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FUZZY PSEUDO-TOPOLOGICAL PROPERTIES AND MAXIMALITY (*)

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0. INTRODUCTION

It is well known that compact Hausdorff topological spaces are maximal compact.

The maximality property with respect to compactness seems to be rather interesting in fuzzy topology where it is possible to find a lot of different formulations of fuzzy compactness axioms and of fuzzy T_2 separation axioms.

In fact, the simultaneous assumption in fuzzy topology of two axioms that generalize compactness and Hausdorfness, respectively, might be made in such a way that any «fuzzy compact» «fuzzy T_2 » fuzzy topological space be maximal «fuzzy compact» (see for example [16] and [10]).

Therefore an intrinsic characterization of maximal «fuzzy compact» fuzzy topologies seems to be useful. In this paper we shall give such a characterization for a large class of properties, namely the fuzzy pseudo-topological properties which verify conditions (1), (2) and (3) given in section 2.

We believe that one of the most important properties of compact (fuzzy) topological spaces is the hereditary property with respect to the closed (fuzzy) subsets and, anyway, such a property plays an important role within the investigation of maximal (fuzzy) compact (fuzzy) topological spaces (see [2] for ordinary topological spaces and section 2 below for fuzzy topological spaces).

In order to formulate the definition of the closed hereditary property of compactlike properties related to a fuzzy topological space we introduce pseudo-topological properties of fuzzy subsets as a non-trivial generalization of fuzzy topological properties (see definition 2.1).

It is obviously meaningful to say that a given (fuzzy) topological property is hereditary with respect to some kind of (crisp) subsets or not, by using the subspace (fuzzy) topology. But this is not the case for fuzzy subsets in a fuzzy topological space, at least in the point-set (or point-free) lattice-theoretic context (in the sense of Rodabaugh [22]).

In fact, if (X, δ) is a fuzzy topological space and $Y \in I^X$ is any fuzzy subset of X, it is not clear which is the induced subtopology on Y.

To our knowledge there are at least two different definitions of subtopologies on a fuzzy subset in a fuzzy topological space (X, δ).

The first one was given by Foster [6] and used by Sarkar [24]: the fuzzy subtopology on any fuzzy subset $Y \subseteq X$ is defined to be the family $\delta_{/Y} = \{A \cap Y : A \in \delta\}$.

This naturally extends the classical definition of a topological subspace obtained by the trace system of open sets, but it has the drawback that the closed fuzzy sets in $\delta_{/Y}$, namely

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the fuzzy sets obtained by setting $Y - (A \cap Y)$ with $A \in \delta$, are generally not the traces on Y of the closed fuzzy sets in δ.

Such a drawback was removed in the second definition, that was given by Erceg [5] and used by Rodabaugh [23].

Erceg modified essentially the notion of the «trace» of a fuzzy set A on a fuzzy set Y; more precisely he called fuzzy subtopology on Y induced by the fuzzy topology δ on L^X the family $\delta_Y = \{A_Y = (A \cap Y) \cup (Y \cap Y') : A \in \delta\}$, which is a fuzzy topology on the lattice $\mathscr{B}_Y = \{A_Y : A \in L^X\}$.

In such a fuzzy subtopology the closed fuzzy sets are the «trace» on Y of the closed fuzzy sets of the whole fuzzy space (X, δ).

Nevertheless, by modifying the definition of the «trace system» of open fuzzy subsets, it is possible to give further definitions of fuzzy subtopologies. We consider such a possibility in another paper [4]. The definition of Erceg can be generalized by using the generalization of «fuzzy subset» introduced by Artico and Moresco in the category of fuzzes (see [1] sect. 3).

Our conclusion is that at this time fuzzy topological properties of fuzzy topological spaces do not concern unequivocally to fuzzy subsets either in the point-set or in the point-free setting.

In a categorical framework with a given mechanism which associates to any fuzzy set a fuzzy topological space (as an object of the category), any fuzzy topological property might be of concern to each fuzzy set too.

But by the constant use of categories of fuzzy topological spaces whose objects have necessarily crisp supports, it often happens that fuzzy topological properties are defined in such a way that they cannot apply to fuzzy (sub)topological spaces on a non-crisp fuzzy (sub)set.

The above seems to be a good motivation to formalize the idea of considering properties of fuzzy subsets in a fuzzy topological space (X, δ) in such a way that they become properties of the fuzzy topological space provided they are verified by the whole set X (in the point-fuzzy set context) or by the maximum $1 \in L$ (in the point-free context).

Keeping in mind the applications and the examples we will give in sections 2, 3 and 4, we formulate our idea in the point-fuzzy set framework using the lattice I, but the same might be done using a different lattice L and a similar formulation might be done in the point-free lattice-theoretic setting.

We call fuzzy pseudo-topological property each property that may be of concern to any fuzzy subset in any fuzzy topological space (X, δ), and that is preserved under fuzzy homeomorphic images.

This idea is not essentially new since it can be found in some way in a paper of Sarkar [24] and, incorrectly from the point of view we present here, in a paper of Lowen [14].

More recently we can refer to Wang [28], whose N-compactness property we consider in section 3, to Li [11] and Luo [18].

With a higher fuzzyness level Šostak generalized classical topological properties i.e. by defining the compactness degree and the connectedness degree of a fuzzy subset in a fuzzy topological space in the sense of Šostak [26].

Here we consider a class of fuzzy pseudo-topological properties including some definitions of compactness and we study the problem of maximality of the fuzzy topologies with respect to the properties of this class.

A new fuzzy pseudo-topological property, S*compactness, is introduced and studied with several examples.

1. PRELIMINARIES

Let X be a non-empty set; we denote by capital letters, for example $A : X \to [0,1]$, the fuzzy sets on X and by I^X, $I = [0,1]$, the family of all fuzzy sets on X.

We indicate by x_α the fuzzy point with support $x \in X$ and value $\alpha \in (0,1]$, that is the fuzzy set such as $x_\alpha(t) = 0$ if $t \neq x$ and $x_\alpha(x) = \alpha$.

The complement A' of a fuzzy set A is defined by $A'(x) = 1 - A(x) \forall x \in X$.

If $\{A_j : j \in J\}$ is a family of fuzzy sets on X, the union and the intersection are defined, respectively, by the formulae $(\cup_{j\in J} A_j)(x) = \sup\{A_j(x) : j \in J\}$ and $(\cap_{j\in J} A_j) = \inf\{A_j(x) : j \in J\}$ $\forall x \in X$.

Moreover, if A and B are two fuzzy sets on X, we call A over B the fuzzy set defined by $(A\Delta B)(x) = A(x)$ if $A(x) > B(x)$ and $(A\Delta B)(x) = 0$ if $A(x) \leq B(x)$. The fuzzy set $A\Delta B$ was denoted by $A \sim B$ in [19].

We say that A is included in B, $A \subseteq B$, if $A(x) \leq B(x) \forall x \in X$. If the fuzzy point x_α is included in A, we say also that x_α belongs to A and write $x_\alpha \in A$.

A fuzzy set A is called a crisp set if $A(x) \in \{0,1\} \forall x \in X$. From now on we identify a crisp set with its support so that all the subsets of X we consider are fuzzy sets included in X.

Let $f : X \to Y$ be a function from X to Y; for any A and B fuzzy sets given in X and Y respectively, we define the image $f(A)$ of A by f to be the fuzzy set in Y such as $f(A)(y) = \sup\{A(x) : f(x) = y\}$ if $y \in f(X)$ and $f(A)(y) = 0$ if $y \notin f(X)$, and call inverse image of B by f the fuzzy set in X $f^{-1}(B) = B \circ f$.

Following Chang [3] we consider a fuzzy topology δ on X to be a family of fuzzy sets on X containing the fuzzy sets ϕ and X and closed under unioins and finite intersections. The fuzzy sets belonging to δ are said to be open, their complements closed.

A fuzzy topology containing all the constant functions of I^X (i.e. a fuzzy topology in the sense of Lowen [14]) is usually called stratified (it was called fuzzy stratified by Pu and Liu [20] and laminated by Šostak [27]).

We say that the fuzzy topology γ is coarser than the fuzzy topology δ, or δ finer than γ, if $\gamma \subseteq \delta$; we shall call δ a refinement of γ too.

If A is a fuzzy set in X and δ is a fuzzy topology on X, we denote by $cl(A)$ the closure of A in δ, that is the smallest closed fuzzy set containing A.

$\delta(A)$ denotes the simple extension of δ by A, that is the fuzzy topology whose elements are the fuzzy sets $O = R \cup (S \cap A)$ whenever $R, S \in \delta$. $\delta(A)$ is the coarsest fuzzy topology containing A and all the members of δ.

Let (X, δ) and (Y, γ) be two fuzzy topological spaces, or fts for short, we say that the function $f : X \to Y$ is fuzzy continuous if $f^{-1}(B) \in \delta\ \forall B \in \gamma$. Moreover we say that f is a fuzzy homeomorphism if f is a fuzzy continuous bijective map of X onto Y such as the inverse map $f^{-1} : Y \to X$ is fuzzy continuous too. f is called fuzzy open if $f(A) \in \gamma\ \forall A \in \delta$.

We say that the fts (X, δ) is T_1 if every fuzzy point in X is closed for δ (see [19], [24] and others).

(X, δ) is to be T_2 if for every pair of fuzzy points x_α and y_β such as $x \neq y$ there exist closed fuzzy sets P_x not containing x_α and P_y not containing y_β such as $P_x \cup P_y = X$ (see for example [19] and [28]).

2. FUZZY PSEUDO-TOPOLOGICAL PROPERTIES AND MAXIMALITY

Definition 2.1. *We call fuzzy pseudo-topological property, $fptp$ for short, a function $\mathscr{P}$ which associates to each fts (X, δ) a function $\mathscr{P}(X, \delta)$ from the set I^X to $\{0, 1\}$ in such a way that, if (X, δ) and (Y, γ) are two fts's and $f : X \to Y$ is a fuzzy homeomorphism, one has $\mathscr{P}(X, \delta)(A) = \mathscr{P}(Y, \gamma)(f(A))$, $\forall A \in I^X$, i.e. the diagram*

$$\begin{array}{ccc} I^X & \xrightarrow{\quad \overline{f} \quad} & I^Y \\ & {\scriptstyle \mathscr{P}(X,\delta)} \searrow \quad \swarrow {\scriptstyle \mathscr{P}(Y,\gamma)} & \\ & \{0, 1\} & \end{array}$$

is commutative, if $\overline{f}$ is the function defined by $\overline{f}(A) = f(A)$ $\forall A \in I^X$.

We shall suggest some generalizations of this definition at the end of this section.

Now we produce trivial examples of $fptp$'s.

Example 2.1. We obtain $fptp$'s $\mathscr{O}, \mathscr{C}, \mathscr{D}$ by setting, respectively, for any $fts(X, \delta)$ and for any fuzzy set A on X,

$$\mathscr{O}(X, \delta)(A) = 1 (= 0) \text{ if } A \in \delta (A \notin \delta),$$

$$\mathscr{C}(X, \delta)(A) = 1 (= 0) \text{ if } A' \in \delta (A' \notin \delta),$$

$$\mathscr{D}(X, \delta)(A) = 1 (= 0) \text{ if } A \text{ is a dense fuzzy set in } X, \text{ i.e. } cl(A) = X$$

(otherwise).

We shall provide in the sequel non-trivial examples.

If $\mathcal{P}$ is an $fptp$ and δ is a fuzzy topology on X, we say that a fuzzy set $A \subseteq X$ satisfies $\mathcal{P}$ for δ, or δ is $\mathcal{P}$ in A, iff $\mathcal{P}(X,\delta)(A) = 1$.

In particular, when X satisfies $\mathcal{P}$ for δ, we say that (X,δ) is a $\mathcal{P}fts$.

Obviously two $fptp's$ $\mathcal{P}_1$ and $\mathcal{P}_2$ coincide (in U) iff $\forall fts(X,\delta)$ and $\forall A \in I^X$ one has $\mathcal{P}_1(X,\delta)(A) = \mathcal{P}_2(X,\delta)(A)$. In other words, $\mathcal{P}_1$ coincides with $\mathcal{P}_2$ iff the fuzzy sets which satisfy $\mathcal{P}_1$ for δ are exactly the ones that satisfy $\mathcal{P}_2$ for δ, whatever may be the $fts(X,\delta)$.

Definition 2.2. *A fuzzy topology δ on a set X is said to be maximal with respect to the $fptp$ $\mathcal{P}$ (for X), or maximal $\mathcal{P}$ (in X), if $\mathcal{P}(X,\delta)(X) = 1$ and $\mathcal{P}(X,\gamma)(X) = 0$ whenever γ is a proper refinement of $\delta(\gamma \supset \delta)$.*

If $\mathcal{P}(X,\delta)(X) = 1$ and from $\mathcal{P}(X,\gamma)(X) = 1$ it follows that $\gamma \subseteq \delta$ for each $fts(X,\gamma)$, then we say that δ is maximum $\mathcal{P}$ (in X).

More briefly we can say that the $fts(X,\delta)$ is maximal $\mathcal{P}$ or maximum $\mathcal{P}$ respectively.

Obviously an $fts(X,\gamma)$ fuzzy homeomorphic to a maximal $\mathcal{P}$ (maximum $\mathcal{P}$) $fts(X,\delta)$ is maximal $\mathcal{P}$ (maximum $\mathcal{P}$) too.

It is easy to show what follows.

Proposition 2.1. *Let (X,δ) be a $\mathcal{P}fts$; the following conditions are equivalent:*

i) (X,δ) is maximum $\mathcal{P}$ (maximal $\mathcal{P}$);

ii) every (fuzzy continuous) bijection $g : (X,\gamma) \to (X,\delta)$ is fuzzy open if (X,γ) satisfies $\mathcal{P}$;

iii) every (fuzzy continuous) bijection $f : (Y,\gamma) \to (X,\delta)$ is fuzzy open if (Y,γ) satisfies $\mathcal{P}$. ∎

Analogous definitions and considerations can be made for the minimality with respect to $\mathcal{P}$.

Moreover it can be shown that there exist two $fptp's$ $\mathcal{P}$ and $\mathcal{P}'$ such as (X,δ) maximum $\mathcal{P}$ and minimum $\mathcal{P}'$ (maximal $\mathcal{P}$ and minimal $\mathcal{P}'$) iff every (fuzzy continuous) bijection $f : (X,\delta) \to (X,\delta)$ is a homeomorfism.

These considerations generalize analogous well known properties of the ordinary topologies (see [2]) and they have been used by Lowen [16] who studied the class of Hausdorff compact stratified fuzzy topologies.

From now on $\mathcal{R}$ will denote an $fptp$ which verifies the following properties (1), (2), (3) and sometime (4) for every $fts(X,\delta)$.

(1) ($\mathscr{R}$ is closed hereditary). If $Y \subseteq X$ is $\mathscr{R}$ for δ and $F \subseteq Y$ is closed in δ, then F is $\mathscr{R}$ for δ.

(2) ($\mathscr{R}$ is contractive). If $Y \subseteq X$ is $\mathscr{R}$ for δ and $\gamma \subseteq \delta$ is another fuzzy topology on X, then Y is $\mathscr{R}$ for γ.

(3) If X is $\mathscr{R}$ for δ and A is a fuzzy set of X such as the complement A' is $\mathscr{R}$ for δ, then X is $\mathscr{R}$ for $\delta(A)$.

(4) If x_α is a fuzzy point of X, then x_α is $\mathscr{R}$ for δ.

Trivially if $\mathscr{P}$ is a contractive $fptp$ and (X,δ) is a $\mathscr{P}fts$, then (X,δ) is maximal $\mathscr{P}$ iff $\mathscr{P}(X,\delta(A))(X) = 0$ whenever $A \notin \delta$.

Furthermore we prove the following results which generalize analogous properties of ordinary topologies (see [2]).

Proposition 2.2. *Let $\mathscr{R}$ be an $fptp$ satisfying conditions (1), (2), (3) and let (X,δ) be an $\mathscr{R}fts$. Then (X,δ) is maximal $\mathscr{R}$ iff the closed fuzzy sets of X in δ are exactly the $\mathscr{R}$ fuzzy sets of X for δ (that is iff $\mathscr{R}(X,\delta) = \mathscr{C}(X,\delta)$).*

Proof. Let (X,δ) be maximal $\mathscr{R}$. If P is closed in δ, by (1) P is also $\mathscr{R}$ for δ. Now suppose P is $\mathscr{R}$ for δ, then X is $\mathscr{R}$ for $\delta(P') \supseteq \delta$ by (3). But since (X,δ) is maximal $\mathscr{R}$, it must be $\delta(P') = \delta$, hence P is closed in δ.

Conversely, let the closed fuzzy sets in δ be the $\mathscr{R}$ fuzzy subsets for δ. If there were an $\mathscr{R}$ fuzzy topology γ strictly finer than δ in X, then we could consider $A \in \gamma - \delta$. Then A' would be closed in γ, hence it would be $\mathscr{R}$ for γ and it should be $\mathscr{R}$ for δ. But from $A \notin \delta$ it would follow that A' is not closed in δ, a contradiction. ∎

Proposition 2.3. *Let $\mathscr{R}$ be an $fptp$ satisfying the conditions (1), (2), (3), (4) and (X,δ) be an fts. If (X,δ) is maximal $\mathscr{R}$, then (X,δ) is T_1.*

Proof. It follows trivially from proposition 2.2. ∎

We shall now define the notion of maximality of a fuzzy subset of X with respect to an $fptp$ $\mathscr{R}$ satisfying (1), (2) and (3).

Definition 2.3. *If (X,δ) is an fts and $Y \subseteq X$ is a fuzzy set, we say that δ is maximal $\mathscr{R}$ in Y if Y is $\mathscr{R}$ for δ and $\forall F \subseteq Y$ one has $F' \in \delta \iff \mathscr{R}(X,\delta)(F) = 1$.*

It is easily seen that if δ is maximal $\mathscr{R}$ in Y and if P is closed in δ, then $P \cap Y$ is closed in δ and $P \cap Y$ is $\mathscr{R}$ for δ. In particular Y is closed in δ and the closed fuzzy sets of δ contained in Y are the traces on Y of the closed fuzzy sets in δ.

It is clear moreover that δ is maximal $\mathscr{R}$ in X according to the definition 2.3 iff (X,δ) is maximal $\mathscr{R}$ according to definition 2.2.

We prove now the following results, where (X,δ) is an fts and Y, T are fuzzy sets on X.

Proposition 2.4. *If δ is maximal $\mathscr{R}$ in Y and $T \subseteq Y$ is $\mathscr{R}$ for δ, then δ is maximal $\mathscr{R}$ in T.*

Proof. From the hypothesis it follows that every fuzzy set $F \subseteq T$ is closed in δ iff F is $\mathscr{R}$ for δ, since $F \subseteq Y$ and δ is maximal $\mathscr{R}$ in Y. ∎

Corollary 2.1. *If (X,δ) is maximal $\mathscr{R}$, δ is maximal $\mathscr{R}$ in each fuzzy subset $Y \subseteq X$ which is $\mathscr{R}$ for δ.* ∎

Proposition 2.5. *Let δ_0, δ be fuzzy topologies on X, $\delta_0 \subseteq \delta$, and let δ_0 be maximal $\mathscr{R}$ in Y and Y be $\mathscr{R}$ for δ. Then δ is maximal $\mathscr{R}$ in Y and δ and δ_0 have the same closed fuzzy sets in Y.*

Proof. We assume $F \subseteq Y$. If F is closed for δ, by (1) F is $\mathscr{R}$ for δ since Y is $\mathscr{R}$ for δ. If F is $\mathscr{R}$ for δ, by (2) it is also $\mathscr{R}$ for δ_0, which is maximal $\mathscr{R}$ in Y; so F is closed for δ_0 and, of course, F is closed for δ. Thus δ is maximal $\mathscr{R}$ in Y.

If F is closed for δ, then F is $\mathscr{R}$ for δ and also fo δ_0, hence F is closed for δ_0 too. Therefore δ and δ_0 have the same closed fuzzy sets in Y. ∎

Now we introduce a new condition attributable to an $fptp\,\mathscr{R}$. Clearly it will be a strenghtening of the condition (3).

(3') If the fuzzy set $Y \subseteq X$ is $\mathscr{R}$ for the fuzzy topology δ on X and A is a fuzzy set in X such as A' is $\mathscr{R}$ for δ and $A' \subseteq Y$, then Y is $\mathscr{R}$ also with respect to the fuzzy topology $\delta(A)$.

Proposition 2.6. *If $\mathscr{R}$ verifies the conditions (1), (2), (3'), if $Y \subseteq X$ is $\mathscr{R}$ for δ and if $\forall F \subseteq Y$ one has that F is $\mathscr{R}$ for δ iff F is the trace on Y of a closed fuzzy set of δ, then the simple extension $\delta(Y')$ is maximal $\mathscr{R}$ in Y.*

Proof. Since Y is $\mathscr{R}$ for δ, it follows from (3') that Y is $\mathscr{R}$ for $\delta(Y')$; furthermore Y is closed for $\delta(Y')$. So, if $F \subseteq Y$ is $\mathscr{R}$ for $\delta(Y')$, it is $\mathscr{R}$ for δ too and there exists P closed in δ such as $F = P \cap Y$. Evidently both P and Y are closed in $\delta(Y')$ hence F is closed in $\delta(Y')$. Since each closed fuzzy set of $\delta(Y')$ contained in Y is $\mathscr{R}$ for $\delta(Y')$, we conclude that $\delta(Y')$ is maximal $\mathscr{R}$ in Y. ∎

Proposition 2.7. *Let $\mathscr{R}$ be an fptp satisfying (1), (2) and (3'), (X,δ) be an fts and $Y \subseteq X$ be a fuzzy set. Then δ is maximal $\mathscr{R}$ in Y iff Y is $\mathscr{R}$ for δ and Y is not $\mathscr{R}$ for $\delta(G)$ whenever $G \notin \delta$ and $G' \subseteq Y$.*

Proof. (Necessity). Consider $G \notin \delta$, $G' \subseteq Y$; if $\delta(G)$ were $\mathscr{R}$ in Y, then, by proposition 2.5, $\delta(G)$ would be maximal $\mathscr{R}$ in Y and it would have the same closed fuzzy sets as δ within Y. Since G' is closed in $\delta(G)$ we would have that G' is closed in δ, which is absurd.

(Sufficiency). Every fuzzy set $F \subseteq Y$ closed in δ is $\mathscr{R}$ for δ, since Y is $\mathscr{R}$ for δ. Conversely, if $F \subseteq Y$ is $\mathscr{R}$ for δ, then F is closed in δ. Otherwise let $G = F'$ be the complement of F; then we should have $G \notin \delta$ and $G' \subseteq Y$, so that Y would not be $\mathscr{R}$ for $\delta(G)$; on the other hand, since G' is $\mathscr{R}$ for δ and $G' \subseteq Y$, from (3') it would follow that Y is $\mathscr{R}$ for $\delta(G)$, which is impossible. ∎

We conclude this section by suggesting some generalizations of the definition 2.1 (for the definitions not given in section 1 we shall refer expressely to some papers listed among the references).

The first one may be given in the category FUZZ considered by Rodabaugh in [22] and including subcategories that can be identified with the categories of Chang-Goguen (point-set lattice-theoretic) $fts's$ or of Lowen (stratified) $fts's$ or of Hutton (point-free) $fts's$ respectively.

Definition 2.4. *A FUZZ pseudo-topological property is a function $\mathscr{P}$ which assignes to each object (X, L, τ) of FUZZ a function from L^X to $\{0,1\}$ in such a way that for every F--homeomorphism (f, φ) from (X, L, τ) to another $fts(Y, M, \sigma)$ the diagram.*

$$\begin{array}{ccc} L^X & \xrightarrow{\ \overline{f}\ } & M^Y \\ & {\scriptstyle \mathscr{P}(X,L,\tau)} \searrow \quad \swarrow {\scriptstyle \mathscr{P}(Y,M,\sigma)} & \\ & \{0,1\} & \end{array}$$

commutes, where $\overline{f}(\mu) = \mu \circ \overline{f}^{-1} \in M^Y$, $\forall \mu \in L^X$ and the minimum 0 and the maximum 1 of the lattices L and M are identified.

When $\mathscr{P}$ is defined in a subcategory $\mathscr{C}$ of FUZZ, it is called a restricted FUZZ pseudo-topological property (relative to $\mathscr{C}$).

The definitions 2.1 and 2.4 may be generalized by using the whole lattices I, L, M instead of their trivial sublattice $\{0,1\}$.

Definition 2.1'. *A generalized fuzzy pseudo-topological property is a function $\mathscr{P}$ which associates to each $fts(X,\delta)$ a function $\mathscr{P}(X,\delta)$ from the set I^X to I in such a way that every fuzzy homeomorphism from (X,δ) to (Y,γ) induces a commutative diagram like in definition 2.1*

$$\begin{array}{ccc} I^X & \xrightarrow{\ \overline{f}\ } & I^Y \\ & {\scriptstyle \mathscr{P}(X,\delta)} \searrow \quad \swarrow {\scriptstyle \mathscr{P}(Y,\gamma)} & \\ & I & \end{array}$$

Definition 2.4'. *A generalized fuzzy pseudo-topological property is a function $\mathscr{P}$ which associates to each object (X, L, τ) of FUZZ a function from L^X to L in such a way that every F-homeomorphism (f, φ) between two arbitrarily given objects (X, L, τ) and (Y, M, σ) induces the below commutative diagram, where $\overline{f}$ is obtained like in definition 2.4*

$$\begin{array}{ccc} L^X & \xrightarrow{\overline{f}} & M^Y \\ \mathscr{P}(X,L,\tau) \downarrow & & \downarrow \mathscr{P}(Y,M,\sigma) \\ L & \xrightarrow[\varphi]{} & M \end{array}$$

Remark. The degree of openess, the degree of closedness, the compactness degree and the connectdness degree considered by Šostak (see [25], [26]) define in a trivial way generalized fuzzy pseudo-topological properties in the category $\mathscr{CS}(I)$ (see [22]) of Chang $fts's$ as well as in the category FT (see [25]) of Šostak $fts's$.

An L-fuzzy topology on X in the sense of Šostak [27] (see also [25]) may be considered as the image of an object (X, L, τ) of FUZZ by a suitable generalized FUZZ pseudo-topological property.

3. SOME KNOWN FUZZY PSEUDO-TOPOLOGICAL PROPERTIES

We recall some definitions given by G. J. Wang [28].

Definition 3.1. *A closed fuzzy set P of an fts is called remoted-neighborhood of the fuzzy point $e = x_\alpha$ if $x_\alpha \notin P$.*

The family of all the remoted-neighborhoods of e is indicated by $\eta(e)$.

Definition 3.2. *If D is the set of all the fuzzy points of X, a fuzzy net $S = \{S(n) : n \in D\}$ is a map $S : D \to E$, where $(D, \geq)$ is a directed set.*

The fuzzy net S is said to be an α-net, $\alpha \in (0, 1]$, if the net $\{\lambda_n : n \in D\}$ converges to α, where λ_n is the value of the fuzzy point $S(n)$.

Definition 3.3. *We say that the fuzzy point e is a cluster point of the fuzzy net S if $\forall P \in \eta(e)$ and $\forall m \in D$ there is $n \geq m$, $n \in D$, s.t. $S(n) \notin P$.*

Definition 3.4. *If (X, δ) is an fts and A is a fuzzy set in X, we assume that A is N-compact for δ if for each $\alpha \in (0, 1]$ and for each α-net S contained in A there exists in A a cluster point S of value α.*

(X, δ) is called N-compact if X is N-compact for δ.

Obviously N-compactness is an $fptp$ which is closed hereditary (see [28]).

Moreover it is plain to show that this $fptp$ is contractive and that every fuzzy point of an $fts(X,\delta)$ is N-compact for δ.

Therefore N-compactness verifies the properties (1), (2) and (4) of the preceding section. We shall prove also that condition (3') is true.

Proposition 3.1. *Let (X,δ) be an fts and $Y \subseteq X$ a fuzzy set which is N-compact for δ. If the fuzzy set $A \subseteq X$ is such as A' is N-compact for δ and $A' \subseteq Y$, then Y is N-compact for $\delta(A)$ too.*

Proof. Take $\alpha \in (0,1]$ and take $S = \{S(n) : n \in D\}$ an α-net contained in Y. We distinguish between two cases.

Case a) Suppose that there exists $n_0 \in D$ so that $\forall n \geq n_0$ one has $S(n) \notin A'$. Since Y is N-compact for δ, there is $x_\alpha \in Y$ cluster point for $S(n)$ with respect to δ. Now let F be closed in $\delta(A)$ and $x_\alpha \notin F$; it is known that it can be written as $F = P \cap (Q \cup A')$, where P and Q are closed in δ. It follows from $x_\alpha \notin F$ that either $x_\alpha \notin P$ or $x_\alpha \notin Q \cup A'$.

If $x_\alpha \notin P$, since x_α is a cluster point for S with respect to δ, one has $S(n) \notin P$ frequently, and hence $S(n) \notin F$ frequently.

If instead $x_\alpha \notin Q \cup A'$ and m is any element of D, take an element $r \in D$ such as $r \geq m$ and $r \geq n_0$. Since $x_\alpha \notin Q$ and x_α is a cluster point for S with respect to δ, there is $n \geq r$ such as $S(n) \notin Q$; on the other hand $\forall n \geq r \geq n_0$ it results clearly $S(n) \notin A'$, then it must be $S(n) \notin Q \cup A'$ and consequently $S(n) \notin F$.

So we verified that x_α is a cluster point for S with respect to $\delta(A)$ in the case a).

Case b) The other possibility is that for every $n \in D$ th ere exists $k \geq n$ such as $S(k) \in A'$. The set $E = \{k \in D : S(k) \in A'\}$ is then a cofinal subset of D and $\{S(k) : k \in E\}$ is a subnet of $\{S(n) : n \in D\}$; we denote this subnet by T, so that $T(k) = S(k)\ \forall k \in E$.

If λ_n is the value of the fuzzy point $S(n)$, it is clear that the net $\{\lambda_k : k \in E\}$ converges to α; thus T is an α-net. Since T is included in A' and A' is N-compact with respect to δ, there is a fuzzy point x_α of value α, $x_\alpha \in A' \subseteq Y$, which is a cluster point for T with respect to δ.

We shall prove that x_α is a cluster point for T with respect to $\delta(A)$ too.

In fact, let $F = P \cap (Q \cup A')$, where P and Q are closed in δ, be a closed fuzzy set in $\delta(A)$ with $x_\alpha \notin F$. It follows from $x_\alpha \in A' \subseteq Q \cup A'$ that $x_\alpha \notin P$. Since x_α is a cluster point for T with respect to δ, we know that $T(k) \notin P$ frequently and hence $T(k) \notin F$ frequently. Now it is clear that a property which is frequently verified in a cofinal subset of a directed set is still frequently true in the directed set, so we conclude that x_α is a cluster point for S with respect to $\delta(A)$ in the case b) too. ∎

We refer now to some definitions given by Z. F. Li [11] and recalled by M. K. Luo [18].

As usual, we consider an ordinary non-empty set X, its fuzzy subsets and a fuzzy topology δ on X, when the need arises.

Definition 3.5. *We say that* $\{U_j : j \in J\}$, $U_j \subseteq X$, *is an* α-*Q-cover of* $Y \subseteq X$, $\alpha \in (0,1]$, *if* $\forall x \in X$ *s.t.* $Y(x) \geq \alpha$ $\exists j \in J$ *s.t.* $U_j(x) > 1-\alpha$.

Y is said to be Q_α-*compact for* δ, $\alpha \in (0,1]$, *if every open* α-*Q-cover of* Y *has a finite* α-*Q-subcover of* Y.

Y is called strong-Q-compact for δ *if* Y *is* Q_α-*compact for each* $\alpha \in (0,1]$.

It is plain that a family of fuzzy sets is an α-Q-cover of X iff it is an $(1-\alpha)$-shading and hence (X,δ) is Q_α-compact (strong-Q-compact) iff it is $(1-\alpha)$-compact (strong fuzzy compact) (see [15] for the definitions we have not given).

It can be verified that the strong-Q-compactness is an $fptp$ and it satisfies the conditions (1), (2), (4) given in the preceding section.

We prove that the condition (3') holds too.

Proposition 3.2. *If* $Y \subseteq X$ *is strong-Q-compact for the fuzzy topology* δ *on* X *and* $A \subseteq X$ *has a strong-Q-compact complement* $A' \subseteq Y$, *then* Y *is strong-Q-compact for* $\delta(A)$.

Proof. Let $\alpha \in (0,1]$ and let $\{V_j \cup (W_j \cap A) : j \in J\}$ be an open α-Q-cover of Y, with $V_j, W_j \in \delta \; \forall j \in J$.

$\{V_j : j \in J\}$ is an α-Q-cover of A'; in fact it follows from $A'(x) \geq \alpha$ that $Y(x) \geq \alpha$ and, since $(A \cap W_j)(x) \leq A(x) \leq 1-\alpha \; \forall j \in J$, some $j \in J$ must be such as $V_j(x) > 1-\alpha$.

By the assumption that A' is strong-Q-compact for δ, we can find a finite subset $J_1 \subseteq J$ such as $\{V_j : j \in J_1\}$ is an α-Q-cover of A'. On the other hand $\{V_j : j \in J\} \cup \{W_j : j \in J\}$ is trivially an open α-Q-cover of the strong-Q-compact subset Y for δ. Hence there exist finite subsets $J_2, J_3 \subseteq J$ such as $\{V_j : j \in J_2\} \cup \{W_j : j \in J_3\}$ is an α-Q-cover of Y.

If we consider the finite set $H = J_1 \cup J_2 \cup J_3$, then we have the finite α-Q-cover of Y $\{V_j \cup (W_j \cap A) : j \in H\}$.

In fact let $x \in X$ be such as $Y(x) \geq \alpha$. Suppose $A'(x) \geq \alpha$, then $j \in J_2$ exists with $V_j(x) > 1-\alpha$ and consequently $(V_j \cup (W_j \cap A))(x) > 1-\alpha$. If otherwise $A'(x) < \alpha$, i.e. $A(x) > 1-\alpha$, then $j \in J_2 \cup J_3 \subseteq H$ exists such as either $V_j(x) > 1-\alpha$ or $W_j(x) > 1-\alpha$ and anyway $(V_j \cup (W_j \cap A))(x) > 1-\alpha$. ∎

Other examples of $fptp's$ can be found in [24], [18] and [14], but none of them verifies all the conditions (1), (2), (3), (4) togheter.

In fact Sarkar showed that proper compactness is not closed hereditary and it is obvious that neither $S*$-paracompactness nor S-paracompactness are contractive (see [18]).

The fuzzy compaactness property of the subsets of an fts given in [14] is not jet closed hereditary. Furthermore we remark that the fuzzy compactness of the $fts(X,\delta)$ defined by Lowen [14] is not equivalent to the fuzzy compactness of the fuzzy set X with respect to δ.

In the next section we are going to study an $fptp$ which does not verify the condition (3') but verifies all the conditions (1), (2), (3), (4).

Zhao [30], Liu and LUo [12], [13] considered N-compactness in L-fts's i.e. essentially in the category $\mathscr{C}\mathscr{S}(L)$ with a subcategory of FUZZ, like in [22], N-compactness can be viewed as a restricted FUZZ pseudo-topological property (see definition 2.4).

4. S*COMPACTNESS

Definition 4.1. *Let (X, δ) be an fts and $A \subseteq X$ a fuzzy set. We say that the family $\{U_j : j \in J\}$ of fuzzy sets in X is an (open in δ) A*shading if $\forall x \in X$ $\exists j \in J$ s.t. $U_j(x) \geq A(x)$ (and $U_j \in \delta$, $\forall j \in J$).*

*In this case we say that $\{U_j : j \in J\}$ is a *shading of A too.*

If U_j is a crisp set $\forall j \in J$ or if J is finite, then an A*shading is nothing else than a cover of A or, as also we will say, an A-cover.

Definition 4.2. *We say that the fuzzy set $Y \subseteq X$ is S*compact for δ if, given an arbitrary closed fuzzy set P in δ, every open *shading in δ of $P \cap Y$ or of $P \Delta Y'$ or of $(P \cap Y) \Delta Y'$ has a finite *subshading of $P \cap Y$ or of $P \Delta Y'$ or of $(P \cap Y) \Delta Y'$ respectively.*

Trivially (X, δ) is S*compact iff for each closed fuzzy set P in δ and for each open P*shading in δ there exists a finite P*subshading.

Furthermore, if (X, δ) is an ordinary topological space and $Y \subseteq X$ is an ordinary subset, then Y is S*compact for δ iff Y is a compact subset in the topological space (X, δ).

It is easily seen that the notion of S*compactness is an $fptp$ fulfilling the conditions (2) and (4). We will show that this $fptp$ verifies also the conditions (1) and (3).

Proposition 4.1. *If $Y \subseteq X$ is S*compact for the fuzzy topology δ on X and $F \subseteq Y$ is closed for δ, then F is S*compact for δ.*

Proof. Let $P \subseteq X$ be closed in δ and δ' be the family of closed subsets in δ. If $\{U_j : j \in J\}$ is an open $F \cap P *$ shading in δ and if we take $Q = F \cap P \in \delta'$, then we have that $F \cap P = Y \cap Q$ and $\{U_j : j \in J\}$ is an $(Y \cap Q) *$ shading open in δ which of course has a finite *subshading of $Y \cap Q$, i.e. of $F \cap P$.

If $\{V_k : k \in K\}$ is an $(P \Delta F') *$ shading open in δ, then $\{V_k : k \in K\} \cup \{F'\}$ is a $(P \Delta Y') *$ shading open in δ. If $\{V_1, \ldots, V_k, F'\}$ is a finite $(P \Delta Y') *$ subshading, then $\{V_1, \ldots, V_k\}$ is a finite $(P \Delta F') *$ subshading.

If eventually $\{W_h : h \in H\}$ is a $((P \cap F) \Delta F') *$ shading open in δ, then by noting that $Q = P \cap F \in \delta'$, one defines, as in the preceding case, a finite $((P \cap F) \Delta F') *$ subshading. ∎

Proposition 4.2. *If (X, δ) is S*compact and the fuzzy set $A \subseteq X$ has its complement A' S*compact for δ, then $(X, \delta(A))$ is S*compact.*

Proof. Let $Q = F \cup (G \cap A')$ be closed in $\delta(A)$, with $F, G \in \delta'$, and take an open Q*shading in $\delta(A)$, $\mathscr{A} = \{U_j \cup (V_j \cap A) : j \in J\}$.

Obviously $\mathscr{A}$ is an open F*shading in $\delta(A)$. In particular $\{U_j : j \in J\}$ is an open $(F \Delta A) *$ shading in δ and, since A' is S*compact for δ, there exists $J_1 \subseteq J$ finite such as $\{U_j : j \in J\}$ is an $(F \Delta A) *$ subshading.

Moreover $\{U_j : j \in J\} \cup \{V_j : j \in J\}$ is an open F*shading in δ and since F is closed thence S*compact for δ, there are finite subsets $J_2, J_3 \subseteq J$ such as $\{U_j : j \in J_2\} \cup \{V_j : j \in J_3\}$ is an F*subshading.

Let $K = J_1 \cup J_2 \cup J_3$; we know that $\{U_k \cup (V_k \cap A) : k \in K\}$ is a finite F*subshading of $\mathscr{A}$.

In fact if $x \in X$ and if $\exists k \in K$ such as $U_k(x) \geq F(x)$, then $(U_k \cup (V_k \cap A))(x) \geq F(x)$; if instead $U_k(x) < F(x)\ \forall k \in K$, then it must be $F(x) \leq A(x)$ and it results that $V_h(x) \geq F(x)$ for some $h \in K$.

Consequently $(V_h \cap A)(x) \geq F(x)$ and so $(U_h \cup (V_h \cap A))(x) \geq F(x)$.

On the other hand $\mathscr{A}$ is an open $(G \cap A') *$ shading in $\delta(A)$ and, in particular, $\{U_j : j \in J\}$ and $\{U_j : j \in J\} \cup \{V_j : j \in J\}$ are respectively a $((G \cap A') \Delta A) *$ shading and a $(G \cap A') *$ shading which are open in δ.

Let J'_1, J'_2, J'_3 be finite subsets of J such as $\{U_j : j \in J'_1\}$, $\{U_j : j \in J'_2\} \cup \{V_j : j \in J'_3\}$ are a $((G \cap A') \Delta A) *$ subshading and a $(G \cap A') *$ subshading respectively.

As in the preceding case it can be verified that $\{U_h \cup (V_h \cap A) : h \in H\}$, $H = J'_1 \cup J'_2 \cup J'_3$, is a finite $(G \cap A') *$ subshading.

It is clear that $\{U_1 \cup (V_1 \cap A) : 1 \in L\}$ is a finite Q*subshading of $\mathscr{A}$ if $L = K \cup H$. ∎

Now we give some examples which show that the notion of S*compactness is indipendent from those we considered in the preceding section and it does not satisfy condition (3').

Example 4.1. Let N be the set of positive integers and denote by δ_0 the fuzzy topology on N whose open sets are the sequences $A : N \to [0, 1]$ converging and having their limit as upper bound. Obviously the fuzzy topology δ_0 is stratified.

Consider the unit interval $I = [0, 1]$ and let δ_1 be the fuzzy topology on I having as open fuzzy sets I and the functions that take their values in the interval $[0, 1/3]$.

Denote by δ_2 the fuzzy topology on I whose open fuzzy sets different from I and ϕ are the functions assuming their values in the interval $[3/4, 4/5]$.

With regard to the examples 5.1 and 5.2 of [28] we observe that the first one provides a space which is neither N-compact nor S*compact, while in the second one a space which is both N-compact and S*compact is given.

The space (N, δ_0) is α-compact for each $\alpha \in [0,1)$, so it is strong fuzzy compact [15] and strong-Q-compact [11]. Since every closed fuzzy set in δ_0 has a maximum it follows from theorem 5.3 of [28] that (N, δ) is N-compact.

Nevertheless (N, δ_0) is not S*compact. In fact, let A be a sequence converging to 1 and such as $A(x) < 1 \ \forall x \in N$, and let, for each $j \in N$, $A_j(j) = 1$ and $A_j(x) = A(x)$ if $x \neq j$; we obtain an open N*shading $\{A_j : j \in N\}$ which has no finite N*subshading.

The spaces (I, δ_1) and (I, δ_2) are S*compact but none of them is strong-Q-compact or strong fuzzy compact and thence they are not N-compact.

In particular (I, δ_1) is not Q_α-compact if $\alpha \in (2/3, 1]$ and it is not α-compact if $\alpha \in (0, 1/3]$.

We remark that a fuzzy set $Y \subseteq I$ is S*compact for δ_2 if $x \in I$ exists such as $Y(x) > 4/5$ or if Y takes its values in the interval $(3/5, 4/5]$ at most in a finite number of elements of I.

Now let Y be the fuzzy set on I such as $Y(1) = 9/10$ and $Y(x) = 7/10$ if $x \neq 1$, and let A be the fuzzy set defined by $A(x) = (2x+3)/5$, $\forall x \in I$.

Obviously $A' \subseteq Y$ and both A' and Y are S*compact for δ_2. But Y is not S*compact for the simple extension $\delta_2(A)$. In fact consider the fuzzy sets A_x, $x \in [0,1)$, defined by $A_x(x) = Y(x)$ and $A_x(t) = 3/5$ if $t \neq x$; the family $\{A\} \cup \{A_x : x \in [0,1)\}$ is an open Y*shading in $\delta_2(A)$ which has no finite Y*subshading.

Therefore the S*compactness property does not satisfy the condition (3').

A drawback of S*compactness is that in general it is not preserved under continuous images as the following example shows.

We shall prove in the sequel that in some meaningful cases, for example when injective functions or crisp subsets are considered, given a fuzzy continuous function f between two $fts's$ (X, δ) and (Z, γ) and considered a fuzzy set $Y \subseteq X$ which is S*compact for δ, then the image $f(Y)$ of Y is S*compact.

Example 4.2. Let δ be the fuzzy topology on the unit interval I whose open fuzzy sets are I and all the functions $A : I \to I$ such as $A(1) = 0$, $A(x) \leq x \ \forall x \neq 1$, and let γ be the fuzzy topology on I having as open fuzzy sets I and the functions $B : I \to I$ such as $B(x) \leq x \ \forall x \in I$.

Furthermore let $f : I \to I$ be the function defined by $f(1) = 0$ and $f(x) = x \ \forall x \neq 1$.

Clearly $f : (I, \delta) \to (I, \gamma)$ is fuzzy continuous and, if $Y : I \to I$ is the fuzzy set on I defined by $Y(x) = x$, then its image, $f(Y) : I \to I$, is defined by $f(Y)(x) = x$ if $x \neq 1$ and $f(Y)(1) = 0$.

However $f(Y)$ is not S*compact for γ although Y is S*compact for δ.

Lemma 4.1. *Let $f : X \to Z$ be a function between ordinary sets. Let $A \subseteq X$, $B \subseteq Z$,*

$W_j \subseteq Z$, $T_h \subseteq Z$ be fuzzy subsets for $j \in J$ and $h \in H$.

If $\{W_j : j \in J\}$ is an $(f(A) \cap B) *$ shading, then $\{f^{-1}(W_j) : j \in J\}$ is an $(A \cap f^{-1}(B)) *$ shading.

If $\{T_h : h \in H\}$ is a $(B \Delta f(A)') *$ shading, then $\{f^{-1}(T_h) : h \in H\}$ is an $(f^{-1}(B) \Delta A') *$ shading.

Proof. Suppose that $\forall z \in Z\ \exists j \in J$ such as $W_j(z) \geq (f(A) \cap B)(z)$ and take $x_0 \in X$.

If $z_0 = f(x_0)$ and $W_{j_o}(z_0) \geq (f(A) \cap B)(z_0)$ where $j_o \in J$, then $f^{-1}(W_{j_o})(x_0) = W_{j_o}(z_0) \geq \inf\{\sup\{A(x) : f(x) = z_0\}, B(f(x_0))\} \geq \inf\{A(x_0), f^{-1}(B)(x_0)\} = (A \cap f^{-1}(B))(x_0)$, hence $\{f^{-1}(W_j) : j \in J\}$ is an $(A \cap f^{-1}(B)) *$ shading.

Now suppose that $\forall z \in Z\ \exists h \in H$ such as $T_h(z) \geq (B \cap f(A)')(z)$ and fix $x_0 \in X$.

If $z_0 = f(x_0)$ and $T_{ho}(z_0) \geq (B \Delta f(A)')(z_0)$ where $ho \in H$, then it must result $f^{-1}(T_h)(x_0) \geq 0 = (f^{-1}(B) \Delta A')(x_0)\ \forall h \in H$ whenever $f^{-1}(B)(x_0) \leq A'(x_0)$.

When instead $f^{-1}(B)(x_0) > A'(x_0)$, then $B(z_0) = B(f(x_0)) = f^{-1}(B)(x_0) > A'(x_0) > \inf\{A'(x) : f(x) = z_o\} = f(A)'(z_0)$; it follows that $f^{-1}(T_{ho})(x_0) = T_{ho}(z_0) \geq (B \Delta f(A)')(z_0) = B(z_0) = B(f(x_0)) = f^{-1}(B)(x_0) = (f^{-1}(B) \Delta A')(x_0)$.

In any case $f^{-1}(T_{ho})(x_0) \geq (f^{-1}(B) \Delta A')(x_0)$, so $\{f^{-1}(T_h) : h \in H\}$ is an $(f^{-1}(B) \Delta A') *$ shading. ■

Lemma 4.2. *Let $f : X \to Z$ be a function between ordinary sets. Let $A \subseteq Z$, $B \subseteq X$, $U_k \subseteq Z$, $V_j \subseteq Z$ be fuzzy subsets with $k \in K$ and $j \in J$.*

If $\{f^{-1}(U_k) : k \in K\}$ is an $(f^{-1}(A) \cap B)$-cover, then $\{U_k : k \in K\}$ is an $(A \cap f(B))$-cover.

If $\{f^{-1}(V_j) : j \in J\}$ is an $(f^{-1}(A) \Delta B')$-cover, then $\{V_j : j \in J\}$ is an $(A \Delta f(B)')$-cover.

Proof. Suppose that $\forall x \in X\ \sup\{f^{-1}(U_k)(x) : k \in K\} \geq (f^{-1}(A) \cap B)(x)$ and fix $z \in Z$.

If $f^{-1}(z) = \phi$ then $0 = f(B)(z) = (A \cap f(B))(z) \leq \sup\{U_k(z) : k \in K\}$.

If instead $f^{-1}(z) \neq \phi$, then, $\forall x \in X$ such as $f(x) = z$, one has $\sup\{U_k)(z) : k \in K\} = \sup\{f^{-1}(U_k)(x) : k \in K\} \geq \inf\{f^{-1}(A)(x), (B)(x)\}$, so we have $\sup\{U_k)(z) : k \in K\} \geq \inf\{A(z), \sup\{B(x) : f(x) = z\}\} = \inf\{A(z), f(B)(z)\} = (A \cap f(B))(z)$.

Anyway $\{U_k : k \in K\}$ is an $(A \cap f(B))$-cover.

Now by supposing that $\sup\{f^{-1}(V_j)(x) : j \in J\} \geq (f^{-1}(A) \Delta B')(x)\ \forall x \in X$ we shall prove that $\{V_j : j \in J\}$ is a cover of $A \Delta f(B)'$.

Excluding the trivial cases when $(A\Delta f(B)')(z) = 0$, it is sufficient to show that sup $\{V_j(z) : j \in J\} \geq A(z)$ $\forall z \in Z$ such as $A(z) > f(B)'(z)$, that is $\forall z \in Z$ such as $f^{-1}(z) \neq \phi$ and $A(z) > \inf\{B'(x) : f(x) = z\}$.

Let z_0 be such an element of Z; we know that there is $x_0 \in X$ such as $f(x_0) = z_0$ and $f^{-1}(A)(x_0) = A(z_0) > B'(x_0)$, hence $(f^{-1}(A)\Delta B')(x_0) = f^{-1}(A)(x_0)$.

Then we have $\sup\{V_j : j \in J\} = \sup\{f^{-1}(V_j)(x_0) : j \in J\} \geq (f^{-1}(A)\Delta B')(x_0) = f^{-1}(A)(x_0) = A(z_0) = (A\Delta f(B)')(z_0)$.

Therefore $\{V_j : j \in J\}$ is a cover of $A\Delta f(B)'$. ∎

Proposition 4.3. *Let (X,δ) and (Z,γ) be two fts's, $f : X \to Z$ a fuzzy continuous function and $Y \subseteq X$ a fuzzy set in X.*

*If Y is S*compact for δ and if $f^{-1}(P \cap f(Y))\Delta Y' = (f^{-1}(P) \cap Y)\Delta Y'$ whenever $P' \in \gamma$, then $f(Y)$ is S*compact for γ.*

Proof. It follows immediately from lemma 4.1 and lemma 4.2 that every open *shading in γ of either $P\cap f(Y)$ or $P\Delta f(Y)'$ has a finite *subshading of either $P\cap f(Y)$ or $P\Delta f(Y)'$, respectively, whenever $P' \in \gamma$.

Moreover the two lemmas 4.1 and 4.2 and the condition given in the assumption allow us to say that every open $((P\cap f(Y))\Delta f(Y)')*$shading in γ has a finite $((P\cap f(Y))\Delta f(Y)')*$ subshading. ∎

Remark. We notice that the hypothesis of the preceding proposition is verified by every S*compact subset $Y \subseteq X$ for δ if f is an injective function and it is verified too, whatever is the fuzzy continuous function f, if the fuzzy S*compact set Y is crisp or it is saturated with respect to f (i.e. $f^{-1}(f(Y)) = Y$).

The theorem can also be proved by assuming that the S*compact fuzzy set Y has its saturated $f^{-1}(f(Y))$ which is S*compact for δ.

In particular we have the following.

Corollary 4.1. *Let f be a continuous function between two fts's (X,δ) and (Z,γ) and let $Y \subseteq X$ be any crisp subset of X.*

*If Y is S*compact for δ then $f(Y)$ is S*compact for γ.* ∎

In a stratified fts (X,δ) the T_2 axiom defined in the first section can be formulated in several equivalent ways according to the definitions given by Pu and Liu [19], Wang [28] and Lowen [16] (see also [17]).

In Sarkar [24] a Hausdorff axiom, or $F - T_2$ axiom, which requires the following two conditions, is defined.

(I) $\forall x \neq y$ in X and $\forall \alpha, \beta \in (0,1]$ $\exists V, W \in \delta$ such as $V(x) > \alpha$, $cl_\delta(V)(y) \leq \beta$ while $W(y) > \beta$, $cl_\delta(W)(x) \leq \alpha$;

(II) $\forall x \in X$ and $\forall 0 < \alpha < \beta \leq 1$ $\exists U \in \delta$ such as $U(x) > \alpha$ and $cl_\delta(U)(y) \leq \beta$.

Clearly a T_2 fts satisfies the condition (I) and a stratified fts satisfies the condition (II).

Sarkar showed that if $Y \subseteq X$ is a properly compact fuzzy set for δ and if (X, δ) verifies the $F - T_2$ axiom, then Y is closed in δ.

Moreover it is plain that in every fts the S*compact fuzzy sets are properly compact and hence it is easy to obtain the following result.

Proposition 4.4. *If (X, δ) verifies the conditions (I) and (II) and $Y \subseteq X$ is S*compact for δ, then δ is maximal S*compact in Y.* ∎

Lowen [16] proved that a stratified fuzzy compact T_2 fts is maximal fuzzy compact.

On the other hand Wang proved in [28] that in a T_2 fts fuzzy compactness, $N-$ compactness, ultrafuzzy compactness and strong fuzzy compactness are equivalent and hence strong-Q-compactness is equivalent to this properties too (all these properties are intended to refer to the whole space (X, δ) of course).

So we have the following.

Proposition 4.5. *A stratified fuzzy compact T_2 fts (X, δ) is maximal with respect to each of the above properties of compactness.* ∎

Notice that strong fuzzy compactness and ultrafuzzy compactness have not been defined as $fptp's$ and that with regard to the fuzzy compactness property we refer to the definition given for the whole fts in [14].

By proposition 2.2 we have the following result.

Proposition 4.6. *If (X, δ) is a stratified N-compact (or equivalently strong-Q-compact) T_2 fts, every N-compact or strong-Q-compact fuzzy subset $Y \subseteq X$ for δ is closed in δ.* ∎

Now we recall the definition of compactness given by Hutton in [9] restating it in the context of the present paper (i.e. point-set context).

Definition 4.3. *A fts (X, δ) is Hutton-compact if every open cover of any closed fuzzy set $F \subseteq X$ has a finite subcover.*

Clearly we have the following non-reversible implication.

Proposition 4.7. (X, δ) *is Hutton-compact* $\Rightarrow$ (X, δ) *is S*compact* ∎

Example 4.3. Let $X \neq \phi$; the family $\kappa = \{K \in I^X : K \text{ is constant}\}$ is a fuzzy topology on X.

The fts (X, κ) is S*compact while it is not Hutton-compact.

The definitions 4.2 and 4.3 can be easily formulated for a whole space of the kind (L^X, δ) in the lattice-theoretic context.

Proposition 4.7 holds in this context too.

Now it is easy to state the following result by using proposition 4.7 and the results given in [21] concerning to the fuzzy unit interval $I(L)$ defined by Hutton [8].

Proposition 4.8. *The fuzzy unit interval $I(L)$ is S*compact if L is a complete boolean algebra.* ∎

Lowen [15] observed that an fts generated by a T_1 ordinary topological space is not α^*-compact for any $\alpha \in (0, 1]$.

Obviously this assertion holds for the S*compact stratified $fts's$ too, since every S* compact stratified fts is α^*-compact if $\alpha \in (0, 1]$.

Actually S*compactness seems to be a strong condition not only for $fts's$ generated by ordinary topologies, but more generally for stratified $fts's$. In such spaces in fact S*compactness may exclude the T_1 axiom, as the following result shows.

Proposition 4.9. *No S*compact (and hence no Hutton-compact) stratified fuzzy topology on the set N of natural numbers can be T_1.*

Proof. It is evident that every sequence which converges and is bounded above by its limit can be obtained as the supremum of a family of sequences which are eventually constant and upper bounded by this constant value.

Now if κ is the fuzzy topology of example 4.3 and $\delta \supseteq \kappa$ is a T_1 fuzzy topology, then δ refines the fuzzy topology δ_0 of the example 4.1, which is not S*compact. ∎

Obviously no stratified fuzzy topology on N is maximal S*compact.

Concluding remarks. The approach we suggest for studying $fptp's$ in $fts's$ does not need an extension of the categorical framework in fuzzy topology including object associated with non-crisp subsets, which is till now a big problem to our knowledge.

Of course, in a richer category of fuzzy topological spaces including objects associated with a non-crisp support, every fuzzy topological property (i.e. property of objects, that are $fts's$) could be considered as an $fptp$.

Nevertheless an $fptp$ does not become necessarily a fuzzy topological property even if a richer category can be considered in the sense above specified.

In this sense $fptp's$ seems to be a non-trivial generalization of fuzzy topological properties.

In another paper we will present an alternative approach to the problem originally motivating the present work, e.g. an internal characterization of maximal «fuzzy compactlike» fuzzy topologies.

This different approach allows us to give an intrinsic characterization of maximality of fuzzy compactlike properties of fuzzy subset in an fts within a category, defined in [4], which is, in some sense, equivalent to $\mathscr{C}\mathscr{G}(I)$ (the category of $fts's$ of Chang-Goguen) but contains objects associated with a non-crisp support.

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