Note di Matematica Vol. X, Suppl. n. 1, 223-227(1990)

ORTHONORMAL SETS IN REPRODUCING KERNEL SPACES AND FUNCTIONAL COMPLETION

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Dedicated to the memory of Professor Gottfried Köthe

Let $f_i(x)$ be a sequence of functions defined on a set S. Suppose the function

$$K(x,y) = \sum_i f_i(x)\overline{f_i(y)}$$

makes sense for all points (x,y) in $S \times S$, i.e., that for every x the sequence $\{f_i(x)\}$ is summable square. Then in a known way the function $K(x,y)$ is a positive matrix and corresponds to a reproducing kernel space $\mathscr{H}_K(S)$ consisting of functions defined on S. The question we ask here is whether the functions $f_i(x)$ form a complete orthonormal system for that space.

It is easy to see that the answer is a negative one if the functions f_i are not linearly independent over S, since every orthogonal system of functions is linearly independent. We suppose in the sequel, therefore, that the system of function $f_i(x)$ is linearly independent.

In the special case that the system of functions is finite it turns out that it indeed is a complete orthonormal set; the problem is trickier in the infinite dimensional case.

If there are only N functions $f_i(x), i = 1,2,\ldots,N$ linearly independent over S we consider the space of all linear combinations

$$g(x) = \sum_{i=l}^{N} a_i f_i(x)$$

and introduce the quadratic norm defined by $||g||^2 = \sum_{i=1}^{N} |a_i|^2$. For this space of functions the point evaluations $L_x(g) = g(x)$ are linear, hence continuous, and there exists a unique element K_x in the space such that $g(x) = (g, K_x)$. Thus our space is a reproducing kernel space, and by the definition of the norm, the $f_i(x)$ are a complete orthonormal set. We can accordingly compute the associated kernel function using that orthonormal set and the kernel turns out to be our initial $K(x,y)$. Hence the $f_i(x)$ are indeed an orthonormal set in $\mathscr{H}_K(S)$, as desired.

A more computational argument can also be given in the finite dimensional case. Since the $f_i(x)$ are linearly independent, an elementary lemma in linear algebra guaranteees the existence of N points $x_1, x_2, \ldots, x_N$ in S such that the matrix

$$F_{ij} = f_i(x_j)$$

is non-singular. Let H_{jk} be the inverse of F; since

$$K_j(x) = K(x, x_j) = \sum_K f_k(x)\overline{f_k(x_j)} = \sum_K \overline{F_{kj}} f_k(x)$$

we have $\sum_j \overline{H}_{jm} K_j = f_m$ and by an easy calculation $(f_m, f_n) = \delta_{mn}$ i.e., the f_k are orthonormal.

Before pursuing our argument for the general case it is worthwhile to recall some elementary facts concerning reproducing kernel spaces. Every such space is obtained from a mapping κ of S into a Hilbert space $\mathscr{H}$

$$\kappa : x \to k_x$$

which gives rise to a kernel function $K(x, y) = (k_y, k_x)$. There is a corresponding linear map κ^* of $\mathscr{H}$ into $\mathscr{H}_K(S)$, a space of functions on S,

$$\kappa^* : f \to f(x) = (f, k_x).$$

The space $\mathscr{H}_K(S)$ is the reproducing kernel space associated with the kernel function $K(x, y)$. [1] The norm is the norm of the quotient $\mathscr{H}/\mathscr{N}$ where $\mathscr{N}$ is the null space of κ^*, a space necessarily closed. The mapping κ^* is an isometry if and only if $\mathscr{N}$ is trivial.

A special case arises in the study of functional completion. Here we suppose that we are given a pre-Hilbert space $\mathscr{H}(S)$ of functions defined on S such that the evaluation functionals L_x are continuous. Here, as before, $L_x(f) = f(x)$, and of course these functionals admit a continuous extension to the (abstract) completion $\mathscr{H}^*$. On $\mathscr{H}^*$ L_x is represented by an element K_x. Thus we have a map κ of S into $\mathscr{H}^*$ and an associated kernel function $K(x, y)$. Now the map κ^* of $\mathscr{H}^*$ into $\mathscr{H}_K(S)$ is or is not an isometry.

If κ^* is an isometry, it is clear that the initial $\mathscr{H}(S)$ was simply a dense subspace of $\mathscr{H}_K(S)$ and has the same norm as that space. The reproducing kernel space is the functional completion of $\mathscr{H}(S)$.

If κ^* is not an isometry it has a null space. Thus there exists a non-trivial element g in $\mathscr{H}^*$ such that $(g, K_x) = 0$ for all x. This g is the limit of a sequence g_n in $\mathscr{H}(S)$ such that $g_n(x)$ converges to 0 for all x, although the norms $||g_n||$ are bounded away from 0. The space $\mathscr{H}(S)$ now appears as a dense subspace of $\mathscr{H}_K(S)$ but the initial norm on $\mathscr{H}(S)$ is not the norm induced on it by the reproducing kernel space; the norm is that of a quotient. In this case no functional completion of $\mathscr{H}(S)$ can exist.

These considerations make it fairly clear how we are to proceed in the general case of our problem. We form the space $\mathscr{F}(S)$ consisting of finite linear combinations of the functions

$f_i(x)$ and note that the representation of such a finite linear combination

$$g(x) = \sum_i a_i f_i(x)$$

is unique, owing to the linear independence of the $f_i(x)$. We again introduce the quadratic norm

$$||g||^2 = \sum_i ||a_i||^2$$

and now $\mathscr{F}(S)$ appears as a pre-Hilbert space. The valuations L_x are continuous linear functionals on $\mathscr{F}(S)$ because of the hypothesis that the sequence $\{f_i(x)\}$ is summable square. With the norm just introduced, the f_i are a complete orthonormal set in the (abstract) completion $\mathscr{F}^*$ of $\mathscr{F}(S)$. Now, either $\mathscr{F}(S)$ has a functional completion or it does not.

If $\mathscr{F}(S)$ has a functional completion then the mapping κ^* from $\mathscr{F}^*$ to $\mathscr{H}_K(S)$ is an isometry, and the orthonormal set f_i maps into an orthonormal set in the reproducing kernel space.

If $\mathscr{F}(S)$ has no functional completion the mapping κ^* is not an isometry, and so the image of the complete orthonormal set f_i cannot be itself an orthonormal set. It follows that the functions $f_i(x)$ are not an orthonormal set in the space $\mathscr{H}_K(S)$.

We see that the $f_i(x)$ are an orthonormal set in the reproducing kernel space if and only if the map κ^* has a trivial null-space. We are therefore able to state a final criterion.

Theorem. *The functions $f_i(x)$ form a complete orthonormal set in $\mathscr{H}_K(S)$ if and only if, for every sequence $\{b_j\}$ summable square the function $B(x) = \sum b_j f_j(x)$ is identically zero on S only when every coefficient b_j vanishes.*

Note that the criterion given in the theorem is a slight strengthening of the hypothesis of linear independence. Note also that our argument applies equally well in the finite-dimensional case.

It is still not clear as to whether or not the case when the initial functions are not an orthonormal set actually occurs. A moment's thought convinces us that it happens just as often as separable spaces $\mathscr{H}(S)$ occur which have no functional completion. For suppose that $\mathscr{H}(S)$ is a separable pre-Hilbert space with continuous evaluation functional L_x which has no functional completion. By the Gram-Schmidt process we can construct an orthonormal set f_i in $\mathscr{H}(S)$ which is complete in the abstract completion $\mathscr{H}^*$. Let $\mathscr{F}(S)$ be the subspace of finite linear combinations of the f_i; it is easy to see that this space has no functional completion either since it contains a Cauchy sequence converging pointwise to 0 not converging to 0 in norm. It follows that the $f_i(x)$ are not an orthonormal set in the corresponding reproducing kernel space, although the kernel function is indeed given by the formula

$$K(x, y) = \sum_i f_i(x)\overline{f}_i(y).$$

The standard example of a functional pre-Hilbert space having no functional completion was given by Aronszajn. [2] For this purpose we consider the reproducing kernel space $\mathscr{H}^2(D)$ consisting of functions analytic in the unit disk $D = [z : |z| < 1]$ which are integrable square; the norm is of course the usual $L^2(D)$ norm. A convenient complete orthonormal set in the space is given by the functions

$$f_n(x) = \sqrt{\frac{n+1}{\pi}} z^n \quad n = 0, 1, 2, \ldots$$

and the corresponding kernel function is

$$K(z, w) = \frac{1}{\pi(1 - z\overline{w})^2}$$

For S we select a sequence $\{z_n\}$ in D with $|z_n|$ converging so rapidly to 1 that a non-trivial Blaschke product $B(z)$ vanishing on S exists. Hence there exists a sequence of polynomials $p_n(z)$ converging to $B(z)$ in $\mathscr{H}^2(D)$ which converges pointwise on S to 0. For $\mathscr{F}(S)$ we take the space of all polynomials restricted to S in the norm of $\mathscr{H}^2(D)$. Manifestly $\mathscr{F}(S)$ has no functional completion and the associated reproducing kernel space $\mathscr{H}_K(S)$ has a different (and smaller) norm than that of $\mathscr{F}(S)$. The $f_n(z)$ are not an orthonormal set in $\mathscr{H}_K(S)$ although the kernel function for that space is the restriction of $K(z, w)$ to $S \times S$.

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Received September 6, 1990
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