\underset{\leftarrow n}{proj}\, Y_{2A,n}$ is continuously embedded in L_{2A}, the proof is complete.

Remark. Lemma 2.3 remains true for $\omega : t \mapsto log(1+t)$. The only change in the proof is that 2.3(1) holds for all B which are sufficiently large.

Lemma 2.4. *(Semi-local to global interpolation). Let $F = (F_1, \dots, F_N)$ be an N-tuple of entire functions which satisfy*

(i) there exist $A_0, B_0 > 0$ with $\sup_{1 \le j \le N} \sup_{z \in \mathbb{C}} |F_j(z)| exp(-A_0(|Im\, z| + \omega(z))) \le B_0$.

(ii) there exist positive numbers ε, C, D such that for each component S of the set

$$S(F, \varepsilon, C) := \left\{ z \in \mathbb{C} \mid \left(\sum_{j=1}^{N} |F_j(z)|^2 \right)^{1/2} < \varepsilon\, exp(-C(|Im\, z| + \omega(z))) \right\}$$

we have

$$\sup_{z \in S}(|Im\, z| + \omega(z)) \le D \left(1 + \inf_{z \in S}(|Im\, z| + \omega(z)) \right).$$

Furthermore, let Q be a set of holomorphic functions defined on $S(F, \varepsilon, C)$ which satisfies *(iii) there exist* $A_1 > 0$ *such that for each* $m \in \mathbf{N}$ *there exists* $B_m > 0$ *:*

$$\sup_{f \in Q} \sup_{z \in S(F,\varepsilon,C)} |f(z)| exp(-A_1 |Im\ z| + m\omega(z)) \le B_m.$$

Then there exist $0 < \varepsilon_1 < \varepsilon$, $C_1 > C$, $M > 0$ *and a sequence* $(E_m)_{m \in \mathbf{N}}$ *of positive numbers, such that the following holds:*

For each $f \in Q$ *there exists* $g \in A(\mathbb{C})$ *and* $\alpha_j \in A(S(F, \varepsilon_1, C_1))$ *for* $1 \le j \le N$, *such that*

$$g(z) = f(z) + \sum_{j=1}^{N} \alpha_j(z) F_j(z) \quad \textit{for all} \quad z \in S(F, \varepsilon_1, C_1)$$

and

$$\sup_{z \in \mathbb{C}} |g(z)|\ exp(-M|Im\ z| + m\omega(z)) \le E_m \quad \text{for each} \quad m \in \mathbf{N}.$$

Proof. From (ii) it follows (see Berenstein and Taylor [1], p. 120) that we can find $0 < \varepsilon_1 < \varepsilon$, $C_1 > C$, $A, B > 0$ and $\chi \in C^\infty(\mathbb{C})$ with $Supp(\chi) \subset S(F,\varepsilon,C)$ and $0 \le \chi \le 1$ so that

$$(1) \qquad \chi|_{S(F,\varepsilon_1,C_1)} \equiv 1, \quad |\overline{\partial}\chi(z)| \le B\ exp(A(|Im\ z| + \omega(z))) \quad \text{for all } z \in \mathbb{C}.$$

Next fix $f \in Q$. Then χf is in $C^\infty(\mathbb{C})$ and $\overline{\partial}(\chi f) = (\overline{\partial}\chi) f$.

Moreover, (1) implies that for $1 \le j \le N$

$$v_j^f := -\overline{F}_j \left(\sum_{k=1}^{N} |F_k|^2 \right)^{-1} \overline{\partial}(\chi f)$$

is in $C^\infty(\mathbb{C})$ and that

$$Supp(v_j^f) \subset S(F,\varepsilon,C) \setminus S(F,\varepsilon_1,C_1).$$

From the hypothesis and (1) we get:

$$|v_j^f| \le B_m B_0 B \frac{1}{\varepsilon^2} exp\left((A + A_0 + A_1 + 2C)|Im\ z| + (A + A_0 + 2C - m)\omega(z)\right)$$

for all $m \in \mathbf{N}$, $z \in \mathbb{C}$, $1 \le j \le N$ and all $f \in Q$. This shows that

$$P := \left\{ v_j^f | f \in Q, \quad 1 \le j \le N \right\}$$

is a bounded set in L_S, $S := A + A_0 + A_1 + 2C$. By Lemma 2.3 we can choose a bounded set R in L_{2S} so that for each $f \in Q$ and $1 \le j \le N$ there exists $v_j^f \in R$ satisfying $\overline{\partial} u_j^f = v_j^f$. Since P is contained in $C^\infty(\mathbb{C})$, u_j^f is in $C^\infty(\mathbb{C})$ for all $f \in Q$, $1 \le j \le N$. For $f \in Q$ we now define

$$(2) \qquad g^f := \chi f + \sum_{j=1}^N u_j^f F_j \quad \text{and} \quad \alpha_j^f := u_j^f|_{S(F,\varepsilon_1,C_1)}, \quad 1 \le j \le N.$$

Then $g^f \in A(\mathbb{C})$ and $\alpha_j^f \in A(S(F, \varepsilon_1, C_1))$ for $1 \le j \le N$, since

$$\overline{\partial} g^f = \overline{\partial}(\chi f) + \sum_{j=1}^N \left(\overline{\partial} u_j^f\right) F_j = \overline{\partial}(\chi f) - \left(\sum_{j=1}^N \overline{F}_j F_j \overline{\partial}(\chi f)\right)\left(\sum_{j=1}^N |F_k|^2\right)^{-1} = 0$$

and

$$\overline{\partial}\alpha_j^f = \overline{\partial} u_j^f|_{S(F,\varepsilon_1,C_1)} = v_j^f|_{S(F,\varepsilon_1,C_1)} = 0.$$

This proves the first assertion. The second one follows by standard arguments from (2), (i), (ii) and the fact that P is bounded in L_{2S}.

In order to apply Lemma 2.4, we introduce the following notation.

Notation 2.5. For an N-tuple $F = (F_1, \ldots, F_N)$ of entire functions let

$$V(F) := \left\{z \in \mathbb{C} \,|\, F_j(z) = 0 \text{ for } 1 \le j \le N\right\}.$$

For $a \in V(F)$ we define $m_\alpha := \min_{1 \le j \le N} \, ordF_j(a)$, where $ordf(a)$ denotes the zero-order of f at a. Then we let

$$I_{loc}(F) := \{f \in A_\omega \,|\, ord\, f(a) \ge m_\alpha \text{ for all } a \in V(F)\},$$

$$I(F) := \left\{ f \in A_\omega \,|\, f = \sum_{j=1}^N g_j F_j, g_j \in A_\omega \text{ for } 1 \le j \le N \right\}.$$

It is easy to see that $I(F)$ and $I_{loc}(F)$ are ideals in A_ω satisfying $I(F) \subset I_{loc}(F)$ and that $I_{loc}(F)$ is closed.

Proposition 2.6. *Assume that $F = (F_1, \ldots, F_N) \in A(\mathbb{C})^N$ satisfies the conditions (i) and (ii) in 2.4. Then we have:*

(a) $\overline{I(F)} = I_{loc}(F)$

(b) If $N = 1$ then $I(F)$ is closed.

Proof. (a) Because of the remark at the end of 2.5 it sufficies to show that $I_{loc}(F) \subset \overline{I(F)}$. To prove this, define $p : \mathbb{C} \to [0, \infty[$ by $p(z) := |Imz| + \omega(z)$ and let

$$A_p := \left\{ f \in A(\mathbb{C}) | \text{ there is } k \in \mathbb{N} : \| f \|_k := \sup_{z \in \mathbb{C}} |f(z)| \, exp(-kp(z)) < \infty \right\}.$$

Endowed with its natural inductive limit topology, A_p is a (DFN)-algebra which contains A_ω as a subalgebra. From 2.4(i) it follows that $F \in (A_p)^N$. Now define $I^p(F)$ and $I^p_{loc}(F)$ as in 2.5, however with A_p instead of A_ω. Then 2.4(ii) together with Kelleher and Taylor [8], Thm. 4.6, implies that $I^p_{loc}(F) = \overline{I^p(F)}$. Now fix $f \in I_{loc}(F) \subset I^p_{loc}(F)$ and note that for each $g \in A_\omega$ the multiplication map $M_g : A_p \to A_\omega$, $M_g(h) := gh$, is continuous and satisfies $M_g(I^p(F)) \subset I(F)$. Hence we get

$$fg = M_g(f) \in M_g(I^p_{loc}(F)) = M_g(\overline{I^p(F)}) \subset \overline{M_g(I^p(F))} \subset \overline{I(F)}.$$

Now we apply this to $g = \widehat{\varphi}_\varepsilon$, where $\varphi_\varepsilon : x \mapsto \frac{1}{\varepsilon}\varphi(\frac{x}{\varepsilon})$, $\varepsilon > 0$, for some function $\varphi \in \mathcal{D}_\omega(\mathbf{R})$ satisfying $\varphi \geq 0$ and $\int_{\mathbf{R}} \varphi d\lambda = 1$. Since it is easy to check that $f = A_\omega - \lim_{\varepsilon \downarrow 0} f\widehat{\varphi}_\varepsilon$, we get $f \in \overline{I(F)}$, which proves (a).

(b) It suffices to show $I_{loc}(F) \subset I(F)$. To prove this, fix $g \in I_{loc}(F)$. Then $\frac{f}{F}$ is in $A(\mathbb{C})$ and it is easy to check that 2.4(ii) and 1.7 together with the maximum principle implies $\frac{f}{F} \in A_\omega$.

Proposition 2.7. *Let $F = (F_1, \ldots, F_N)$ be an N-tuple of entire functions which satisfies the conditions 2.4(i) and (ii). If $V(F)$ is an infinite set then $A_\omega / I_{loc}(F)$ is linearly isomorphic to $\lambda(\alpha, \beta)'$, where $\alpha = (|Im\ a_j|)_{j \in \mathbf{N}}$ and $\beta = (\omega(a_j))_{j \in \mathbf{N}}$ and where the sequence $(a_j)_{j \in \mathbf{N}}$ counts the elements of $V(F)$ according to their multiplicities (m_a at $a \in V(F)$).*

Proof. Fix $\varepsilon, C, D > 0$ as in 2.4(ii) and choose an enumeration $(S_j)_{j \in \mathbf{N}}$ of those components S of $S(F, \varepsilon, C)$ which satisfy $S \cap V(F) \neq \emptyset$. Then define the sequence $\gamma = (\gamma_j)_{j \in \mathbf{N}}$ and $\delta = (\delta_j)_{j \in \mathbf{N}}$ by

$$\gamma_j := \sup_{z \in S_j} |Im\ z|, \qquad \delta_j := \sup_{z \in S_j} \omega(z).$$

Next define for $j \in \mathbf{N}$:

$$A^\infty(S_j) := \left\{ f \in A\left(S_j\right) | \, \| f \|_j := \sup_{z \in S_j} |f(z)| < \infty \right\}$$

$$E_j := \prod_{a \in V(F) \cap S_j} \mathbb{C}^{m_a}$$

$$\rho_j : A^\infty(S_j) \to E_j, \quad \rho_j(f) := \left(f^{(k)}(a)\right)_{a \in V(F) \cap S_j, 0 \le k < m_a}.$$

It is easy to see that ρ_j is surjective. Hence we can endow E_j with the corresponding quotient norm, i.e.

$$\| \mu \|_j := inf\{\| f \|_j \, | f \in A^\infty(S_j), \quad \rho_j(f) = \mu\}, \quad \text{for } \mu \in E_j.$$

By $\mathbf{E}$ we denote the sequence $\mathbf{E} = (E_j, \| \cdot \|_j)_{j \in \mathbf{N}}$. Then we remark that by the proof of Meise, Momm and Taylor [11], 3.5 (which is almost the same as of Meise, Taylor and Vogt [13], 2.3) property 1.7(*) also holds for the components S of $S(F, \varepsilon, C)$ provided that ε and C are chosen appropriately. From this and 2.4(i) it follows easily that the map

$$\rho : A_\omega \to K(\gamma, \delta, \mathbf{E}), \quad \rho(f) := \left(\rho_j(f | S_j)\right)_{j \in \mathbf{N}}$$

is defined linear and continuous and that $ker\, \rho = I_{loc}(F)$. To prove the surjectivity of ρ, fix $\mu \in K(\gamma, \delta, \mathbf{E})$. Then there exists $k \in \mathbf{N}$ so that for each $l \in \mathbf{N}$ there exists $C_l > 0$ so that

$$\| \mu_j \|_j \le C_l \, exp\left(k\gamma_j - l\delta_j\right) \quad \text{for all} \quad j \in \mathbf{N}.$$

By the definition of $\| \cdot \|_j$ we can choose $f_j \in A^\infty(S_j)$ with $\rho_j(f_j) = \mu_j$ and $\| f_j \|_j \le 2 \, \| \mu_j \|_j$. Now define $f : S(F, \varepsilon, C) \to \mathbb{C}$ by

$$f(z) := \begin{cases} f_j(z) & \text{if } z \in S_j \\ 0 & \text{if } z \in S(F, \varepsilon, C) \setminus \bigcup_{k=1}^\infty S_k. \end{cases}$$

From 2.4(ii) we get for each $l \in \mathbf{N}$:

$$|f(z)| \le 2 C_l e^{Dk} \, exp(Dk|Im\, z| - (l - Dk)\omega(z)) \quad \text{for all } z \in S(F, \varepsilon, C).$$

Hence f satisfies 2.4(iii). Therefore, Lemma 2.4 implies the existence of $g \in A_\omega$ with $\rho(g) = \mu$. Thus ρ is surjective. Now the open mapping theorem for (LF)-spaces together with $ker\, \rho = I_{loc}(F)$ implies

$$A_\omega / I_{loc}(F) \cong K(\gamma, \delta, \mathbf{E}).$$

Next let $n_j := dim E_j$ for $j \in \mathbf{N}$ and note that by Remark b) to Cor. 3.8 in Meise [9], there exists $l \in \mathbf{N}$ so that

$$\sup_{j\in\mathbf{N}} n_j \, exp\left(-l\left(\gamma_j + \delta_j\right)\right) < \infty.$$

Hence Lemma 1.10 and Lemma 1.9 imply

$$A_\omega / I_{loc}(F) \cong K(\gamma, \delta, \mathbf{E}) = \lambda(\gamma, \delta, \mathbf{E}')' = \lambda(\overline{\gamma}, \overline{\delta})',$$

where $\overline{\gamma}$ (resp. $\overline{\delta}$) is obtained by repeating γ_j (resp. δ_j) n_j-times. Now the conclusion follows easily from this and 2.4(ii) together with the definition of γ and δ.

Remark 2.8. If we identify in Proposition 2.7 the quotient $A_\omega / I_{loc}(F)$ with $\lambda(\alpha, \beta)'$ then a set G in $\lambda(\alpha, \beta)'$ is equicontinuous if and only if there exist $k \in \mathbf{N}$ and a bounded set M in $\underset{\leftarrow m}{proj}\, A(\omega, k, m)$ so that $G = \rho(M)$. This follows from the characterization in 1.9(3) and the proof of Proposition 2.7 together with Lemma 2.4.

Exponential solutions 2.9. For $a \in \mathbb{C}$ and $k \in \mathbf{N}_0$ we define

$$e_{a,k} : x \mapsto (ix)^k e^{ixa}, \quad x \in \mathbf{R}.$$

It is easy to check that $e_{a,k}$ is in $\mathcal{E}_\omega(\mathbf{R})$ for each weight function ω. Now fix ω and $\mu \in \mathcal{E}_\omega(\mathbf{R})'$. Then the elements of

$$span\left\{e_{a,k} | a \in V(\widehat{\mu}), \quad 0 \le k < m_a\right\}$$

are called exponential solutions of the convolution operator S_μ. This notation is justified by the following identity which is a consequence of the definitions and remarks in 1.6:

$$\begin{aligned} S_\mu(e_{a,k})[x] &= \langle \mu_y, e_{a,k}(x-y)\rangle = \langle \mu_y, (i(x-y))^k e^{i(x-y)a}\rangle \\ &= \sum_{j=0}^{k} \binom{k}{j} (ix)^{k-j} e^{ixa} \langle \mu_y, (-iy)^j e^{-iya}\rangle \\ &= \sum_{j=0}^{k} \binom{k}{j} (ix)^{k-j} e^{ixa} \widehat{\mu}^{(j)}(a) = 0. \end{aligned}$$

Theorem 2.10. *Let S_μ be a convolution operator on $\mathcal{D}_\omega(\mathbf{R})'$ which admits a fundamental solution and assume that $ker\ S_\mu$ is infinite dimensional. Then $ker\ S_\mu$ admits an absolute basis of exponential solutions, with respect to which $ker\ S_\mu$ is isomorphic to $\lambda(\alpha,\beta)$, where $\alpha = (|Im\ a_j|)_{j\in\mathbf{N}}$ and $\beta = (\omega(a_j))_{j\in\mathbf{N}}$ and where the sequence $(a_j)_{j\in\mathbf{N}}$ counts the zeros of $\widetilde{\mu}$ with multiplicities.*

Proof. By 1.6(2) we know that $F := \widetilde{\mu}$ satisfies condition 2.4(i). Since S_μ admits a fundamental solution, Proposition 1.7 shows that F also satisfies condition 2.4(ii). Therefore Proposition 2.7 together with 1.6(4) and Proposition 2.6 implies that for $F = \widetilde{\mu}$ we have the following isomorphisms

$$(ker\ S_\mu)' \cong A_\omega/\overline{\widetilde{\mu}A_\mu} = A_\omega/I_{loc}(\widetilde{\mu}) \cong \lambda(\alpha,\beta)'.$$

Moreover, 1.7(5) and Remark 2.8 imply that the resulting isomorphism induces a bijection between the equicontinuous sets in $(ker\ S_\mu)'$ and the equicontinuous sets in $\lambda(\alpha,\beta)'$. Hence it is the adjoint of an isomorphism Φ between $ker\ S_\mu$ and $\lambda(\alpha,\beta)$. If one computes $\Phi : \lambda(\alpha,\beta) \to ker\ S_\mu$ explicitly (see the proof of Meise, Schwerdtfeger and Taylor [12], 2.6), then it follows that Φ maps the canonical basis vectors of $\lambda(\alpha,\beta)$ into exponential solutions.

More generally one can prove (using [12], 2.4):

Theorem 2.11. *Let $S_{\mu_1},\dots,S_{\mu_N}$ be convolution operators on $\mathcal{D}'_\omega(\mathbf{R})$ and assume that $F = (\widetilde{\mu}_1,\dots,\widetilde{\mu}_N)$ satisfies the conditions 2.4(i) and 2.4(ii). If $\cap_{j=1}^N ker\ S_{\mu_j}$ is infinite dimensional then $\cap_{j=1}^N ker\ S_{\mu_j} = ker(S_{\mu_1},\dots,S_{\mu_N})$ admits an absolute basis of exponential solutions with respect to which $ker(S_{\mu_1},\dots,S_{\mu_N})$ is isomorphic to $\lambda(\alpha,\beta)$, where $\alpha = (|Im\ a_j|)_{j\in\mathbf{N}}$ and $\beta = (\omega(a_j))_{j\in\mathbf{N}}$ and where the sequence $(a_j)_{j\in\mathbf{N}}$ counts $V(\widetilde{\mu}_1,\dots,\widetilde{\mu}_N)$ with multiplicities.*

Remark 2.12. Theorem 2.10 and Theorem 2.11 also hold for $\mathcal{D}'(\mathbf{R})$ instead of $\mathcal{D}'_\omega(\mathbf{R})$ as their proofs show.

3. SURJECTIVITY OF CONVOLUTION OPERATORS ON $\mathcal{D}_\omega(\mathbf{R})'$

Following the arguments in section 3 of Braun, Meise and Vogt [4] we use results of Palamodov [15] and Vogt [17], [18] together with Theorem 2.10 to show that a convolution operator S_μ acts sufjectively on $\mathcal{D}_\omega(\mathbf{R})'$ if and only if S_μ admits a fundamental solution. To do this, we recall the following notions concerning projective spectra from Vogt [17], [18].

Projective Spectra 3.1. (1) A sequence $\mathcal{X} = (X_n, \iota^n_{n+1})_{n\in\mathbf{N}}$ of linear spaces X_n and linear maps $\iota^n_{n+1} : X_{n+1} \to X_n$ is called a projective spectrum. We define ι^n_m for $n \le m$ by $\iota^n_n := \mathrm{id}_{X_n}$ and $\iota^n_m := \iota^n_{n+1} \circ \dots \circ \iota^{m-1}_m$ for $m > n$.

(2) For a projective spectrum $\mathcal{X} = (X_n, \iota^n_{n+1})_{n\in\mathbf{N}}$ we define the linear spaces $Proj^0\mathcal{X}$ and $Proj^1\mathcal{X}$ by

$$Proj^0\mathcal{X} := \left\{(x_n)_{n\in\mathbf{N}} \in \prod_{n\in\mathbf{N}} X_n \middle| \iota^n_{n+1}(x_{n+1}) = x_n \text{ for all } n\in\mathbf{N}\right\}$$

$$Proj^1\mathcal{X} := \left(\prod_{n\in\mathbf{N}} X_n\right)/B(X),$$

where

$$B(\mathcal{X}) := \left\{(a_n)_{n\in\mathbf{N}} \in \prod_{n\in\mathbf{N}} X_n \middle| \text{ there is } (b_n)_{n\in\mathbf{N}} \in \prod_{n\in\mathbf{N}} X_n \text{ with } a_n = \iota^n_{n+1}(b_{n+1}) - b_n \text{ for all } n\in\mathbf{N}\right\}.$$

(3) For projective spectra $\mathcal{X} = (X_n, \iota^n_{n+1})$ and $\mathcal{Y} = (Y_n, \iota^n_{n+1})$ a map $\Phi : \mathcal{X} \to \mathcal{Y}$ is a sequence $\varphi^n_{k(n)} : X_{k(n)} \to \mathcal{Y}_n$ of linear maps which satisfies for all $n\in\mathbf{N}$:

$$k(n) \le k(n+1) \text{ and } \varphi^n_{k(n)} \circ \iota^{k(n)}_{k(n+1)} = j^n_{n+1} \circ \varphi^{n+1}_{k(n+1)}.$$

For $m \ge k(n)$ we put $\varphi^n_m := \varphi^n_{k(n)} \circ \iota^{k(n)}_m$.

(4) Let the maps $\Phi : \mathcal{X} \to \mathcal{Y}$ and $\Psi : \mathcal{Y} \to \mathcal{Z}$ be defined by $(\varphi^n_{k(n)})_{n\in\mathbf{N}}$ and $(\psi^n_{l(n)})_{n\in\mathbf{N}}$. Then their composition $\Psi\circ\Phi : \mathcal{X} \to \mathcal{Z}$ is defined by $\chi^n_{k(l(n))} := \psi^n_{l(n)} \circ \varphi^{l(n)}_{k(l(n))}$.

(5) Two maps $\Phi, \Psi : \mathcal{X} \to \mathcal{Y}$ defined by $(\varphi^n_{k(n)})_{n\in\mathbf{N}}$ and $(\psi^n_{l(n)})_{n\in\mathbf{N}}$ respectively, are called equivalent, if for each $n\in\mathbf{N}$ there exists $m(n) \ge \max(k(n), l(n))$ with

$$\varphi^n_{m(n)} = \psi^n_{m(n)}.$$

(6) Two projective spectra $\mathcal{X}$ and $\mathcal{Y}$ are called equivalent, if there exist maps $\Phi : \mathcal{X} \to \mathcal{Y}$ and $\Psi : \mathcal{Y} \to \mathcal{X}$ such that $\Phi\circ\Psi$ is equivalent to $id_{\mathcal{Y}} := (j^n_{n+1})_{n\in\mathbf{N}}$ and $\Psi\circ\Phi$ is equivalent to $id_{\mathcal{X}} := (\iota^n_{n+1})_{n\in\mathbf{N}}$.

Example 3.2. Let ω be a fixed weight function.

(1) For $n\in\mathbf{N}$ we let

$$\mathcal{D}'_\omega(\mathbf{R},]-n,n[) := \{\nu\in\mathcal{D}'_\omega(\mathbf{R}) | Supp\,\nu \subset \mathbf{R}\setminus]-n,n[\}.$$

Then $\mathcal{D}'_\omega(\mathbf{R},]-n,n[)$ is a closed linear subspace of $\mathcal{D}'_\omega(\mathbf{R})$. Hence, we can define

$$\mathcal{D}'_n := \mathcal{D}'_\omega(\mathbf{R})/\mathcal{D}'_\omega(\mathbf{R},]-n,n[).$$

By $q_n : \mathcal{D}'_\omega(\mathbf{R}) \to \mathcal{D}'_n$ we denote the corresponding quotient map. Note that $\mathcal{D}'_n$ is a (DFN)-space, since $\mathcal{D}'_n$ can also be described as

$$\mathcal{D}'_n = \mathcal{D}'_\omega[-m,m]/(\mathcal{D}'_\omega(\mathbf{R},]-n,n[) \cap \mathcal{D}'_\omega[-m,m]),$$

for each $m > n$. This follows from the fact that $\mathcal{D}'_\omega(\mathbf{R})$ and hence $\mathcal{D}'_n$ is ultrabornologic by [3], 5.6, while the second quotient is a (DFN)-space. It is easy to check that for each $n \in \mathbf{N}$ the map

$$\iota^n_{n+1} : \mathcal{D}'_{n+1} \to \mathcal{D}'_n, \quad \iota^n_{n+1}(q_{n+1}(\nu)) := q_n(\nu)$$

is well-defined, continuous and linear.

Let $\mathcal{D}'_\omega$ denote the projective spectrum $(\mathcal{D}'_n, \iota^n_{n+1})_{n\in\mathbf{N}}$ of (DFN)-spaces.

(2) From the definition in (1) it follows easily that the map $Q : \mathcal{D}_\omega(\mathbf{R})' \to \prod_{n\in\mathbf{N}} \mathcal{D}'_n$, $Q(\nu) := (q_n(\nu))_{n\in\mathbf{N}}$ induces a linear bijection between $\mathcal{D}_\omega(\mathbf{R})'$ and $Proj^0\mathcal{D}'_\omega$.

(3) $Proj^1\mathcal{D}'_\omega = 0$.

To see this, let $(a_n)_{n\in\mathbf{N}} \in \prod_{n\in\mathbf{N}} \mathcal{D}'_n$ be given. By the definition of $\mathcal{D}'_n$ we can choose $\nu_n \in \mathcal{D}_\omega(\mathbf{R})'$ with $a_n = q_n(\nu_n)$ for each $n \in \mathbf{N}$. If we let $b_n := q_n(\sum_{j=1}^{n-1} \nu_j)$ then the definitions in (1) imply

$$\iota^n_{n+1}(b_{n+1}) - b_n = q_n\left(\sum_{j=1}^{n} \nu_j - \sum_{j=1}^{n-1} \nu_j\right) = q_n(\nu_n) = a_n$$

for each $n \in \mathbf{N}$. Hence $(a_n)_{n\in\mathbf{N}}$ is in $B(\mathcal{D}'_\omega)$, which implies $B(\mathcal{D}'_\omega) = \prod_{n\in\mathbf{N}} \mathcal{D}'_n$.

(4) For $\mu \in \mathcal{E}_\omega(\mathbf{R})'$ with $Supp(\mu) \subset]-k,k[$ for some $k \in \mathbf{N}$ we define the map $S_\mu : \mathcal{D}'_\omega \to \mathcal{D}'_\omega$ by $S_\mu = (\sigma^n_{n+k})_{n\in\mathbf{N}}$, where $\sigma^n_{n+k} : \mathcal{D}'_{n+k} \to \mathcal{D}'_n$ is defined by

$$\sigma^n_{n+k}(q_{n+k}(\nu)) = q_n(\mu * \nu), \quad \nu \in \mathcal{D}'_\omega(\mathbf{R}).$$

Because of $Supp(\mu * \nu) \subset Supp(\mu) + Supp(\nu)$, this is a reasonable definition. Obviously, σ^n_{n+k} is a continuous linear map.

(5) For μ and k as in (4), define the projective spectrum $\mathcal{K}(\omega,\mu) := (K_n, i^n_{n+1})_{n\in\mathbf{N}}$ by $K_n = 0$ and , $i^n_{n+1} = 0$ for $1 \le n \le k$ and for $n > k$ by

$$K_n := \{\nu \in \mathcal{D}'_n | \sigma^{n-k}_n(\nu) = 0\}, \quad i^n_{n+1} = \iota^n_{n+1}|K_{n+1}.$$

Furthermore, we define $J : \mathcal{K}(\omega,\mu) \to \mathcal{D}'_\omega$ by $J = (j^n_n)_{n\in\mathbf{N}}$, where $j^n_n : K_n \to \mathcal{D}'_n$ denotes the inclusion map.

(6) For $\mu \in \mathcal{E}_\omega(\mathbf{R})'$ assume that S_μ admits a fundamental solution and that $ker\ S_\mu$ is infinite dimensional. Let the sequences α and β be defined as in Theorem 2.10. Using the notation from 1.8 we then let

$$\lambda_k(\alpha,\beta) := \underset{m\to}{ind}\, \lambda(k,m), \quad k \in \mathbf{N}$$

and we denote by $\iota^k_{k+1} : \lambda_{k+1}(\alpha,\beta) \to \lambda_k(\alpha,\beta)$ the obvious inclusion map. Moreover, we denote the projective spectrum $(\lambda_k(\alpha,\beta), \iota^k_{k+1})_{k\in\mathbf{N}}$ of (DFS)-sequence spaces by $\Lambda(\alpha, \beta)$.

If $\mu \in \mathcal{E}_\omega(\mathbf{R})'$ satisfies the hypothesis of 3.2(6) then Theorem 2.10 shows that $ker\ S_\mu$ is linearly isomorphic to $Proj^0\Lambda(\alpha,\beta)$. On the other hand it is easy to check that $ker\ S_\mu$ is linearly isomorphic to $Proj^0\mathcal{K}(\omega,\mu)$. It is not quite evident that the projective spectra $\mathcal{K}(\omega,\mu)$ and $\Lambda(\alpha,\beta)$ are equivalent. However, this is the case by the following lemma, the proof of which is an adaptation of the one of Braun, Meise and Vogt [4], 3.6.

Lemma 3.3. *Assume that the convolution operator S_μ on $\mathcal{D}_\omega(\mathbf{R})'$ admits a fundamental solution and that $ker\ S_\mu$ is infinite dimensional. Then the projective spectra $\mathcal{K}(\omega,\mu)$ and $\Lambda(\alpha,\beta)$ are equivalent.*

Proof. Assume that $Supp(\mu) \subset]-k,k[$ for some $k \in \mathbf{N}$ and let the projective spectrum $\mathcal{X} = (X_n, \xi^n_{n+1})_{n\in\mathbf{N}}$ be defined in the following way: $X_0 = 0$ and $\xi^n_{n+1} = 0$ for $1 \le n \le k$ and

$$X_n := \overline{q_n(ker\ S_\mu)}^{\mathcal{D}'_n} \subset K_n \text{ and } \xi^n_{n+1} := i^n_{n+1}|X_{n+1} \text{ for } n > k.$$

We claim that $\mathcal{X}$ and $\mathcal{K}(\omega,\mu)$ are equivalent. By the definition of $\mathcal{X}$ this follows from

$(*)$ For each $l \in \mathbf{N}$ with $l > k$ there exists $m \in \mathbf{N}, m > l$ so that $i^l_m(K_m) \subset X_l$.

To prove this, note that by [3], 6.2, and by Braun, Meise and Vogt [4], 2.7, the hypothesis implies that S_μ maps $\mathcal{E}_\omega(\mathbf{R})$ onto $\mathcal{E}_\omega(\mathbf{R})$. Therefore, Meise, Taylor and Vogt [13], 3.8, shows that for each $\rho > 0$ there exists $r = r(\rho) > \rho$ such that for each $R \ge r + k$ and each $g \in \mathcal{E}_\omega(\mathbf{R})$ satisfying $S_\mu(g)|[-R,R] \equiv 0$ we have

$$g(x) = \sum_{j=1}^{\infty} \lambda_j e_j(x) \text{ for all } x \in [-\rho,\rho],$$

where the series converges in $\mathcal{E}_\omega(]-\rho,\rho[)$ and where $e_j \in ker\ S_\mu$ for all $j \in \mathbf{N}$.

Now fix $l \in \mathbf{N}$, put $\rho := l + 2$, choose $r = r(\rho)$ according to the above and choose $m \in \mathbf{N}$, $m \geq r + 2k + 1$. Next choose $\chi \in \mathcal{D}_\omega[-1,1]$ with $\chi \geq 0$ and $\int_{\mathbf{R}} \chi d\lambda = 1$ and define $\chi_\varepsilon \in \mathcal{D}_\omega(\mathbf{R})$ by $\chi_\varepsilon(x) := \frac{1}{\varepsilon}\chi(\frac{x}{\varepsilon})$ for $0 < \varepsilon < 1$. Then fix $\nu \in K_m$, choose $\tau \in \mathcal{D}'_\omega(\mathbf{R})$ with $q_m(\tau) = \nu$ and choose a zero-neighbourhood U in $\mathcal{D}'_l$ arbitrarily. Since $\tau * \chi_\varepsilon$ tends to τ in $\mathcal{D}'_\omega(\mathbf{R})$ as ε tends to zero, we can choose $0 < \varepsilon < 1$ so that $q_l(\tau * \chi_\varepsilon) - i^l_m(\nu) \in \frac{1}{2}U$. Next note that $S_\mu(\tau)|_{[-m+k,m-k]} \equiv 0$ since ν is in K_m. Hence we have $S_\mu(\tau * \chi_\varepsilon)|_{[-R,R]} \equiv 0$ for $R := m - k - 1 \geq r + k$. Therefore, we get from the above

$$\tau * \chi_\varepsilon(x) = \sum_{j=1} \lambda_j e_j(x) \text{ for all } x \in [-l-1, l+1],$$

where the series converges in $\mathcal{E}_\omega(]-l-1, l+1[)$. Consequently, we can choose $n \in \mathbf{N}$ so large that

$$q_l\left(\tau * \chi_\varepsilon - \sum_{j=1}^{n} \lambda_j e_j\right) \in \frac{1}{2}U.$$

Since $\sum_{j=1}^n \lambda_j e_j$ is in $ker\ S_\mu$, we have shown that (*) holds. Knowing that $\mathcal{X}$ and $\mathcal{K}(\omega,\mu)$ are equivalent, the proof can now be completed as the one of Braun, Meise and Vogt [4], 3.6.

Theorem 3.4. *Let ω be a weight function and let $\mu \in \mathcal{E}_\omega(\mathbf{R})'$, $\mu \not\equiv 0$, be given. Then the following assertions are equivalent:*

(1) $S_\mu : \mathcal{D}_\omega(\mathbf{R})' \to \mathcal{D}_\omega(\mathbf{R})'$ is surjective

(2) S_μ admits a fundamental solution.

Proof. It is obvious that (1) implies (2). To show the converse implication, assume that S_μ admits a fundamental solution. Then we choose $k \in \mathbf{N}$ with $Supp(\mu) \subset]-k, k[$ and note that

$$0 \to K_{n+k} \overset{j^{n+k}_{n+k}}{\to} \mathcal{D}'_{n+k} \overset{\sigma^n_{n+k}}{\to} \mathcal{D}'_n \to 0$$

is an exact sequence for each $n \in \mathbf{N}$. Hence

$$0 \to \mathcal{K}(\omega,\mu) \overset{J}{\to} \mathcal{D}'_\omega \overset{S_\mu}{\to} \mathcal{D}'_\omega \to 0$$

is an exact sequence of projective spectra. Consequently, we get from Palamodov [15], p. 542, (see also Vogt [17], 1.5) the exactness of the following sequence:

$$(*) \qquad \begin{array}{cccccccc} 0 & \to & Proj^0\mathcal{K}(\omega,\mu) & \overset{J^0}{\to} & Proj^0\mathcal{D}'_\omega & \overset{S^0_\mu}{\to} & Proj^0\mathcal{D}'_\omega & \overset{\delta^*}{\to} \\ & \to & Proj^1\mathcal{K}(\omega,\mu) & \overset{J^1}{\to} & Proj^1\mathcal{D}'_\omega & \overset{S^1_\mu}{\to} & Proj^1\mathcal{D}'_\omega & \to \quad 0. \end{array}$$

By 3.2(2) we can identify $Proj^0\mathcal{D}'_n$ with $\mathcal{D}'_\omega(\mathbb{R})$. If we do this then S^0_μ coincides with S_μ. By 3.2(3) we have $Proj^1\mathcal{D}'_\omega = 0$. Hence (*) gives the exact sequence

$$0 \to ker\ S_\mu \to \mathcal{D}'_\omega \overset{S_\mu}{\to} \mathcal{D}'_\omega \to Proj^1\mathcal{K}(\omega,\mu) \to 0.$$

If $kerS_\mu$ is infinite dimensional then we get from Lemma 3.3 and Vogt [18], 1.4, that $Proj^1\mathcal{K}(\omega,\mu) = Proj^1\Lambda(\alpha,\beta)$. By Vogt [17], 4.3(i) and 4.2 we have $Proj^1\Lambda(\alpha,\beta) = 0$. Hence S_μ is surjective. If $kerS_\mu$ is finite dimensional then it is easily seen that $Proj^1\mathcal{K}(\omega,\mu) = 0$.

Remark 3.5. In Theorem 2.10 the hypothesis «S_μ admits a fundamental solution» can be replaced by «S_μ is surjective».

Added in proof. Note that Thm. 2.7 above extends Prop. 2.4 and Thm. 2.6 of I. Cioranescu: *Convolution equations in ω-ultradistribution spaces*, Rev. Roum. Math. Pures et Appl. **25** (1980), 719-737. Note also that S. Abdullah: *Convolution equations in Beurling's distributions*, Acta Math. Hung **52** (1988), 7-20, has characterized by different methods the surjective convolution operators on $\mathcal{D}'_\omega(\mathbb{R}^n)$, where $\mathcal{D}_\omega(\mathbb{R}^n)$ is defined in the sense of Beurling-Björck (see [4], 8.4 (2)).

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Received July 6, 1990

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