Note di Matematica ISSN 1123-2536, e-ISSN 1590-0932
Note Mat. **33** (2013) no. 2, 35–42. doi:10.1285/i15900932v33n2p35

On some continued fraction expansions for the ratios of the function $\rho(a,\ b)$

D. D. Somashekara
Department of Studies in Mathematics, University of Mysore, Manasagangotri, Mysore 570 006, India.
dsomashekara@yahoo.com

S. L. Shalini
Department of Studies in Mathematics, University of Mysore, Manasagangotri, Mysore 570 006, India.
shalinisl.maths@gmail.com

K. Narasimha Murthy
Department of Studies in Mathematics, University of Mysore, Manasagangotri, Mysore 570 006, India.
simhamurth@yahoo.com

Received: 7.1.2013; accepted: 1.4.2013.

Abstract. In his lost notebook, Ramanujan has defined the function $\rho(a,\ b)$ by

$$\rho(a,\ b) := \left(1+\frac{1}{b}\right)\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n+1)/2} a^n b^{-n}}{(-aq)_n},$$

where $|q|<1$, and $(a;q)_n = \prod_{k=0}^{n-1}(1-aq^k)$, $\quad n=1,2,3,\ldots$, and has given a beautiful reciprocity theorem involving $\rho(a,\ b)$. In this paper we obtain some continued fraction expansions for the ratios of $\rho(a,\ b)$ with some of its contiguous functions. We also obtain some interesting special cases of our continued fraction expansions which are analogous to the continued fraction identities stated by Ramanujan.

Keywords: Basic hypergeometric series, q-continued fractions.

MSC 2010 classification: 2010 Mathematics Subject Classification: primary 33D05, secondary 11A55.

1 Introduction

Ramanujan, a pioneer in the theory of continued fractions has recorded scores of continued fraction identities in chapter 12 of his second notebook [23] and in his lost notebook [24]. This part of Ramanujan's work has been treated and developed by several authors including Andrews [4], Hirschhorn [19], Carlitz [12], Gordon [18], Al-Salam and Ismail [3], Ramanathan [21], [22], Denis

[13], [14], [15], Bhargava and Adiga [8], [9], Bhargava, Adiga and Somashekara [10], [11], Adiga and Somashekara [2], Verma, Denis and Srinivasa Rao [29], Singh [26], Bhagirathi [5], [6], [7], Adiga, Denis and Vasuki [1], Denis, Singh and Bhagirathi [17], Denis and Singh [16], Vasuki [27], Vasuki and Madhusudan [28], Somashekara and Fathima [25], Mamta and Somashekara [20].

The main purpose of this paper is to establish continued fraction expansions for the ratios $\rho(aq,\ b)/\rho(a,\ b)$ and $\rho(a,\ bq)/\rho(a,\ b)$, where

$$\rho(a,\ b)=\left(1+\frac{1}{b}\right)\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n+1)/2}a^n b^{-n}}{(-aq)_n}, \tag{1.1}$$

which was given by Ramanujan in his lost notebook [24]. In fact Ramanujan has given a beautiful reciprocity theorem for the function $\rho(a,\ b)$ in his lost notebook.

In section 2, we prove some key functional relations satisfied by $\rho(a,\ b)$, which will be used in the development of continued fractions. In section 3, we prove our main results and in section 4 we obtain some special cases of our continued fractions which are analogous to the continued fractions of Ramanujan.

2 Some functional relations satisfied by $\rho(a,\ b)$

In this section, we prove some functional relations satisfied by $\rho(a,\ b)$.

Lemma 1. *$\rho(a,\ b)$ satisfies the following functional relations.*

$$(1+aq)\frac{\rho(a,\ b)}{\left(1+\frac{1}{b}\right)}-aq\frac{\rho(aq,\ b)}{\left(1+\frac{1}{b}\right)}=\frac{\rho(aq,\ bq)}{\left(1+\frac{1}{bq}\right)}, \tag{2.1}$$

$$(1+aq)\frac{\rho(a,\ bq)}{\left(1+\frac{1}{bq}\right)}-(1+aq)\frac{\rho(a,\ b)}{\left(1+\frac{1}{b}\right)}=\frac{aq}{b}\frac{\rho(aq,\ b)}{\left(1+\frac{1}{b}\right)}-\frac{a}{b}\frac{\rho(aq,\ bq)}{\left(1+\frac{1}{bq}\right)}, \tag{2.2}$$

$$\frac{\rho(a,\ bq)}{\left(1+\frac{1}{bq}\right)}=\left(1-\frac{a}{b}\right)\frac{\rho(a,\ b)}{\left(1+\frac{1}{b}\right)}+\frac{\frac{aq}{b}(1+a)}{(1+aq)}\frac{\rho(aq,\ b)}{\left(1+\frac{1}{b}\right)}, \tag{2.3}$$

$$\rho(a,\ b)=\left(\frac{1-\frac{aq}{b}+aq}{1+aq}\right)\rho(aq,\ b)+\left(\frac{aq^2/b}{1+aq^2}\right)\rho(aq^2,\ b), \tag{2.4}$$

$$(1+aq)\frac{\rho(a,\ bq)}{\left(1+\frac{1}{bq}\right)}-aq\frac{\rho(aq,\ bq)}{\left(1+\frac{1}{bq}\right)}=\frac{\rho(aq,\ bq^2)}{\left(1+\frac{1}{bq^2}\right)}, \tag{2.5}$$

$$\rho(a,\ b)=\left[\frac{a+bq(a-1)}{a(1+bq)}\right]\rho(a,\ bq)+\left[\frac{bq^2}{a(1+bq^2)}\right]\rho(a,\ bq^2) \tag{2.6}$$

Proof. Using (1.1), the left side of (2.1) can be written as

$$(1+aq)+(1+aq)\sum_{n=1}^{\infty}\frac{(-1)^n q^{n(n+1)/2}a^n b^{-n}}{(-aq)_n}$$

$$-aq-aq\sum_{n=1}^{\infty}\frac{(-1)^n q^{n(n+1)/2}(aq)^n b^{-n}}{(-aq^2)_n}$$

$$=1+\sum_{n=1}^{\infty}\frac{(-1)^n q^{n(n+1)/2}a^n b^{-n}}{(-aq^2)_{n-1}}\left\{1-\frac{aq^{n+1}}{1+aq^{n+1}}\right\}=\frac{\rho(aq,\ bq)}{\left(1+\frac{1}{bq}\right)},$$

which is the right side of (2.1).

Using (1.1), the left side of (2.2) can be written as

$$(1+aq)\sum_{n=1}^{\infty}\frac{(-1)^n q^{n(n+1)/2}a^n (bq)^{-n}}{(-aq)_n}-(1+aq)\sum_{n=1}^{\infty}\frac{(-1)^n q^{n(n+1)/2}a^n b^{-n}}{(-aq)_n}$$

$$=\frac{-a}{b}\sum_{n=1}^{\infty}\frac{(-1)^{n-1} q^{n(n-1)/2}a^{n-1} b^{-n+1}}{(-aq^2)_{n-1}}$$

$$+\frac{aq}{b}\sum_{n=1}^{\infty}\frac{(-1)^{n-1} q^{n(n-1)/2}(aq)^{n-1} b^{-n+1}}{(-aq^2)_{n-1}}$$

$$=\frac{aq}{b}\frac{\rho(aq,\ b)}{\left(1+\frac{1}{bq}\right)}-\frac{a}{b}\frac{\rho(aq,\ bq)}{\left(1+\frac{1}{bq}\right)}.$$

This proves (2.2).

Substituting for $\rho(aq,\ bq)/(1+1/bq)$ in (2.2) from (2.1), we obtain (2.3) on some simplifications.

Changing a to aq in (2.3), then adding resulting equation to (2.1), we obtain (2.4).

Using (1.1), the left side of (2.5) can be written as

$$(1+aq)+(1+aq)\sum_{n=1}^{\infty}\frac{(-1)^n q^{n(n+1)/2}a^n (bq)^{-n}}{(-aq)_n}$$

$$-aq-aq\sum_{n=1}^{\infty}\frac{(-1)^n q^{n(n+1)/2}(aq)^n (bq)^{-n}}{(-aq^2)_n}$$

$$=1+\sum_{n=1}^{\infty}\frac{(-1)^n q^{n(n+1)/2}a^n (bq)^{-n}}{(-aq^2)_{n-1}}\left\{1-\frac{aq^{n+1}}{1+aq^{n+1}}\right\}=\frac{\rho(aq,\ bq^2)}{\left(1+\frac{1}{bq^2}\right)}.$$

which is the right side of (2.5).

Adding (2.1), (2.2) and the negative of (2.5), we obtain (2.6) on some simplifications. QED

3 Main results

In this section, we deduce the continued fraction expansions for the ratios $\rho(aq,\ b)/\rho(a,\ b)$ and $\rho(a,\ bq)/\rho(a,\ b)$.

Theorem 1. *We have*

$$\frac{\rho(aq,\ b)}{\rho(a,\ b)} = \frac{(1+aq)}{N_1+}\ \frac{M_1}{N_2+}\ \frac{M_2}{N_3+\cdots}\ \frac{M_n}{N_{n+1}\cdots}, \tag{3.1}$$

where

$$M_n = \frac{aq^{n+1}}{b}(1+aq^n),$$

and

$$N_n = \left(1 - \frac{aq^n}{b} + aq^n\right), \quad n = 0, 1, 2, \ldots.$$

Proof. Changing a to aq^n in (2.4), we obtain

$$\rho(aq^n,\ b) = \left(\frac{1 - \frac{aq^{n+1}}{b} + aq^{n+1}}{1+aq^{n+1}}\right)\rho(aq^{n+1},\ b) + \left(\frac{aq^{n+2}/b}{1+aq^{n+2}}\right)\rho(aq^{n+2},\ b).$$

This can be written as

$$T_n \equiv \frac{\rho(aq^n,\ b)}{\rho(aq^{n+1},\ b)} = \left(\frac{1 - \frac{aq^{n+1}}{b} + aq^{n+1}}{1+aq^{n+1}}\right) + \frac{\left(\frac{aq^{n+2}/b}{1+aq^{n+2}}\right)}{T_{n+1}}. \tag{3.2}$$

Iterating (3.2) with $n = 0, 1, 2, \ldots$, and then taking reciprocals, we obtain (3.1) after some simplifications. QED

Theorem 2. *We have*

$$\frac{\rho(a,\ bq)}{\rho(a,\ b)} = \frac{\left(1-\frac{a}{b}\right)(1+bq)}{q(1+b)} + \frac{(1+bq)M_0}{q(1+b)N_1+}\ \frac{q(1+b)M_1}{N_2+}\ \frac{M_2}{N_3+\cdots}\ \frac{M_n}{N_{n+1}\cdots}, \tag{3.3}$$

where M_n and N_n are as in theorem (3.1).

Proof. Equation (2.3) can be written as

$$\frac{\rho(a,\ bq)}{\rho(a,\ b)} = \frac{\left(1-\frac{a}{b}\right)(1+bq)}{q(1+b)} + \frac{\frac{aq}{b}(1+a)(1+bq)}{q(1+b)(1+aq)\ \frac{\rho(a,\ b)}{\rho(aq,\ b)}}. \tag{3.4}$$

Iterating (3.2) with $n = 0, 1, 2, \ldots$, and substituting the resulting identity in (3.4), we obtain (3.3) after some simplifications. QED

Theorem 3. *We have*

$$\frac{\rho(a,\ bq)}{\rho(a,\ b)}=\frac{a(1+bq)}{A_0+}\ \frac{B_0}{A_1+}\ \frac{B_1}{A_2+\cdots}\ \frac{B_n}{A_{n+1}\cdots}, \tag{3.5}$$

where

$$A_n=[a+bq^{n+1}(a-1)],$$

and

$$B_n=\left[abq^{n+2}(1+bq^{n+1})\right], \qquad n=0,1,2,\ldots.$$

Proof. Changing b to bq^n in (2.6), we obtain on some simplifications

$$\rho(a,\ bq^n)=\left[\frac{a+bq^{n+1}(a-1)}{a(1+bq^{n+1})}\right]\rho(a,\ bq^{n+1})+\left[\frac{bq^{n+2}}{a(1+bq^{n+2})}\right]\rho(a,\ bq^{n+2}).$$

This can be written as

$$F_n\equiv\frac{\rho(a,\ bq^n)}{\rho(a,\ bq^{n+1})}=\left[\frac{a+bq^{n+1}(a-1)}{a(1+bq^{n+1})}\right]+\frac{\left[\frac{bq^{n+2}}{a(1+bq^{n+2})}\right]}{F_{n+1}}. \tag{3.6}$$

Iterating (3.6) with $n=0,1,2,\ldots$, and then taking reciprocals, we obtain (3.5) after some simplifications. QED

4 Some special cases

In this section, we derive the following special cases of (3.1), (3.3) and (3.5).

$$\frac{\sum_{n=0}^{\infty}\frac{(-1)^nq^{n(n+3)/2}}{(-q^2)_n}}{\sum_{n=0}^{\infty}\frac{(-1)^nq^{n(n+1)/2}}{(-q)_n}}=\frac{1+q}{1+}\ \frac{q^2(1+q)}{1+}\ \frac{q^3(1+q^2)}{1+\cdots}, \tag{4.1}$$

$$\frac{\sum_{n=0}^{\infty}\frac{q^{n(n+3)/2}}{(q^2)_n}}{\sum_{n=0}^{\infty}\frac{q^{n(n+1)/2}}{(q)_n}}=\frac{1-q}{1-}\ \frac{q^2(1-q)}{1-}\ \frac{q^3(1-q^2)}{1-\cdots}, \tag{4.2}$$

$$\frac{\sum_{n=0}^{\infty}\frac{q^{n(n+1)/2}}{(q^2)_n}}{\sum_{n=0}^{\infty}\frac{q^{n(n-1)/2}}{(q)_n}}=\frac{1-q}{(2-q)-}\ \frac{q(1-q)}{(1+q-q^2)-\cdots}, \tag{4.3}$$

$$\frac{\sum_{n=0}^{\infty}\frac{(-1)^nq^{n(n+5)/2}}{(-q^3)_n}}{\sum_{n=0}^{\infty}\frac{(-1)^nq^{n(n+3)/2}}{(-q^2)_n}}=\frac{1+q^2}{1+}\ \frac{q^3(1+q^2)}{1+}\ \frac{q^4(1+q^3)}{1+\cdots}, \tag{4.4}$$

$$\frac{\sum_{n=0}^{\infty}\frac{q^{n(n-1)/2}}{(q^2)_n}}{\sum_{n=0}^{\infty}\frac{q^{n(n+1)/2}}{(q^2)_n}} = 2 - \frac{q(1-q)}{(1+q-q^2)-}\ \frac{q^2(1-q^2)}{(1+q^2-q^3)-\cdots}, \tag{4.5}$$

$$\frac{\sum_{n=0}^{\infty}\frac{q^{n(n+1)/2}}{(q^2)_n}}{\sum_{n=0}^{\infty}\frac{q^{n(n+3)/2}}{(q^2)_n}} = (1+q) - \frac{q^2(1-q)}{1-}\ \frac{q^3(1-q^2)}{1-\cdots}, \tag{4.6}$$

$$\frac{\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n-3)/2}}{(-q^2)_n}}{\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n-1)/2}}{(-q^2)_n}} = \frac{q-1}{q} + \frac{(1+q)}{q^2+\cdots}, \tag{4.7}$$

$$\frac{\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n+1)/2}}{(-q^3)_n}}{\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n+3)/2}}{(-q^3)_n}} = (1-q) + \frac{q^2(1+q^2)}{(1-q^2+q^3)+\cdots}, \tag{4.8}$$

$$\frac{\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n-1)/2}}{(-q)_n}}{\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n+1)/2}}{(-q)_n}} = \frac{2q}{1+}\ \frac{q^2(1+q)}{1+}\ \frac{q^3(1+q^2)}{1+\cdots}, \tag{4.9}$$

$$\frac{\sum_{n=0}^{\infty}\frac{q^{n(n-1)/2}}{(q)_n}}{\sum_{n=0}^{\infty}\frac{q^{n(n+1)/2}}{(q)_n}} = \frac{2q}{(1+2q)-}\ \frac{q^2(1+q)}{(1+2q^2)-\cdots}, \tag{4.10}$$

$$\frac{\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n+1)/2}}{(-q^2)_n}}{\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n+3)/2}}{(-q^2)_n}} = \frac{2}{1+}\ \frac{(1+q)}{(1-q+q^2)+}\ \frac{q^2(1+q^2)}{(1-q^2+q^3)+\cdots}, \tag{4.11}$$

$$\frac{\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n-3)/2}}{(-q)_n}}{\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n-1)/2}}{(-q)_n}} = \frac{q(1+q)}{1+}\ \frac{q^3(1+q^2)}{1+}\ \frac{q^4(1+q^3)}{1+}\cdots. \tag{4.12}$$

Proof. Setting $a = 1 = b$ in (3.1) and using the definition (1.1) of $\rho(a,\ b)$ we obtain (4.1) after some simplifications. Similarly we obtain (4.2), (4.3) and (4.4) from (3.1) for $a = -1,\ b = 1;\ \ a = -1,\ b = q$ and $a = q,\ b = 1$ respectively.

Setting $a = -q,\ b = q$ in (3.3) and using the definition (1.1) of $\rho(a,\ b)$ we obtain (4.5) after some simplifications. Similarly we obtain (4.6), (4.7) and (4.8) from (3.3) for $a = -q,\ b = 1$; $a = q,\ b = q^2$ and $a = q^2,\ b = q$ respectively.

Setting $a = 1 = b$ in (3.5) and using the definition (1.1) of $\rho(a,\ b)$ we obtain (4.9) after some simplifications. Similarly we obtain (4.10), (4.11) and (4.12) from (3.5) for $a = -1,\ b = 1$; $a = q,\ b = 1$ and $a = 1,\ b = q$ respectively. QED

Acknowledgements. The first author is thankful to University Grants Commission (UGC), India for the financial support under the grant SAP-DRS-

1-NO.F.510/2/DRS/2011. The second author is thankful to UGC for awarding the award of Rajiv Gandhi National Fellowship, No. F1-17.1/2011-12/RGNF-SC-KAR-2983/(SA-III/Website) and the third author is thankful to University Grants Commission, India, for the award of Teacher Fellowship under the grant No. KAMY074-TF01-13112010.

References

[1] C. Adiga, R. Y. Denis and K. R. Vasuki: On some continued fraction expansions for the ratios of ${}_2\psi_2$, Special Functions: Selected Articles, P. K. Banerji (Ed.), Scientific Publishers (India), 2001, 1–16.

[2] C. Adiga and D. D. Somashekara: *On some Rogers-Ramanujan type continued fraction identities*, Mathematical Balcanika, New series, **12** (1998), 37–45.

[3] W. A. Al-Salam and M. E. H. Ismail: *Orthogonal polynomials associated with the Rogers-Ramanujan continued fraction*, Pacific J. Math., **104** (1983), 269–283.

[4] G. E. Andrews: *Ramanujan's 'lost' notebook III. The Rogers-Ramanujan continued fraction*, Adv. Math., **41** (1981), 186–208.

[5] N. A. Bhagirathi: *On certain continued fractions and q-series*, Math. Student, **56** (1988), 97–104.

[6] N. A. Bhagirathi: *On basic bilateral hypergeometric series and continued fraction*, Math. Student, **56** (1988), 135–141.

[7] N. A. Bhagirathi: *On certain investigations in q-series and continued fraction*, Math. Student, **56** (1988), 158–170.

[8] S. Bhargava and C. Adiga: *On some continued fraction identities of Srinivasa Ramanujan*, Proc. Amer. Math. Soc., **92** (1984), 13–18.

[9] S. Bhargava and C. Adiga: Two generalizations of Ramanujan's continued fraction identities, Number Theory, K. Alladi (Ed.), Lecture Note in Math., No. 1122, Springer-Verlag, Berlin, (1985), 56–62.

[10] S. Bhargava, C. Adiga and D. D. Somashekara: *On some generalizations of Ramanujan's continued fraction identities*, Proc. Indian Acad. Sci., **97** (1987), 31–43.

[11] S. Bhargava, C. Adiga and D. D. Somashekara: *On certain continued fraction related to ${}_3\phi_2$ basic hypergeometric function*, J. Math. Phys. Sci., **21** (1987), 613–629.

[12] L. Carlitz: *Note on some continued fraction of the Rogers-Ramanujan type*, Duke Math. J., **32** (1965), 713–720.

[13] R. Y. Denis: *On certain q-series and continued fractions*, Math. Student, **44** (1984), 70–76.

[14] R. Y. Denis: *On basic hypergeometric functions and continued fractions*, Math. Student, **52** (1984), 129–136.

[15] R. Y. Denis: *On certain q-series and continued fraction identities*, Math. Student, **53** (1985), 243–248.

[16] R. Y. Denis and S. N. Singh: Generalized hypergeometric functions and continued fractions, Selected Topics in Special Functions, R. P. Agarwal, H. L. Manocha and K. Srinivasa Rao, (Eds.), 2001, 173–207.

[17] R. Y. Denis, S. N. Singh and N. A. Bhagirathi, On certain bilateral q-series and Ramanujan's continued fractions, Special Functions: Selected Articles, P. K. Banerji (Ed.), Scintific Publishers (India), 2001, 149–165.

[18] B. Gordon, *Some continued fractions of Rogers-Ramanujan type*, Duke Math. J., **32** (1965), 741–748.

[19] M. D. Hirschhorn, *A continued fraction of Ramanujan*, J. Austral. Math. Soc., Ser. A, **29** (1980), 80–86.

[20] D. Mamta and D. D. Somashekara, *On some continued fraction expansions for the ratios* ${}_2\psi_2$, Far East J. Math. Sci., **23**, n. 1, (2006), 65–80.

[21] K. G. Ramanathan, *On Ramanujan's continued fraction*, Acta Arith., **43** (1984), 209–226.

[22] K. G. Ramanathan, *On the Roger's-Ramanujan continued fraction*, Proc. Indian Acad. Sci., (Math. Sci.), **93** (1984), 67–77.

[23] S. Ramanujan, Notebooks (2 Volumes), Tata Institute of Fundamental Research, Bombay, 1957.

[24] S. Ramanujan, The Lost Notebook and Other Unpublished Papers, Narosa, New Delhi, 1988.

[25] D. D. Somashekara and Syeda Noor Fathima, *On continued fraction expansions for the ratios* ${}_2\psi_2$, Indian J. Math., **45**, n. 3, (2003), 333–355.

[26] S. N. Singh, *On q-hypergeometric functions and continued fractions*, Math. Student, **56** (1988), 81–84.

[27] K. R. Vasuki, *On some continued fractions related to* ${}_2\psi_2$ *basic bilateral hypergeometric series*, Mathematical Forum, **12** (1998), 31–43.

[28] K. R. Vasuki and H. S. Madhusudan, *On certain continued fractions related to* ${}_2\psi_2$ *basic bilateral hypergeometric functions*, Indian J. Pure Appl. Math., **33**, n. 10, (2002), 1563–1573.

[29] A. Verma, R. Y. Denis and K. Srinivasa Rao, *New continued fractions involving basic hypergeometric* ${}_3\phi_2$ *functions*, J. Math. Phys. Sci., **21** (1987), 585–592.