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IRREGULAR SAMPLING AND THE THEORY OF FRAMES, I

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Dedicated to the memory of Professor Gottfried Köthe

Abstract. *Irregular sampling expansions are proved in an elementary way by an analysis of the inverse frame operator. The expansions are of two dual types: in the first, the sampled values at irregularly spaced points are the coefficients; in the second, the sequence of sampling functions are irregularly spaced translates of a single sampling function. The results include regular sampling theory as well as the irregular sampling theory of Paley-Wiener, Levinson, Beutler, and Yao-Thomas. The use of frames also gives rise to a new interpretation of aliasing.*

1. INTRODUCTION

The subject of sampling, whether as method, point of view, or theory, weaves its fundamental ideas through a panorama of engineering, mathematical, and scientific disciplines. Sampling is so pervasive that excellent expositions and surveys abound; [BSS] and [Hi2] are two such papers that are particularly appropriate for our perspective. Alas, our contributions focus on an important result by Köthe [K] (1936), on a new look at Duffin and Schaeffer's theory of frames [DS] (1952) in light of the emergence of wavelet theory, and on effective, elementary, and unifying methods for irregular sampling in terms of frames.

Köthe was the first to prove that all bounded unconditional bases are equivalent in a given separable Hilbert space. An explanation of this result and its relationship with the theory of frames are the content of Theorem 2.5. Section 2 presents a crisp compendium of frame theory with Theorem 2.5 as its focal point. To titillate the reader during this dry compilation, we've pointed out yet another «first» by Vitali [V] in Remark 2.3. The technical device we extricate from Section 2 is the inverse frame operators S^{-1} for weighted Fourier frames associated with the lattice $\{(na, mb)\}$, e.g., Definition 2.6 and Theorems 2.7 and 2.8; and this operator is our basic tool in proving regular sampling theorems.

Section 3 is devoted to classical regular sampling expansions of the form,

$$f(t) = \sum_{n=-\infty}^{\infty} f(nT)\, s(t - nT), \tag{1}$$

where $T > 0$ is the sampling rate, $\{f(nT)\}$ is the set of regularly sampled values of the signal f, and s is the sampling function. The point of Section 3 is to prove (1) quickly in terms

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of frame decompositions. Regular sampling involves orthonormal bases of exponentials, and S^{-1} is used as a multiplier, e.g., Theorem 3.1 and Theorem 3.3. An important consequence of this line of thinking is a new interpretation and explanation of aliasing in terms of frames, e.g., Section 3.2.

Since our main goal is to prove irregular sampling expansions analogous to (1), we develop the theory of weighted Fourier frames associated with irregular «lattices» $\{(a_n, b_m)\}$ in Section 4.

Our results on irregular sampling are the subject of Sections 5 and 6. In Section 5, irregularly sampled values of f are used in the expansions analogous to (1). In Section 6, irregular translates of a single sampling function are used in the expansions analogous to (1). These two expansions are dual in the context of frame theory in a way that is explained in the text. The results in Section 5 use special frames associated with Köthe's work, and include the completeness and sampling theory of Paley-Wiener, Levinson, Beutler, and Yao-Thomas. The results in Section 6 use ordinary frames, and lead to an algorithm providing insight into the role of irregularly sampled values for the expansions of this section. The irregular sampling of Section 5 involves bounded unconditional bases, and S^{-1} is used in terms of biorthonormality, cf., our remark above, about Sections 3, on the role of S^{-1}.

We indicated at the outset that sampling ideas have diverse theoretical foundations and catholic applicability. As such, the sequel to this paper has two components. First, there is a critical comparison in Part II of other approaches to irregular sampling, cf., the analysis by Feichtinger and Gröchenig [FG]. Second, as regards applicability, Part II contains results dealing with aliasing, the algorithm, stability, and higher dimensions, all in the context of our frame theoretic approach. We have already indicated our technical direction for aliasing and the algorithm in Section 3 and 6, respectively. In Part II, the aliasing method is fully developed for the irregular sampling case, and an error analysis is conducted on the algorithm for various truncations of the inverse frame operator. Our approach to stability builds on the ideas of Yao and Thomas [YT], and ties in with the results of Beurling and Malliavin [BM] and Landau [La]. Our approach to higher dimensions is direct.

Besides the usual notation in analysis as found in the books by Hörmander [Hö], Schwartz [S], and Stein and Weiss [SW], we shall use the conventions and notation described at the end of the paper.

Finally, in this paper we have only proved convergence in the L^2 norm. All of our results have been proved for other modes of convergence, and details are found in [H]. Also, we have deals exclusively with bandlimited sampling functions.

2. RIESZ BASES AND FRAMES

Definition 2.1. *a) A sequence $\{g_n\} \subseteq H$, a separable Hilbert space, is a* **frame** *if there exist*

$A, B > 0$ such that

$$\forall f \in H, \quad A\|f\|^2 \le \sum |\langle f, g_n\rangle|^2 \le B\|f\|^2,$$

where $\langle , \rangle$ is the inner product on H and the norm of $f \in H$ is $\|f\| = \langle f, f\rangle^{1/2}$. A and B are the **frame bounds,** *and a frame $\{g_n\}$ is* **tight** *if $A = B$. A frame $\{g_n\}$ is* **exact** *if it is no longer a frame when any one of its elements is removed. Clearly, if $\{g_n\}$ is an orthonormal basis of H then it is a tight exact frame with $A = B = 1$.*

b) The **frame operator** *of the frame $\{g_n\}$ is the function $S : H \to H$ defined as $Sf = \sum \langle f, g_n\rangle g_n$.*

The theory of frames is due to Duffin and Schaeffer [DS] in 1952. Expositions include [Y] and [HW], the former presented in the context of non-harmonic Fourier series and the latter in the setting of wavelet theory.

Theorem 2.2. *Let $\{g_n\} \subseteq H$ be a frame with frame bounds A and B.*

a) S is a topological isomorphism with inverse $S^{-1} : H \to H$. $\{S^{-1}g_n\}$ is a frame with frame bounds B^{-1} and A^{-1}, and

$$\forall f \in H, \quad f = \sum \langle f, S^{-1}g_n\rangle g_n = \sum \langle f, g_n\rangle S^{-1}g_n.$$

The first expansion is the **frame expansion** *and the second is the* **dual frame expansion.**

b) If $\{g_n\}$ is tight, $\|g_n\| = 1$ for all n, and $A = B = 1$, then $\{g_n\}$ is an orthonormal basis of H.

c) If $\{g_n\}$ is exact, then $\{g_n\}$ and $\{S^{-1}g_n\}$ are biorthonormal, i.e.

$$\forall m, n, \quad \langle g_m, S^{-1}g_n\rangle = \delta_{mn}.$$

2.3 Remark. We comment on part b) because it is surprisingly useful and because of a stronger result by Vitali (1921) [V].

To prove b) we first use tightness and $A = 1$ to write,

$$\|g_m\|^2 = \|g_m\|^4 + \sum_{n \neq m} |\langle g_m, g_n\rangle|^2;$$

and obtain that $\{g_n\}$ is orthonormal since each $\|g_n\| = 1$. To conclude the proof we then invoke the well-known result: if $\{g_n\} \subset H$ is orthonormal then it is an orthonormal basis of H if and only if

$$\forall f \in H, \quad \|f\|^2 = \sum |\langle f, g_n\rangle|^2.$$

In 1921,Vitali proved that an orthonormal sequence $\{g_n\} \subset L^2[a,b]$ is complete, and so $\{g_n\}$ is an orthonormal basis, if and only if

$$(2.1) \qquad \forall t \in [a,b], \sum \left| \int_a^t g_n(u)\,du \right|^2 = t - a.$$

For the case $H = L^2[a,b]$, Vitali's result is stronger than part b) since (2.1) is tightness with $A = 1$ for functions $f = \mathbf{1}_{[a,t]}$.

Other remarkable and important contributions by Vitali are highlighted in [B].

Definition 2.4.Let H be a separable Hilbert space. *A sequence $\{g_n\} \subseteq H$ is a* **Schauder basis** *or* **basis** *of H if each $f \in H$ has a unique decomposition $f = \sum c_n(f) g_n$. A basis $\{g_n\}$ is an* **unconditional basis** *if*

$$\exists C \text{ such that } \forall F \subseteq Z, \text{ where card } F < \infty, \text{ and}$$

$$\forall b_n, c_n \in Z, \text{ where } n \in F \text{ and } |b_n| \leq |c_n|,$$

$$\left\| \sum_{n \in F} b_n g_n \right\| \leq C \left\| \sum_{n \in F} c_n g_n \right\|.$$

An unconditional basis $\{g_n\}$ is **bounded** *if*

$$\exists A, B > 0 \text{ such that } \forall n, A \leq \|g_n\| \leq B.$$

Separable Hilbert spaces have orthonormal bases, and orthonormal bases are bounded unconditional bases.

Köthe's result mentioned in Section 1 is the implication, b) implies c), of the following theorem. The implication, c) implies b), is straightforward; and the equivalence of a) and c) is found in [Y, pp. 188-189].

Theorem 2.5.Let H be a separable Hilbert space and let $\{g_n\} \subseteq H$ be a given sequence. *The following are equivalent:*

a) $\{g_n\}$ is an exact frame for H;

b) $\{g_n\}$ is a bounded unconditional basis of H;

c) $\{g_n\}$ is a **Riesz basis,** *i.e., there is an orthonormal basis $\{u_n\}$ and a topological isomorphism $T : H \to H$ such that $Tg_n = u_n$ for each n.*

Definition/Remark 2.6. *a) Given $g \in L^2(\mathbf{R})$ and sequences $\{a_n\}, \{b_m\} \subseteq \mathbf{R}$. Define $(T_{a_n}g)(t) = g(t - a_n)$ and $E_{b_m}(t) = e^{2\pi i t b_m}$. If $\{E_{b_m}T_{a_n}g\}$ is a frame for $L^2(\mathbf{R})$ it is called a* **weighted Fourier frame** *with weight g.*

b) **Fourier frames** *$\{E_{b_m}\}$ were defined in [DS] for $L^2[-T,T]$. Gabor's seminal paper [G] deals with «regular latticed» systems $\{E_{mb}T_{na}g\}$, where g is the Gaussian; and it turns out that the Heisenberg group is fundamental in analyzing the structure of modulations and translations. As such, the names «Gabor» and «Weyl-Heisenberg» have also been associated with these systems in the case of regular lattices.*

c) $\{E_{b_m}T_{a_n}g\}$ is a frame for $L^2(\mathbf{R})$ if and only if $\{T_{a_n}(E_{b_m}g)\}$ is a frame for $L^2(\mathbf{R})$.

Also, our weighted Fourier frames will often be defined for $L^2(\widehat{\mathbf{R}})$. As such we note that

$$(E_{a_n}T_{b_m}\widehat{g})^{\vee} = e^{2\pi i a_n b_m}E_{b_m}T_{-a_n}g.$$

Theorem 2.7. *Given $g \in L^2(\mathbf{R})$ and $a, b > 0$. Define*

$$G(t) = \sum |g(t - na)|^2.$$

Assume that there exist $A, B > 0$ such that

$$0 < A \leq G(t) \leq B < \infty \quad a.e.\ on\ \mathbf{R}, \tag{2.2}$$

and that $supp g \subseteq I$ where I is an interval of length 1/b. Then $\{E_{mb}T_{na}g\}$ is a frame for $L^2(\mathbf{R})$, with frame bounds $b^{-1}A$ and $b^{-1}B$, and

$$\forall f \in L^2(\mathbf{R}), \quad S^{-1}f = \frac{bf}{G}. \tag{2.3}$$

Theorem 2.8. *Given $g \in L^2(\mathbf{R})$ and $a, b > 0$. Assume $\{E_{na}T_{mb}\widehat{g}\}$ is a frame for $L^2(\widehat{\mathbf{R}})$. Then*

$$S^{-1}(E_{na}T_{mb}\widehat{g}) = E_{na}T_{mb}S^{-1}\widehat{g}. \tag{2.4}$$

Example 2.9. a) Given $g \in L^2(\mathbf{R})$ and $a, b > 0$ for which $ab = 1$. *If $\{E_{mb}T_{na}g\}$ is a frame then it is an exact frame.* This remarkable fact (for $ab = 1$) can be proved using properties of the Zak transform which we now define.

b) The **Zak transform** of $f \in L^2(\mathbf{R})$ is

$$Zf(x, \omega) = a^{1/2}\sum f(xa + ka)e^{2\pi i k\omega}$$

for $(x,\omega) \in \mathbf{R} \times \widehat{\mathbf{R}}$ and $a > 0$. It turns out that the Zak transform is a unitary map of $L^2(\mathbf{R})$ onto $L^2(Q), Q = [0,1) \times [0,1)$.

c) If $\{E_{mb}T_{na}g\}$ is a frame for $ab = 1$, it is a bounded unconditional basis (part a) and Theorem 2.5); and, in particular, the frame decomposition

$$\forall f \in L^2(\mathbf{R}), \quad f = \sum c_{m,n} E_{mb} T_{na} g$$

(Theorem 2.2.a) is unique. We shall verify that

$$(2.5) \qquad c_{m,n} = \langle f, S^{-1} E_{mb} T_{na} g \rangle$$

$$= \int_0^1 \int_0^1 \frac{Zf(x,\omega)}{Zg(x,\omega)} e^{-2\pi i m x} e^{-2\pi i n \omega} \, dx d\omega .$$

First, with the hypotheses that $\{E_{mb}T_{na}g\}$ is a frame for $L^2(\mathbf{R})$ and $ab = 1$, we compute

$$\forall F \in L^2(Q), \quad S_Z F = F |Zg|^2,$$

where $S_Z : L^2(Q) \to L^2(Q)$ is the frame operator for the frame $\{Z(E_{mb}T_{na}g)\}$. Thus,

$$(2.6) \qquad \forall f \in L^2(\mathbf{R}), \quad S_Z^{-1}(Zf) = \frac{Zf}{|Zg|^2}.$$

Next, using (2.6), we compute

$$\forall f \in L^2(\mathbf{R}), \quad Zf = S_Z S_Z^{-1}(Zf)$$

$$= \sum \left\langle \frac{Zf}{Zg}, E_{m,n} \right\rangle E_{m,n} Zg,$$

where $E_{m,n}(x,\omega) = e^{2\pi i m x} e^{2\pi i n \omega}$. Consequently,

$$\forall f \in L^2(\mathbf{R}), \quad f = \sum \left\langle \frac{Zf}{Zg}, E_{m,n} \right\rangle Z^{-1}(E_{m,n} Zg)$$

$$= \sum \left\langle \frac{Zf}{Zg}, E_{m,n} \right\rangle E_{mb} T_{na} g,$$

so that (2.5) is obtained by the uniqueness of the representation.

3. REGULAR SAMPLING AND WEIGHTED FOURIER FRAMES

The theme of this section is to prove classical sampling results by frame methods in the case that the inverse frame operator S^{-1} is a multiplier.

The **Paley-Wiener space,** PW_Ω, is the subset of $L^2(\mathbf{R})$ whose elements are Ω-bandlimited, i.e.

$$PW_\Omega = \{f \in L^2(\mathbf{R}) : supp\, \hat{f} \subseteq [-\Omega, \Omega]\}.$$

Clearly the elements of PW_Ω are entire functions.

Theorem 3.1. *Given* $T, \Omega > 0$ *for which* $0 < T \leq \frac{1}{2\Omega}$. *Then*

$$\forall f \in PW_\Omega, \quad f = T \sum f(nT) T_{nT} d_{2\pi\Omega} \ in\ L^2(\mathbf{R}), \tag{3.1}$$

where $d_{2\pi\Omega}$ *is the* $2\pi\Omega$ *dilation of the Dirichlet function*

$$d(t) = \frac{\sin t}{\pi t},$$

where $f(nT)$ *is the value of* f *at* $nT \in \mathbf{R}$, *and where* $T_{nT} d_{2\pi\Omega}$ *is the translation*

$$T_{nT} d_{2\pi\Omega}(t) = d_{2\pi\Omega}(t - nT) = \frac{\sin 2\pi\Omega(t - nT)}{\pi(t - nT)}.$$

Proof. Let $g = \frac{1}{(2\Omega)^{1/2}} d_{2\pi\Omega}$ so that $\hat{g} = \frac{1}{(2\Omega)^{1/2}} \mathbf{1}_{(\Omega)}$ and $\|g\|_2 = 1$. Set $a = T$ and $b = 2\Omega$ so that $ab = 2T\Omega \leq 1$. Note that

$$\sum |\hat{g}(\gamma - mb)|^2 = \frac{1}{2\Omega} \ a.e.,$$

$supp\, \hat{g} \subseteq [-\Omega, \Omega]$, and $|[-\Omega, \Omega]| \leq 1/a$. Thus, by Theorem 2.7, $\{E_{na} T_{mb} \hat{g}\}$ is a frame. Consequently, by Theorem 2.2.a and Theorem 2.8,

$$\forall f \in L^2(\mathbf{R}), \quad \hat{f} = \sum \langle \hat{f}, E_{na} T_{mb} S^{-1} \hat{g} \rangle E_{na} T_{mb} \hat{g} \ in\ L^2(\hat{\mathbf{R}}). \tag{3.2}$$

Since $supp\, \hat{g}$ is compact, we have

$$\forall h \in L^2(\hat{\mathbf{R}}), \quad S^{-1} \hat{h} = 2T\Omega \hat{h}$$

by Theorem 2.7; and, hence, (3.2) becomes

$$(3.3) \qquad \forall f \in L^2(\mathbf{R}), \quad f = 2T\Omega \sum \langle \widehat{f}, E_{na} T_{mb}\widehat{g}\rangle T_{-na} E_{mb} g \ in\ L^2(\mathbf{R}).$$

If $f \in PW_\Omega$ then

$$(3.4) \qquad \langle \widehat{f}, E_{na} T_{mb}\widehat{g}\rangle = \begin{cases} \frac{1}{(2\Omega)^{1/2}} f(-nT), & \text{for } m = 0 \\ 0, & \text{for } m \neq 0. \end{cases}$$

The sampling formula, (3.1), follows from (3.3) and (3.4). ∎

The hypothesis, that $f \in PW_\Omega$, was essential in both parts of (3.4); and the above proof shows that only «t-information» (i.e., $m = 0$) is required in this case. When f is not Ω-bandlimited so that aliasing occurs, phase information contributed by $m \neq 0$ is required in the frame decomposition of a signal. To quantify this remark, we define the **aliasing pseudo-measure,** $\alpha_{t,\Omega}$, on $\mathbf{R}$ as the distributional Fourier transform, $\alpha_{t,\Omega} = A^{\vee}_{t,\Omega}$, where each t is fixed and

$$A_{t,\Omega} \equiv \sum (e^{2\pi i(2m\Omega t)} - 1)(T_{2m\Omega}\mathbf{1}_{(\Omega)}) \in L^\infty(\widehat{\mathbf{R}}).$$

Calculation/Definition 3.2. Let $f \in L^2(\mathbf{R})$ and assume $2T\Omega = 1$. Writing (3.3) as a sum, $\sum_{m=0,n} + \sum_{m\neq 0,n}$, we compute

$$(3.5) \qquad f(t) = T\sum f(nT)T_{nT} d_{2\pi\Omega}(t) + T\sum (f * \alpha_{t,\Omega})(nT)T_{nT} d_{2\pi\Omega}(t).$$

The **aliasing error** of f at t for the low pass filter $d_{2\pi\Omega}$ is

$$ae(f,t) = T\sum (f * \alpha_{t,\Omega})(nT)T_{nT} d_{2\pi\Omega}(t).$$

Formally, standard calculations give

$$(3.6) \qquad \|ae(f,\cdot)\|_\infty \leq 2\int_{|\gamma|\geq\Omega} |\widehat{f}(\gamma)|\,d\gamma.$$

In the following result we use sampling kernels s with more rapid decay than $d_{2\pi\Omega}$. The goal is better computational efficiency for low pass filters; the price to be paid is more sampling.

Theorem 3.3. *Given* $T, \Omega > 0$, *for which* $0 < T < \frac{1}{2\Omega}$, *and* $g \in S(\mathbf{R})$ *with the properties that* $supp \widehat{g} \subseteq \left[\frac{-1}{2T}, \frac{1}{2T}\right]$, $\widehat{g} = 1$ *on* $[-\Omega, \Omega]$, *and* $\widehat{g} > 0$ *on* $\left(\frac{-1}{2T}, -\Omega\right] \cup \left[\Omega, \frac{1}{2T}\right)$. *Set*

$$G(\gamma) = \sum |\widehat{g}(\gamma - mb)|^2 \quad and \quad s(t) = \left(\frac{\widehat{g}}{G}\right)^{\vee}(t),$$

where $\Omega + \frac{1}{2T} \leq b < \frac{1}{T}$. *Then* $0 < A \leq G(\gamma) \leq B < \infty, s \in S(\mathbf{R}), supp\ \widehat{s} = supp\ \widehat{g}, \widehat{s} = \frac{1}{G}$ *on* $[-\Omega, \Omega]$, *and*

$$\forall f \in PW_\Omega, \quad f = T \sum f(nT) \tau_{nT} s \quad in \quad L^2(\mathbf{R}). \tag{3.7}$$

Proof. The assertion about G and s follow from our choice of b.

Set $a = T$ so that $|supp\ \widehat{g}| = 1/a$. Thus, using the fact, $A \leq G(\gamma) \leq B$, and Theorem 2.7, we see that $\{E_{na} \tau_{mb} \widehat{g}\}$ is a frame. Since $supp\ \widehat{g}$ is compact, we have

$$\forall h \in L^2(\mathbf{R}), \quad S^{-1}\widehat{h} = T\frac{\widehat{h}}{G}$$

by Theorem 2.7; and, hence, we have the frame decomposition

$$\forall f \in L^2(\mathbf{R}), \quad \widehat{f} = T \sum_{m,n} \langle \widehat{f}, E_{na} \tau_{mb} \widehat{g} \rangle E_{na} \tau_{mb} \widehat{s}, \tag{3.8}$$

where we have used the fact that $S^{-1}(E_{na} \tau_{mb} \widehat{g}) = E_{na} \tau_{mb} S^{-1} \widehat{g}$ (Theorem 2.8).

If $f \in PW_\Omega$, then (3.4) is again valid since $\widehat{g} = 1$ on $[-\Omega, \Omega]$. The sampling formula (3.7) follows from (3.4) and (3.8). ∎

Example 3.4. a) In Theorem 3.1, $\{E_{na} \tau_{mb} \widehat{g}\}$ is a tight frame with frame bounds $A = B = 1$ in the case $2T\Omega = 1$, where $a = T$ and $b = 2\Omega$. Clearly, $\langle E_{na} \tau_{mb} \widehat{g}, E_{qa} \tau_{pb} \widehat{g} \rangle$ is 1 if $(m, n) = (p, q)$ and is 0 if $m \neq p$. If $m = p$ and $n \neq q$ then this inner product is

$$\frac{e^{2\pi i(2T\Omega)m(n-q)}}{2T\Omega\pi(n-q)} \sin(2T\Omega\pi(n-q)).$$

Thus, $\{E_{na}T_{mb}\widehat{g}\}$ is an orthonormal sequence if and only if $2T\Omega = 1$. Consequently, by Theorem 2.2.b, $\{E_{na}T_{mb}\widehat{g}\}$ is an orthonormal basis if and only if $2T\Omega = 1$.

b) Suppose $2T\Omega < 1$. To construct $g \in S(\mathbf{R})$ satisfying the conditions of Theorem 3.3 we proceed as follows, cf., [H] for a different construction depending on the Pythagorean theorem.

We begin in the standard «distributional way» by defining

$$\psi_\varepsilon(\gamma) = \frac{\phi(\varepsilon - |\gamma|^2)}{\int \phi(\varepsilon - |\gamma|^2)\, d\gamma},$$

where $\phi \in C^\infty(\widehat{\mathbf{R}})$ vanishes on $(-\infty, 0]$ and equals $e^{-1/\gamma}$ on $[0, \infty)$. Thus, $\psi_\varepsilon \in C_c^\infty(\widehat{\mathbf{R}})$ is an even function satisfying the conditions, $supp\psi_\varepsilon = [-\varepsilon, \varepsilon]$ and $\int \psi_\varepsilon(\gamma)\, d\gamma = 1$. Next set

$$\psi_{U,V} = \frac{1}{|V|}\mathbf{1}_V * \mathbf{1}_{U-V}, \quad U, V \subseteq \widehat{\mathbf{R}},$$

so that $\psi_{U,V}$ is 1 on U and vanishes off of $U + V - V$. The function g will be defined in terms of $\widehat{g}$ as $\widehat{g} = \psi_{U,V} * \psi_\varepsilon$, where we shall now specify ε, U, and V given $2T\Omega < 1$. Let $U = [-u, u]$, where $u \in \left(\Omega, \frac{1}{2T}\right)$ is arbitrary, and let $\varepsilon = u - \Omega$. Choose $V = [-v, v]$ by setting $v = \frac{w-u}{2}$, where $w = \frac{1}{2T} + \varepsilon$. These choices are necessitated by a simple geometrical argument, and the resulting function $\widehat{g}$ satisfies the desired properties.

4. WEIGHTED FOURIER FRAMES FOR IRREGULAR LATTICES

In the case of irregular lattices, the following result is the analogue of Theorem 2.7 for $\widehat{\mathbf{R}}$.

Theorem 4.1. *Given $\Omega > 0$ and let $g \in PW_\Omega$. Assume that $\{a_n\}, \{b_m\}$ are real sequences for which*

$$(4.1) \qquad \{E_{a_n}\} \;\; is\ a\ frame\ for\ L^2[-\Omega, \Omega],$$

and that there exist $A, B > 0$ such that

$$(4.2) \qquad 0 < A \leq G(\gamma) \leq B < \infty \; a.e.\ on\, \widehat{\mathbf{R}},$$

where

$$G(\gamma) = \sum |\widehat{g}(\gamma - b_m)|^2.$$

Then $\{E_{a_n}T_{b_m}\widehat{g}\}$ is a frame for $L^2(\widehat{\mathbf{R}})$; and $\{E_{a_n}T_{b_m}\widehat{g}\}$ is a tight frame for $L^2(\widehat{\mathbf{R}})$ if and only if $\{E_{a_n}\}$ is a tight frame for $L^2[-\Omega,\Omega]$ and G is a constant a.e. on $\widehat{\mathbf{R}}$.

Proof. $I = [-\Omega,\Omega]$ and set $I_m = I + b_m$. For fixed m, $\{T_{b_m}E_{a_n}\}$ is a frame for $L^2(I_m)$ with frame bounds A_I, B_I independent of m. Thus, for all $h \in L^2(\mathbf{R})$ for which $supp\,\widehat{h} \subseteq I_m$, we have

$$A_I\|\widehat{h}\|^2_{L^2(I_m)} \leq \sum_n |\langle \widehat{h}, T_{b_m}E_{a_n}\rangle_{I_m}|^2 \leq B_I\|\widehat{h}\|^2_{L^2(I_m)}, \tag{4.3}$$

Take any $f \in L^2(\mathbf{R})$. Because of (4.2), $\widehat{g} \in L^\infty(\widehat{\mathbf{R}})$; and, hence, $\widehat{h}_{m,f} = \widehat{f}T_{b_m}\overline{\widehat{g}} \in L^2(I_m)$. Also, since g is Ω-bandlimited, $\widehat{h}_{m,f}$ vanishes off of I_m. Substituting $\widehat{h}_{m,f}$ into (4.3) and summing over m, we obtain

$$A_I\sum_m \|\widehat{f}T_{b_m}\overline{\widehat{g}}\|^2_{L^2(I_m)} \leq \sum_m\sum_n |\langle \widehat{f}, (T_{b_m}\widehat{g})T_{b_m}E_{a_n}\rangle|^2 \leq B_I\sum_m \|\widehat{f}T_{b_m}\overline{\widehat{g}}\|^2_{L^2(I_m)}. \tag{4.4}$$

We now compute

$$\langle \widehat{f}, (T_{b_m}\widehat{g})T_{b_m}E_{a_n}\rangle = \langle \widehat{f}, T_{b_m}(\widehat{g}E_{a_n})\rangle$$

and, using the fact that g is Ω-bandlimited,

$$\sum_m \|\widehat{f}T_{b_m}\overline{\widehat{g}}\|^2_{L^2(I_m)} = \int |\widehat{f}(\gamma)|^2 \left(\sum |\widehat{g}(\gamma - b_m)|^2\right) d\gamma.$$

By these calculations, as well as (4.2) and (4.4), we obtain

$$AA_I\|\widehat{f}\|_2^2 \leq \sum_m\sum_n |\langle \widehat{f}, T_{b_m}E_{a_n}\widehat{g}\rangle|^2 \leq BB_I\|\widehat{f}\|_2^2. \tag{4.5}$$

Thus, $\{E_{a_n}T_{b_m}\widehat{g}\}$ is a frame for $L^2(\widehat{\mathbf{R}})$. The characterization of $\{E_{a_n}T_{b_m}\widehat{g}\}$ as a tight frame follows immediately from (4.5). ∎

Corollary 4.2. *Given the hypotheses of Theorem 4.1 and set $I_m = [-\Omega,\Omega] + b_m$. For each fixed m, $\{T_{b_m}E_{a_n}\}$ is a frame for $L^2(I_m)$ with frame operator S_m, cf., (4.3), $\{E_{a_n}T_{b_m}\widehat{g}\}$ is a frame for $L^2(\widehat{\mathbf{R}})$ with frame operator S, and*

$$\forall h \in L^2(\mathbf{R}), \qquad S\widehat{h} = \sum T_{b_m}\widehat{g}S_m(\widehat{h}T_{b_m}\overline{\widehat{g}}).$$

Proof. We compute

$$\begin{aligned} S\widehat{h} &= \sum_m \sum_n \langle \widehat{h}, E_{a_n} T_{b_m} \widehat{g} \rangle E_{a_n} T_{b_m} \widehat{g} \\ &= \sum_m T_{b_m} \widehat{g} \left(\sum_n \langle \widehat{h}, E_{a_n} T_{b_m} \widehat{g} \rangle_{\widehat{\mathbf{R}}} E_{a_n} \right) \mathbf{1}_{I_m} \\ &= \sum_m T_{b_m} \widehat{g} \left(\sum_n \langle \widehat{h} T_{b_m} \overline{\widehat{g}}, E_{a_n} \rangle_{I_m} E_{a_n} \mathbf{1}_{I_m} \right) \\ &\equiv \sum_m T_{b_m} \widehat{g} S_m (\widehat{h} T_{b_m} \overline{\widehat{g}}). \end{aligned}$$

■

If «$\widehat{g}$» is any Borel measurable function for which $G(\gamma) \leq B$ a.e. on $\widehat{\mathbf{R}}$, then $\widehat{g} \in L^\infty(\widehat{\mathbf{R}})$. The converse is a part of the following result.

Theorem 4.3. *Given $\Omega > 0$. Assume that $\{a_n\}, \{b_m\}$ are real sequences for which $\{E_{a_n}\}$ is a frame for $L^2[-\Omega, \Omega]$, and that there exist $d, D > 0$ such that*

$$(4.6) \qquad \forall m, \quad 0 < d \leq b_{m+1} - b_m \leq D < 2\Omega,$$

where $\lim_{m\to\pm\infty} b_m = \pm\infty$. Suppose $g \in PW_\Omega$ has the properties that $\widehat{g} \in L^\infty(\widehat{\mathbf{R}})$ and $A = inf\{|\widehat{g}(\lambda)|^2 : \lambda \in I\} > 0$ for some interval $I \subseteq [-\Omega, \Omega]$ having measure $|I| = D$. Then $\{E_{a_n} T_{b_m} \widehat{g}\}$ is a frame for $L^2(\widehat{\mathbf{R}})$.

Proof. It suffices to verify condition (4.2) of Theorem 4.1.

For each $\gamma, G(\gamma)$ is a finite sum; and, in fact, this sum has at most $\left[\frac{2\Omega}{d}\right] + 1$ terms. Thus,

$$\forall \gamma, \quad G(\gamma) \leq \left(\left[\frac{2\Omega}{d} \right] + 1 \right) ||\widehat{g}||_\infty \equiv B < \infty,$$

and the upper bound is obtained.

For each $\gamma \in \widehat{\mathbf{R}}$ there is a b_m such that $\gamma - b_m \in I$. Thus,

$$G(\gamma) \geq |\widehat{g}(\gamma - b_m)|^2 \geq A > 0,$$

and the lower bound is obtained. ■

Remark 4.4. a) Consider condition (4.1), used in both Theorem 4.1 and 4.3.

a.i) A sequence $\{a_n\} \subseteq \mathbf{R}$ has **uniform density** $\Delta > 0$ if there exist constants L and d such that

$$\forall n, \quad \left|a_n - \frac{n}{\Delta}\right| \leq L$$

and

$$\forall n \neq m, \quad |a_n - a_m| \geq d > 0.$$

Duffin and Schaeffer [DS] proved that *if* $\{a_n\}$ *has uniform density* $\Delta > 0$ *and* $0 < 2\Omega < \Delta$ *then* $\{E_{a_n}\}$ *is a frame for* $L^2[-\Omega, \Omega]$. For a given sequence $\{a_n\} \subseteq \mathbf{R}$ let $\Omega_\mathbf{R}$ be the least upper bound of all Ω for which $\{E_{a_n}\}$ is a frame for $L^2[-\Omega, \Omega]$; $\Omega_\mathbf{R}$ is the **frame radius** of $\{a_n\}$. Duffin and Schaeffer's theorem can be rephrased and refined as follows: if $\{a_n\}$ has uniform density $\Delta > 0$ then $\Omega_\mathbf{R} \geq \frac{\Delta}{2}$.

Important work on this topic is due to [La; J], cf., [H]. We mention the following fact which follows from [DS; J]. *Suppose* $\{E_{a_n}\}$ *is an exact frame for* $L^2[-\Omega, \Omega]$. *Then* $\{E_{a_n}\}$ *is not a frame for* $L^2[-\Omega_1, \Omega_1]$ *for any* $\Omega_1 > \Omega$, *and* $\{E_{a_n}\}$ *is an inexact frame for* $L^2[-\Omega_1, \Omega_1]$ *for every* $0 < \Omega_1 < \Omega$. In this latter case we can remove any finite number of arbitrarily selected elements of $\{a_n\}$ and still have a frame for $L^2[-\Omega_1, \Omega_1]$.

a.ii) If $a_n = na$ and $a = \frac{1}{2\Omega}$ then $\{E_{na}\}$ is an orthonormal basis of $L^2[-\Omega, \Omega]$. The sequence $\{na\}$ has uniform density $\Delta = \frac{1}{a}$.

b.i) *Given the hypotheses of Theorem 4.1 in the case* $a_n = na$ *and* $a = \frac{1}{2\Omega}$. *Then*

$$\forall f \in L^2(\mathbf{R}), \quad S^{-1}\widehat{f} = \frac{1}{2\Omega}\frac{\widehat{f}}{G}. \tag{4.7}$$

To verify (4.7) note that $\left\{\frac{1}{(2\Omega)^{1/2}}E_{na}\right\}$ is an orthonormal basis of each $L^2(I_m)$ and that $\widehat{f}T_{b_m}\overline{\widehat{g}} \in L^2(I_m)$. Since $\{E_{na}T_{b_m}\widehat{g}\}$ is a frame for $L^2(\widehat{\mathbf{R}})$ we have

$$\begin{aligned} S\widehat{f} &\equiv \sum_m \sum_n \langle \widehat{f}, E_{na}T_{b_m}\widehat{g}\rangle E_{na}T_{b_m}\widehat{g} \\ &= \sum_m (T_{b_m}\widehat{g}) \left(\sum_n \langle \widehat{f}T_{b_m}\overline{\widehat{g}}, E_{na}\rangle E_{na}\right) \\ &= 2\Omega \widehat{f} G. \end{aligned} \tag{4.8}$$

Using $S^{-1}\widehat{f}$ instead of $\widehat{f}$ in (4.8) we obtain (4.7).

b.ii) In Theorem 3.3 we used the commutativity of the operators S^{-1} and $E_{na}T_{mb}$ in proving the sampling formula.

Now suppose we have the hypotheses of Theorem 4.1 in the case $a_n = na$ and $a = \frac{1}{2\Omega}$. Then by part b.i) we have (4.7), so that

$$S^{-1}(E_{na}T_{b_m}\widehat{g}) = \frac{1}{2\Omega}\frac{E_{na}T_{b_m}\widehat{g}}{G}.$$

On the other hand,

$$E_{na}T_{b_m}S^{-1}\widehat{g} = \frac{1}{2\Omega}\frac{E_{na}T_{b_m}\widehat{g}}{T_{b_m}G},$$

so that *the operators* S^{-1} *and* $E_{na}T_{b_m}$ *are not commutative for irregular sequences* $\{b_m\}$.

Example 4.5. Given the hypotheses of Theorem 4.1. Then

$$\forall f \in L^2(\mathbf{R}), \quad \widehat{f} = \sum \langle \widehat{f}, S^{-1}(E_{a_n}T_{b_m}\widehat{g})\rangle E_{a_n}T_{b_m}\widehat{g} \quad in\ L^2(\widehat{\mathbf{R}}),$$

and so

$$\forall f \in L^2(\mathbf{R}), \quad f(t) = \sum c_n(t) T_{-a_n} g(t) \quad in\ L^2(\mathbf{R}), \tag{4.9}$$

where

$$c_n(t) = \sum_m \langle \widehat{f}, e^{-2\pi i a_n b_m} S^{-1}(E_{a_n}T_{b_m}\widehat{g})\rangle E_{b_m}(t).$$

With various further hypotheses, (4.9) will be a «sampling» formula, cf., Theorem 6.2. The point we make now is that *the frequencies for Fourier frames on* $\widehat{\mathbf{R}}$ *provide the translation points on* $\mathbf{R}$ *for sampling formulas.*

5. IRREGULAR SAMPLING - SAMPLED COEFFICIENTS AND EXACT FRAMES

The theory of non-harmonic Fourier series was developed by Paley and Wiener [PW, Chapters 6 and 7] and Levinson [L, Chapter 4]. Related work preceding [PW] is due to G. D. Birkhoff (1917), J.L. Walsh (1921), and Wiener (1927). The Paley-Wiener and Levinson theory has been reformulated and analyzed in terms of irregular sampling by Beutler [Be1; Be2] for completeness and Yao and Thomas [YT] for expansions. The Yao and Thomas expansion was discovered independently by Higgins [Hi1] using reproducing kernels; there is also the interesting new work by Rawn [R]. In this section we shall state and prove this irregular sampling expansion by frame methods. The coefficients in the expansion are the values of the given signal at the given irregularly spaced sampling points, cf., Section 6.

Whereas we implemented S^{-1} as a multiplier in Section 3, in this section we shall invoke a formula, viz., (5.1), related to the fact that $\{S^{-1}g_n\}$ is the unique biorthonormal sequence associated to a given exact frame $\{g_n\}$, cf., Theorem 2.2.c.

Proposition 5.1. *Let H be a separable Hilbert space and let $\{g_n\} \subseteq H$ be an exact frame with inverse frame operator S^{-1}. Then*

$$\forall f \in H, \quad S^{-1}f = \sum \langle f, h_n \rangle h_n \quad in\, H, \tag{5.1}$$

where $\{h_n\}$ is the unique biorthonormal sequence associated with $\{g_n\}$. In particular, $\{S^{-1}g_n\} = \{h_n\}$, and so S^{-1} is the frame operator of the dual frame $\{S^{-1}g_n\}$.

Proof. Since $\{g_n\}$ is exact, $\{g_n\}$ and $\{S^{-1}g_n\}$ are biorthonormal (Theorem 2.2.c); and since $\{g_n\}$ is complete, we see that $\{S^{-1}g_n\}$ is the unique biorthonormal sequence associated with $\{g_n\}$. (5.1) follows immediately from Theorem 2.2.a. ∎

Theorem 5.2. *Given $\Omega > 0$ and $\{a_n\} \subseteq \mathbf{R}$, let $t_n = -a_n$, and assume $\{E_{a_n}\}$ is an exact frame for $L^2[-\Omega, \Omega]$. Define $s_n(t)$ in terms of its involution $\tilde{s}_n(t) = \overline{s_n(-t)}$, where*

$$\forall t \in \mathbf{R}, \quad \tilde{s}_n(t) = \int_{-\Omega}^{\Omega} \overline{h}_n(\gamma) e^{2\pi i t\gamma} d\gamma, \tag{5.2}$$

and where $\{h_n\}$ is the unique biorthonormal sequence associated with $\{E_{a_n}\}$. (In particular, $\tilde{s}_n \in PW_\Omega$). Then

$$\forall f \in PW_\Omega, \quad f = \sum f(t_n) s_n \quad in\, L^2(\mathbf{R}). \tag{5.3}$$

where $s_n(t) = \overline{\tilde{s}_n(-t)} \in PW_\Omega$.

Proof. Let $g = \frac{1}{(2\Omega)^{1/2}} d_{2\pi\Omega}$ and set $b_m = 2\Omega m$. Note that

$$G(\gamma) \equiv \sum |\hat{g}(\gamma - b_m)|^2 = \frac{1}{2\Omega} \quad a.e.$$

and $supp\hat{g} \subseteq [-\Omega, \Omega]$. Thus, since $\{E_{a_n}\}$ is a frame, we can apply Theorem 4.1 to obtain that $\{E_{a_n} T_{b_m} \hat{g}\}$ is a frame for $L^2(\widehat{\mathbf{R}})$ with frame operator S. In particular,

$$\forall h \in L^2(\mathbf{R}), \quad \hat{h} = \sum \langle \hat{h}, E_{a_n} T_{b_m} \hat{g} \rangle S^{-1}(E_{a_n} T_{b_m} \hat{g}) \quad in \quad L^2(\widehat{\mathbf{R}}). \tag{5.4}$$

Similarly to (3.4), we obtain

$$\langle \hat{f}, E_{a_n} T_{2\Omega m} \hat{g} \rangle = \begin{cases} \frac{1}{(2\Omega)^{1/2}} f(-a_n), & \text{if } m = 0 \\ 0, & \text{if } m \neq 0 \end{cases}$$

for $f \in PW_\Omega$.

Let S_m be the frame operator for the frame $\{T_{b_m} E_{a_n}\}$ for $L^2(I_m)$, where $I_m = [-\Omega, \Omega] + b_m$. By Corollary 4.2, we have

$$(5.6) \qquad \forall h \in L^2(\mathbf{R}), \quad S\widehat{h} = \sum_m T_{b_m}\widehat{g} S_m(\widehat{h} T_{b_m}\overline{\widehat{g}}) \quad in \quad L^2(\mathbf{R}).$$

From (5.6) and the definition of g we compute

$$S\widehat{f} = \frac{1}{2\Omega} \sum_m \mathbf{1}_{(\Omega)}(\gamma - 2\Omega m) S_m(\widehat{f}(\cdot)\mathbf{1}_{(\Omega)}(\cdot - 2\Omega m))(\gamma)$$

$$= \frac{1}{2\Omega} S_0 \widehat{f}(\gamma)$$

for $f \in PW_\Omega$, where the second equality follows since $supp \widehat{f} \subseteq [-\Omega, \Omega]$. Thus,

$$(5.7) \qquad \forall f \in PW_\Omega, \quad S\widehat{f} = \frac{1}{2\Omega} S_0 \widehat{f},$$

i.e., the action of S on $L^2[-\Omega, \Omega]$ can be realized by the action of $\frac{1}{2\Omega} S_0$ on that same subspace of $L^2(\widehat{\mathbf{R}})$.

Using (5.7), we can write

$$\forall f \in PW_\Omega, \quad \widehat{f} = S^{-1} S \widehat{f}$$

$$= \frac{1}{2\Omega} S^{-1} S_0 \widehat{f},$$

so that, if we replace $\widehat{f}$ by $S_0^{-1}\widehat{f}$, we obtain

$$S^{-1} = 2\Omega S_0^{-1} \quad on \quad L^2[-\Omega, \Omega].$$

Therefore, since $g \in PW_\Omega$,

$$S^{-1}(E_{a_n} T_{b_0} \widehat{g}) \equiv S^{-1}(E_{a_n} \widehat{g})$$

$$= 2\Omega S_0^{-1}(E_{a_n} \widehat{g})$$

$$= (2\Omega)^{1/2} S_0^{-1}(E_{a_n} \mathbf{1}_{(\Omega)})$$

$$= (2\Omega)^{1/2} \sum_m \langle E_{a_n}, h_m \rangle_{[-\Omega, \Omega]} h_m$$

$$= (2\Omega)^{1/2} h_n,$$

where the penultimate equality depends on the exactness hypothesis and Proposition 5.1. Substituting this information into (5.4) and (5.5) gives the reconstruction,

$$\forall f \in PW_\Omega, \quad \hat{f} = \sum_n \frac{1}{(2\Omega)^{1/2}} f(-a_n)(2\Omega)^{1/2} h_n \quad in\, L^2(\hat{\mathbf{R}}),$$

which, in turn, yields (5.3). ∎

Remark 5.3. Levinson [L, Theorem 18] proved that if $\Omega > 0$ and $\{a_n\} \subseteq \mathbf{R}$ satisfy

$$\sup_n |n - 2\Omega a_n| < \frac{1}{4}, \tag{5.8}$$

then $\{E_{a_n}\}$ is complete in $L^2[-\Omega,\Omega]$ and has a unique biorthonormal sequence $\{h_n\}$. Kadec (1964) [Ka] provided the direct calculation proving that $\{E_{a_n}\}$ is a Riesz basis, i.e., exact frame, if (5.8) holds, cf., [Y, pp. 34-36] for a characterization to ensure that complete sets with associated biorthonormal sequences are Riesz bases.

The bound «1/4» in (5.8) is best possible [L, Theorem 19].

The explicit formulas in the following result are proved in [PW, pp. 89-90 and pp. 114-116] and [L, pp. 48 ff]. The calculations by Paley and Wiener were refined by Young (1979), e.g., [Y, pp. 148-150]. The remainder of the proof is referenced in Remark 5.3.

Theorem 5.4. *Given $\Omega > 0$ and $\{a_n\} \subseteq \mathbf{R}$, and assume (5.8). Then $\{E_{a_n}\}$ is an exact frame for $L^2[-\Omega,\Omega]$ with unique biorthonormal sequence $\{h_n\}$; and $\tilde{s}_n$, defined by (5.2), is*

$$\tilde{s}_n(t) = \frac{\tilde{s}(t)}{\tilde{s}'(a_n)(t-a_n)} \tag{5.9}$$

where

$$\tilde{s}(t) = (t-a_0)\prod_{n=1}^{\infty}\left(1-\frac{t}{a_n}\right)\left(1-\frac{t}{a_{-n}}\right). \tag{5.10}$$

5.5 Example. Note that the sampling functions s_n, defined in Theorem 5.2, are given by

$$s_n(t) = \frac{s(t)}{s'(t_n)(t-t_n)},$$

where

$$s(t) = (t - t_0) \prod_{n=1}^{\infty} \left(1 - \frac{t}{t_n}\right)\left(1 - \frac{t}{t_{-n}}\right);$$

and they have the property that

$$\forall m, n \quad s_n(t_m) = \langle h_n, E_{a_m} \rangle_\Omega = \delta_{mn}. \tag{5.11}$$

This property of sampling functions is usually described by saying that $\{s_n\}$ is a sequence of Lagrangia interpolating functions.

6. IRREGULAR SAMPLING - IRREGULAR TRANSLATES OF A SAMPLING FUNCTION

Our basic result in this section, Theorem 6.2, is dual to the sampling theorem, Theorem 5.2, in the following way. Exact frames were required in Section 5 and the sampled values of the signal were explicit in the dual frame expansion. Theorem 6.2 will use general frames, and the frame expansion will only require the irregular translates of a single sampling function. The dual frame expansion was global in Section 5 and the frame expansion is local in this section.

The following fact is clear.

Lemma 6.1. *Given $f, f_n \in L^2(\mathbf{R})$, and assume $f = \sum f_n$ in $L^2(\mathbf{R})$. If $g \in L^\infty(\mathbf{R})$ then $fg = \sum f_n g$ in $L^2(\mathbf{R})$.*

Theorem 6.2. *Given $\Omega > 0$ and $\{a_n\} \subseteq \mathbf{R}$, let $t_n = -a_n$, and assume $\{E_{a_n}\}$ is a frame for $L^2[-\Omega_1, \Omega_1]$, for some $\Omega_1 > \Omega$, with frame operator S. Let $g \in S(\mathbf{R})$ have the properties that $supp \widehat{g} \subseteq [-\Omega_1, \Omega_1]$ and $\widehat{g} = 1$ on $[-\Omega, \Omega]$. Then*

$$\forall f \in PW_\Omega, \quad f = \sum \langle S^{-1}(\widehat{f}\mathbf{1}_{(\Omega_1)}), E_{-t_n} \rangle_{[-\Omega_1, \Omega_1]} T_{t_n} s \in L^2(\mathbf{R}), \tag{6.1}$$

where $s \equiv g$. (We choose «s» since it represents the «sampling» function).

Proof. Since $\{E_{a_n}\}$ is a frame for $L^2[-\Omega_1, \Omega_1]$ and $supp \widehat{f} \subset [-\Omega, \Omega]$, we have

$$\widehat{f} = \widehat{f}\mathbf{1}_{(\Omega_1)} \tag{6.2}$$

$$= \sum \langle S^{-1}(\widehat{f}\mathbf{1}_{(\Omega_1)}), E_{a_n} \rangle_{[-\Omega_1, \Omega_1]} E_{a_n} \mathbf{1}_{(\Omega_1)} \quad in\ L^2(\widehat{\mathbf{R}}).$$

In this expression, we note that S^{-1}, being positive, is self-adjoint so that the frame expansion in Theorem 2.2.a gives rise to (6.2). Also, the $L^2[-\Omega_1,\Omega_1]$ convergence from our frame hypothesis can be taken to be in $L^2(\widehat{\mathbf{R}})$ by extending all functions to be zero outside $[-\Omega_1,\Omega_1]$.

We have $\widehat{f}=\widehat{f}\widehat{g}$ on $\widehat{\mathbf{R}}$ since $\widehat{g}=1$ on $[-\Omega,\Omega]$ and $\widehat{f}=0$ off of $[-\Omega,\Omega]$. Also,

$$\widehat{g}\sum\langle S^{-1}(\widehat{f}\mathbf{1}_{(\Omega_1)}),E_{a_n}\rangle_{[-\Omega_1,\Omega_1]}E_{a_n}\mathbf{1}_{(\Omega_1)}$$

$$=\sum\langle S^{-1}(\widehat{f}\mathbf{1}_{(\Omega_1)}),E_{a_n}\rangle_{[-\Omega_1,\Omega_1]}E_{a_n}\mathbf{1}_{(\Omega_1)}\widehat{g}\quad in\ L^2(\widehat{\mathbf{R}})$$

by Lemma 6.1. Thus, since $supp\widehat{g}\subseteq[-\Omega_1,\Omega_1]$, we obtain

$$\widehat{f}=\widehat{f}\widehat{g}$$

$$=\sum_n\langle S^{-1}(\widehat{f}\mathbf{1}_{(\Omega_1)}),E_{a_n}\rangle_{[-\Omega_1,\Omega_1]}E_{a_n}\widehat{g}\quad in\ L^2(\widehat{\mathbf{R}}).$$

Taking the inverse transform gives (6.1). ∎

The following result allows us to be more explicit about the coefficients in (6.1) in the case of exact frames and the Levinson (and Kadec) condition (5.8).

Theorem 6.3. *Given $\Omega>0$ and $\{a_n\}\subseteq\mathbf{R}$, let $t_n=-a_n$, and assume*

$$\sup_n|n-2\Omega_1 a_n|<\frac{1}{4}$$

for some $\Omega_1>\Omega$. Then $\{E_{a_n}\}$ is an exact frame for $L^2[-\Omega_1,\Omega_1]$ with frame operator S and unique biorthonormal sequence $\{h_n\}$. Further, if we define $\widetilde{s}_n$ and $\widetilde{s}$ on $[-\Omega_1,\Omega_1]$ by (5.9) and (5.10), then $(\widetilde{s}_n)^\wedge=\overline{h}_n$ (where $h_n\equiv 0$ off of $[-\Omega_1,\Omega_1]$) and the coefficients of (6.1) are

$$\forall n,\quad\langle S^{-1}(\widehat{f}\mathbf{1}_{(\Omega_1)}),E_{-t_n}\rangle_{[-\Omega_1,\Omega_1]}=\langle f(t),s_n(t)\rangle,$$

where $f\in PW_\Omega$.

Proof. The exact frame and biorthonormal sequence conclusions follow from Theorem 5.4, as well as (5.9), (5.10), and the relation $(\widetilde{s}_n)^\wedge=\overline{h}_n$. Letting $H=L^2[-\Omega_1,\Omega_1]$ in Proposition 5.1, we have

$$\forall F\in L^2[-\Omega_1,\Omega_1],\quad S^{-1}(F)=\sum\langle F,h_m\rangle_{[-\Omega_1,\Omega_1]}h_m.$$

Consequently, by orthonormality,

$$\forall n, \quad S^{-1}(E_{a_n}) = \sum \langle E_{a_n}, h_m\rangle_{[-\Omega_1,\Omega_1]} h_m$$
$$= \langle E_{a_n}, h_n\rangle_{[-\Omega_1,\Omega_1]} h_n$$
$$= h_n.$$

Therefore, setting $h_n = 0$ off of $[-\Omega_1, \Omega_1]$ and noting that $\tilde{s}_n$ is real-valued, we compute

$$\langle S^{-1}(\widehat{f}\mathbf{1}_{(\Omega_1)}), E_{-t_n}\rangle_{[-\Omega_1,\Omega_1]} = \langle \widehat{f}, S^{-1}(E_{a_n})\rangle_{[-\Omega_1,\Omega_1]}$$
$$= \langle \widehat{f}, h_n\rangle_{[-\Omega_1,\Omega_1]}$$
$$= \langle \widehat{f}, h_n\rangle_{\hat{\mathbf{R}}}$$
$$= \langle f, \check{h}_n\rangle_{\mathbf{R}}$$
$$= \langle f(t), \overline{\tilde{s}_n(-t)}\rangle$$
$$= \langle f(t), s_n(t)\rangle,$$

for each $f \in PW_\Omega$. ∎

Algorithm 6.4. It is possible to estimate the coefficients in (6.1) without dealing with exact frames. In so doing, we shall see to what extent these coefficients contain information from the sampled values $f(t_n)$.

Let $\{E_{a_n}\}$ be a frame for $L^2[-\Omega_1, \Omega_1]$ with frame operator S and frame bounds A and B. Since

$$\left\| I - \frac{2}{A+B} S \right\| \leq \frac{B-A}{A+B} < 1, \tag{6.3}$$

we have

$$S^{-1} = \frac{2}{A+B} \sum_{k=0}^{\infty} \left(I - \frac{2}{A+B} S\right)^k, \tag{6.4}$$

where $I : L^2[-\Omega_1, \Omega_1] \rightarrow L^2[-\Omega_1, \Omega_1]$ is the identity map, the norm in (6.3) is the operator norm, and the convergence in (6.4) is the operator norm topology on the space of continuous linear operators on $L^2[-\Omega_1, \Omega_1]$ (into itself).

Setting $t_n = -a_n$ and using (6.4), the coefficients in (6.1) become

$$c_n \equiv \langle S^{-1}(\widehat{f}\mathbf{1}_{(\Omega_1)}), E_{-t_n}\rangle_{[-\Omega_1,\Omega_1]} \tag{6.5}$$

$$= \frac{2}{A+B}\sum_{k=0}^{\infty}\left\langle \left(I - \frac{2}{A+B}S\right)^k (\widehat{f}\mathbf{1}_{(\Omega_1)}), E_{-t_n}\right\rangle_{[-\Omega_1,\Omega_1]}$$

for $f \in PW_\Omega, \Omega < \Omega_1$. If we truncate the expansion (6.5) after the $k = 0$ term, then

$$c_n = \frac{2}{A+B} f(t_n).$$

NOTATION

The Fourier transform $\widehat{f}$ of $f \in L^1(\mathbf{R})$ is defined as $\widehat{f}(\gamma) = \int f(t)e^{-2\pi i t\gamma}dt$, where «$\int$» designates integration over the real line $\mathbf{R}$; $\widehat{f}$ is defined on $\widehat{\mathbf{R}}(= \mathbf{R})$ and $\check{f}$ is the inverse Fourier transform of f. The Fourier transform is defined on $L^2(\mathbf{R})$, and, for fixed $\Omega > 0$,

$$PW_\Omega \equiv \{f \in L^2(\mathbf{R}) : supp\widehat{f} \subseteq [-\Omega,\Omega]\},$$

where $supp\widehat{f}$ is the support of $\widehat{f}$. A function (or distribution) f, whose Fourier transform exists, is Ω -bandlimited if $supp\widehat{f} \subseteq [-\Omega,\Omega]$.

Besides the $L^p(\mathbf{R})$-spaces and the Schwartz space $\mathscr{S}(\mathbf{R})$, we deal with the space $C^\infty(\mathbf{R})$ of infinitely differentiable functions and its subspace $C_c^\infty(\mathbf{R})$ whose elements have compact support.

«$\sum$» designates summation over the whole discrete group in question, e.g., over $\mathbf{Z} \times \mathbf{Z}$ where $\mathbf{Z}$ is the group of integers. The function $\mathbf{1}_S$ is the characteristic function of $S \subseteq \mathbf{R}$, $|S|$ is the Lebesgue measure of S, and $\mathbf{1}_{(\Omega)} \equiv \mathbf{1}_{[-\Omega,\Omega]}$. The function δ_{mn} is defined as 0 if $m \neq n$ and as 1 if $m = n$. The dilation f_λ of the function f is $f_\lambda(t) = \lambda f(\lambda t)$, and the translation $T_{t_0}f$ is $T_{t_0}f(t) = f(t - t_0)$. Finally, the exponential function E_a is $E_a(t) = e^{2\pi i a t}$.

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