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CHARACTERIZATIONS OF ALMOST SHRINKING BASES

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1.1. INTRODUCTION AND MOTIVATION

The material of this paper depends upon the theory of locally convex spaces, sequence spaces and Schauder bases in topological vector spaces and as such we refer to [2] (cf. also [9]), [6] and [7] respectively for several unexpalined definitions, results and terms prevalent in the sequel. However, we do recall a few definitions and terms relevant to the present paper. So, we write thorughout $X \equiv (X,T)$ for an arbitrary Hausdorff locally convex space (l.c.s.) with X^* denoting the topological dual of X and D_T representing the saturated collection of all T-continuous seminorms generating the locally convex (l.c.) topology T on X. Also we write the pair of sequences $\{x_n; f_n\}$ for an arbitrary Schauder basis (S.b.) for X where $x_n \in X$, $f_n \in X^*$ and $f_m(x_n) = \delta_{mn}; m,n \geq 1$. An S.b. $\{x_n; f_n\}$ for (X,T) is called *shrinking* if $\{f_n; \Psi x_n\}$ is an S.b. for the strong dual $(X^*, \beta(X^*,X))$, Ψ being the usual canonical embedding from X into $X^{**} \equiv (X^*, \beta(X^*,X))^*$.

Shrinking bases were essentially introduced for two reasons: (i) to find applications of Schauder bases in the structural study of locally convex spaces, and (ii) to answer a natural question concerning the S.b. character of $\{f_n; \Psi x_n\}$ for the space $(X^*, \beta(X^*,X))$ in case X has already an S.b. $\{x_n; f_n\}$. Kalton made an interesting observation in [3]: indeed, there are spaces with S.b.'s for which the strong duals are not even separable (for instance, consider the space $(\ell^1, \sigma(\ell^1, \ell^\infty))$ equipped with its S.b. $\{e^n; e^n\}$, $e^n = \{\delta_{nj} : j = 1,2,\ldots\}$ where δ_{nj} is the Kronecker delta) and so it appears that it is too much to ask the S.b. character of the s.a.c.f. $\{f_n\}$ for the strong dual $(X^*, \beta(X^*,X))$ of an l.c.s. X containing an S.b. $\{x_n; f_n\}$. On the other hand, if an l.c.s. X has an S.b., $\{x_n; f_n\}$, then $(X^*, \tau(X^*,X))$ is always separable (cf. [8]) and hence we are more justified in asking the following property of an S.b. contained in [3]:

Definition 1.1.1. *An S.b.* $\{x_n; f_n\}$ *for an l.c.s.* (X,T) *is called* almost shrinking *(an a.s.S.b.) provided* $\{f_n; \Psi x_n\}$ *is an S.b. for the Mackey space* $(X^*, \tau(X^*,X))$.

The next two examples further justify the introduction of almost shrinking bases.

Example 1.1.2. Consider the space $(\ell^1, \tau(\ell^1, \ell^\infty))$ having the S.b. $\{e^n; e^n\}$. Then $\{e^n; e^n\}$ is an S.b. for $(\ell^\infty, \tau(\ell^\infty, \ell^1))$; for details, see Section 8, Chapter 2 of [6].

Example 1.1.3. The space $(\ell^1, \sigma(\ell^1, k))$ has an S.b. $\{e^n; e^n\}$. However, $\{e^n; e^n\}$ is not an S.b. for $(k, \tau(k, \ell^1))$; cf. [6], p. 123 or [7], p. 90.

Remark. $\mathfrak{G}$ -uniform bases were introduced in [5] and if $\mathfrak{G}$ denotes the collection of all balanced, convex and $\sigma(X,X^*)$-compact subsets of an l.c.s. X having an S.b. $\{x_n; f_n\}$, then the concepts of an $\mathfrak{G}$-uniform base and a.s.S.b. are equivalent.

In this paper, we primarily concentrate on two more characterizations of almost shrinking bases and these results extend those given in [3], which we follow in spirit throughout the sequel.

2.1. CHARACTERIZATIONS

Our first characterization of almost shrinking bases depend upon three celebrated results of Eberlein, Krein and Šmulian in the theory of locally convex spaces and runs as follows.

Theorem 2.1.1. *Let $(X,\tau(X,X^*))$ be quasi-complete. Then an S.b. $\{x_n; f_n\}$ for (X,T) is an a.s.S.b. if and only if for any $y_k \to y$ in $(X,\sigma(X,X^*))$ and $\{n_k\}$ in $\mathfrak{g}$, the collection of all infinite subsequences of $\mathbf{N} = \{1,2,\ldots\}$, the following relation holds.*

$$(2.4.2) \qquad S_{n_k}(y_k) \to y \quad in \quad \sigma(X,X^*)$$

where

$$S_n(x) = \sum_{i=1}^{n} f_i(x) x_i, \qquad x \in X.$$

Proof. Let $\{x_n; f_n\}$ be an a.s.S.b. for (X,T). If $K = \{y_k\} \cup \{y\}$, then by Krein's Theorem (cf. [9], p. 325), K^{00} is $\sigma(X,X^*)$-compact. Hence for $\varepsilon > 0$ and f in X^*, there exists k_0 such that

$$\sup_{j\geq 1}\left|\left(f - \sum_{i=1}^{n_k} \Psi x_i(f) f_i\right)(y_j)\right| \leq \varepsilon, \qquad \forall k \geq k_0.$$

In particular,

$$|f(y_k) - f(S_{n_k}(y_k))| \leq \varepsilon, \qquad \forall k \geq k_0$$

and this proves the necessity.

The converse makes repeated use of the theorems of Eberlein and Krein. Let W be a $\sigma(X,X^*)$-compact subset of X and put

$$K \equiv K(W) = \bigcup_{n\geq 1} S_n[W].$$

We first show that K is $\sigma(X,X^*)$-relatively compact. Since sequential compactness implies countable compactness, appealing to Eberlein's Theorem (cf. [9], p. 313) it is enough to show that K is $\sigma(X,X^*)$-relatively sequentially compact. Consider, therefore, an arbitrary sequence $\{y_k\}$ in K. Then $y_k = S_{n_k}(u_k), u_k \in W, k \geq 1$. Since $(X^*, \sigma(X^*, X))$ is separable, $(X, \sigma(X, X^*))$ admits a weaker metrizable locally convex topology by the Hahn-Banach theorem. Hence by the extended form of Šmulian's theorem (cf. [9], p. 311) we may assume that $\{u_k\}$ has a subsequence which we denote by itself such that $u_k \to u$ in $(X, \sigma(X, X^*))$.

If $\{n_k\}$ is bounded, then $\{y_k\} \subset U\{S_n[W] : n = 1, \ldots, N\}$ and since the latter set is relatively compact (being finite dimensional) $\{y_k\}$ has a convergent subsequence. On the other hand, assume that $n_k \to \infty$, then by the hypothesis, $S_{n_k}(u_k) \to u$. Thus, in any case, K is $\sigma(X, X^*)$-relatively sequentially compact.

Replacing W by a balanced, convex and $\sigma(X, X^*)$-compact subset of X and noting that K^{00} is $\sigma(X, X^*)$-compact along with the fact that $f[S_n[W]] \subset f[K^{00}]$ for each $n \geq 1$ and f in X^*, one easily verifies that $\{S_n^*\}$ is equicontinuous on $(X^*, \tau(X^*, X))$. Further, $[f_n]^\sigma = [f_n]^\tau = X^*$, where $\sigma \equiv \sigma(X^*, X)$ and $\tau \equiv \tau(X^*, X)$. The required result now easily follows, for instance, one may use Theorem 2.1 of [4].

Another characterization

The next main theorem of this section depends upon the characterization of compact subsets of X^* under different polar topologies and these results are derived with the help of several important theorems of Grothendieck. We thus pass on to a few preparatory lemmas. At the outset, we mention that $X \equiv (X, T)$ stands for an arbitrary l.c.s. with or without an S.b. and to avoid repetition of bigger symbols, let us abbreviate hereafter the symbols $\sigma(X, X^*), \sigma(X^*, X), \tau(X, X^*)$ and $\tau(X^*, X)$ as σ, σ_*, τ and τ_* respectively.

In order to prove the intermediate lemmas, we need the concept of T-limited sets of X^+ (the sequential dual of an l.c.s. (X, T)) and related results. A subset K of X^+ is called T-limited if for each null sequence in (X, T),

$$\lim_{n\to\infty} \sup_{f \in K} |f(x_n)| = 0.$$

Corresponding to an l.c.s. (X, T), let T^+ denote the finest Hausdorff locally convex topology on X such that T and T^+ have the same convergent sequences. Then $X^+ = (X, T^+)^*$ and T^+ is the topology of uniform convergence of all T-limited subsets of X^+; cf. [11] for all relevant details.

Given an l.c.s. (X, T), let $\mathscr{S}_c^T$ (resp. $\mathscr{S}_n^T$) denote the family of all subsets A of X with the property that every sequence in A has a T-Cauchy subsequence (resp. the family of all T-null sequences). Further, let us write T_c (resp. T_n) for the locally convex topology on X^+ generated by th polars of $\mathscr{S}_c^T$ (resp. $\mathscr{S}_n^T$). Then following [1], Exercise 2.2), p. 214, we have

Proposition 2.1.3. *Let (X,T) be an l.c.s. and $K \subset X^+$. Then the following statements are equivalent: (i) K is T-limited, (ii) K is T_c-precompact and (iii) K is T_n-precompact.*

The next lemma is a simple consequence of Grothendieck's completion theorem.

Lemma 2.1.4. *Let (X,T) be a Mazur space $(X^* = X^+)$. Then (X^*, σ_n) is complete; in addition if (X,τ) is quasi-complete, then (X^*, τ_*) is complete.*

Lemma 2.1.5. *Let (X,T) be a Mazur space such that (X,τ) is quasi-complete. Let $K \subset X^*$, then the following statements are equivalent:*

(i) K is σ-limited

(ii) K is σ_n-relatively compact.

If (X,T) also satisfies the condition that X^* is σ_*-separable or alternatively, (X,T) admits a weaker metrizable locally convex topology, then (i) and (ii) are equivalent to

(iii) K is τ_*-relatively compact.

Proof. For the equivalence of (i) and (ii), use Proposition 2.1.3 and Lemma 2.1.4.

(iii) $\Rightarrow$ (ii). By Krein's Theorem, $\sigma_n \subset \tau_*$.

(ii) $\Rightarrow$ (iii). By Proposition 2.1.3, K is σ_c-precompact. Hence from Šmulian's Theorem, K is τ_*-precompact and now use Lemma 2.1.4.

Lemma 2.1.5 and the external construction of σ^+ (cf. second paragraph of this subsection) now immediately yield.

Lemma 2.1.6. *Let (X,T) be a Mazur space such that (X,T) is quasi-complete and X^* is σ_*- separable. Then σ^+ is equivalent to the locally convex topology generated by the polars of all τ_*-compact subsets of X^*.*

Finally, we pass on to the main

Theorem 2.1.7. *Let (X,T) be a Mazur space having a S.b. $\{x_n; f_n\}$ such that (X,τ) is complete. Then the following statements are equivalent:*

(i) $\{x_n; f_n\}$ is an a.s.S.b.

(ii) $\{f_n; \Psi x_n\}$ is an e-Schauder base for (X^*, τ_*).

(iii) $\{x_n; f_n\}$ is e-Schauder for (X, σ^+).

Proof. The implications (i) $\Leftrightarrow$ (ii), especially the first one are contained in the proof of the second half of Theorem 2.1.1.

(ii) $\Rightarrow$ (iii). It is enough to show that $\{S_n\}$ is σ^+-equicontinuous. In other words, by Lemma 2.1.6, given a τ_*-compact subset K of X^*, we have to find a τ_*-compact subset J of X^* such that

$$(*) \qquad \sup_{f\in K} |\langle S_n(x), f\rangle| \le \sup_{g\in J} |\langle x, g\rangle|, \qquad \forall x \in X.$$

To prove (*), let us observe that

$$P(S_n^*(f)) \leq q(f); \qquad \forall f \in X^*, n \geq 1$$

where p and q are τ_*-continuous seminorms on X^* such that p is arbitrary and q depends upon p. If $r = \max(p, q)$, then for each $\varepsilon > 0$ we can find $g_1, \ldots, g_m$ in K so that $K \subset \{g_1, \ldots, g_m\} + \{g \in X^* : r(g) < \varepsilon/3\}$. Further, there exists N so that $p(S_n^*(g_j) - g_j) < \varepsilon/3$ for all $n \geq N$ and each j with $1 \leq j \leq m$. Also for f in $K, r(f - g_{j_0}) < \varepsilon/3$ for some j_0 in $\{1, \ldots, m\}$. Hence $p(S_n^*(f) - f) < \varepsilon$ for all $n \geq N$ uniformly in $f \in K$. Thus, if

$$H = \bigcup_{m \geq 1} S_m^*[K],$$

it follows that $S_n^*(f) \to f$ uniformly on H relative to τ_*. Since $S_n^*[H]$ is τ_*-precompact for each $n \geq 1$, H is τ_*-precompact by a result of [10]. Therefore by Lemma 2.1.4, $J = \overline{H}$ is τ_*-compact. Finally, for x in X,

$$\sup_{f \in K} |\langle S_n(x), f \rangle| = \sup_{f \in K} |\langle x, S_n^*(f) \rangle| \leq \sup_{g \in J} |\langle x, g \rangle|.$$

(iii) $\Rightarrow$ (i). This follows from Theorem 2.1.1.

3.1. CONCLUDING REMARKS

It is clear that a shrinking basis is always an a.s.S.b. and the converse is known only for Banach spaces in an elegant result of [3], where one only, requires a justifiable restriction on the dual X^*, i.e. that it is norm separable; that is, in such Banach spaces, both notions of a basis are equivalent. However, this result does not seem to have an analogue for more general spaces.

There exists a good relationship between unconditional and shrinking bases (cf. [7] and [8], Chapter 9) and so one would be tempted to know a similar relationship between an a.s.S.b. and one of the bases of the types – unconditional, subseries or bounded multiplier. In this direction, we offer the following simple.

Proposition 3.1.1. *Every subseries base $\{x_n; f_n\}$ for an S-space (X,T) is an a.s.S.b.*

Proof. If $f \in X^*$, then

$$f = \sum_{n \geq 1} f(x_n) f_n,$$

the series being subseries convergent in $(X^*, \sigma(X^*, X))$ and the required result follows by the well-known Orlicz-Pettis theorem (cf. [6]).

Remark. The condition that $(X^*, \sigma(X^*, X))$ is sequentially complete in Proposition 3.1.1. cannot be dropped, for consider

Example 3.1.2. This is the space $(\ell^1, \sigma(\ell^1, k))$ considered in Example 1.1.3 and observe that this space is not an S-space.

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