Note di Matematica Vol. XII, 69-75 (1992)

ON THE SPACE $\mathscr{K}(P, P^*)$ OF COMPACT OPERATORS ON PISIER SPACE P

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Dedicated to the memory of Professor Gottfried Köthe

By Pisier space we will mean an infinite dimensional Banach space P such that

(i) On $P \otimes P$ the extremal ε-and π-tensor norms are equivalent.

(ii) P and P^* are both cotype 2 spaces.

Such a space was constructed by Pisier [8]. It is not difficult to see then [3] that P is a Hilbert-Schmidt space in the sense of [2]. This means that $\mathscr{L}(P, l_2) = \mathscr{P}_2(P, l_2)$ where $\mathscr{P}_2$ denotes the ideal of absolutely 2-summing operators.

J. Johnson [5] proved the following result: For two Banach spaces E and F, the latter with the λ-bounded approximation property, there is a projection $p : \mathscr{L}(E, F)^* \to \mathscr{L}(E, F)^*$ satisfying

$$\|p\| \le \lambda$$

$$\ker p = \mathscr{K}(E, F)^0$$

$$\operatorname{Im} p = \mathscr{K}(E, F)^* \qquad \lambda\text{-isomorphically}.$$

Hence we show this statement (3. Proposition) for the space $E = P$ and $F = P^*$ where P is the Pisier space. This result cannot be obtained by Johnson's statement since Pisier spaces by Pisier's factorization theorem never have the approximation property.

Our proof depends on a compactness argument different from the one used in Johnson's paper. Next we list some of many properties equivalent to the fact that each bounded operator $f : P \to P^*$ is compact (1. and 2. Proposition). Finally we observe that $\mathscr{L}(P, P^*)$ may always be embedded into $\mathscr{K}(P, P^*)^{**}$, which is a result also similar to the corresponding result of J. Johnson.

In the following P_1^*, P_1^{**} denote the closed unit balls of P^*, P^{**} in its w^*-topologies. By measure we will mean any positive Radon measure on P_1^* and the set of all asuch measures will be denoted by M^+. By $\mathscr{K}(P, P^*) = \mathscr{K}$ or $\mathscr{L}(P, P^*) = \mathscr{L}$ we denote the space of all compact or bounded operators $f : P \to P^*$ respectively.

For the sake of simplicity we will suppose that the Pisier space P is separable in Propositions 3 and 4, so that $C(P_1^*)$ is separable.

We start with the following observation (cf. also [7]):

Proposition 1. *Let P be a Pisier space. The following are equivalent*

a) $\mathscr{L}(P, P^*) = \mathscr{K}(P, P^*)$.

b) $\mathscr{L}(P, l_2) = \mathscr{K}(P, l_2)$.

c) There is no surjection of P onto l_2.

d) P does not contain isomorphically the sequence space l_1.

Proof. Having in mind that $\mathscr{L}(P,l_2) = \mathscr{P}_2(P,l_2)$ the equivalence of b), c) and) follows immediately from [7, Proposition 3].

For the convenience of the reader we give here one of possible proofs: The implication a) $\Rightarrow$ b) follows from the known fact (cf. e.g. [4]) that for any operator $A : P \to l_2$ we have

$$A \text{ is compact iff } A^*A : P \to P^* \text{ is compact}.$$

b) $\Rightarrow$ c) is trivially always true; c) $\Rightarrow$ d) suppose that P contains a copy of l_1. Let A be a surjection of l_1 onto l_2. The operator A being absolutely summing it allows a continuous extension onto the whole space. We denote this extension again by A. Thus $A : P \to l_2$ is a surjection. d) $\Rightarrow$ a): the assumption d) implies by Rosenthal's l_1-theorem that each bounded sequence in P has a weak Cauchy subsequence. Now let $f \in \mathscr{L}(P,P^*)$.

Then f is fully complete. Indeed, (i) is equivalently expressed by the statement that every operator $f : P \to P^*$ is integral. Thus f is fully complete (cf. e.g. [1, 19.6.2]). This means that f takes weak Cauchy sequences into norm convergent ons. This finishes the proof.

Our aim is to show an analogy of a result of J. Johnson [5] namely that $\mathscr{K}(P,P^*)^*$ is a complemented subspace of $\mathscr{L}(P,P^*)^*$. We complement this result (and 1. Proposition) by

Proposition 2. *The following are equivalent*

a) $\mathscr{K}(P,P^*) \neq \mathscr{L}(P,P^*)$

b) $\mathscr{K}(P,P^*)$ *is not complemented in* $\mathscr{L}(P,P^*)$

c) $\mathscr{K}(P,l^2)$ *is not complemented in* $\mathscr{L}(P,l^2)$.

Proof. If a) is satisfied then by the preceding proposition there exists a surjection $A : P \to l_2$. Then $\mathscr{L}(P,l^2) \neq \mathscr{K}(P,l^2)$ and [9, Theorem 6 or its Corollary] implies c). To show b) let $j : l_\infty \to \mathscr{L}(P,l_2)$ be the isomorphism defined in the proof of that result [9, Proposition 4]. By construction the operator j maps c_0 into $\mathscr{K}(P,P^*)$. Let $\imath$ be the embedding of $\mathscr{L}(P,l_2)$ into $\mathscr{L}(P,P^*)$ given by $\imath(f) = A^*f$ where the dual A^* of A is evidently an embedding fo l_2 into P^*. Let us suppose now that p is a continuous projection of $\mathscr{L}(P,P^*)$ onto $\mathscr{K}(P,P^*)$. Then $S = p \circ \imath \circ j : l_\infty \to \mathscr{L}(P,P^*)$ is weakly compact since P does not contain complemented copy of l_1 and P^* does not contain a copy of l_∞ (cf. [6, Corollary to Theorem 4]). Then the restriction of S to c_0 is again weakly compact, but evidently $S|_{c_0} = \imath j|_{c_0}$ is norm isomorphism which cannot be weakly compact - a contradiction.

Proposition 3. *There is a projection* p *on* $L(P,P^*)^*$ *such that* $\|p\| \leqq c$ *for some constant* c*, the range of* p *is* c*-isomorphic to* $\mathscr{K}(P,P^*)^*$ *and the kernel of* P *is the anihilator of* $\mathscr{K}(P,P^*)$. *Thus* $\mathscr{L}^*$ *is the topological direct sum* $\mathscr{L}^* = \mathscr{K}^0 + J_K(\mathscr{K}^*)$ *where* $J_K : \mathscr{K}^* \to \mathscr{L}^*$ *is the isomorphic embedding.*

The proof will be contained in the following observations:

1) (cf. e.g. [6]). Let X,Y be Banach spaces, $K = X_1^{**} \times Y_1^*$ (the cartesian product of the unit balls in their w^*-topologies).

Then any compact operator $f \in \mathscr{K}(X,Y)$ may be identified with $\tilde{f} \in C(K)$, where $\tilde{f}(x^{**},y^*) = x^{**}(f^*(y^*))$ for $x^{**} \in X^{**}$ and $y^* \in Y^*$. This identification is an isometric embedding of $\mathscr{K}(X,Y)$ into $C(K)$.

2) Let $\mu \in M^+$, i.e. μ is a positive Radon measure on P_1^*. Let us denote by A_μ the canonical mapping $A_\mu : P \to L_2(P_1^*,\mu) = H_\mu$ given by $A_\mu x(x^*) = x^*(x)$. Observe that if $k\mu_2 \leqq \mu_1$, $K > 0$, then $A_\mu = A_{\mu_1\mu_2} A_{\mu_2}$ where $A_{\mu_1\mu_2} : H_{\mu_2} \to H_{\mu_1}$ is induced by the identity embedding and $\|A_{\mu_1\mu_2}\| \leqq \sqrt{k}$.

In this notation we now state

3) Every $f : P \to P^*$ may be expressed as a composition $f = A_\mu^* f_\mu A_\mu$ for suitable probability measure $\mu \in M^+$ and suitable $f_\mu \in \mathscr{L}(H_\mu, H_\mu^*)$. Moreover a constant $c = c(P)$ exists depending only on the space P and not $f \in \mathscr{L}(P,P^*)$ such that $\|f_\mu\| \leqq c\|f\|$.

Indeed, the mapping f being integral by (i) we get the factorization $f = BA$ through a Hilbert space l_2 and the estimate $\|A\|\cdot\|B\| \leqq c_1\|f\|$ with some constant c_1. Now P being a Hilbert-Schmidt space, the Pietsch factorization theorem gives probability measure $\mu_1 \in M^+$ and the factorization $A = S_1 A_{\mu_1}$ where $S_1 : H_{\mu_1} \to l_2$ and $\|S_1\| \leqq P_2(A) \leqq c_2\|A\|$. Similarly $B^*|_P = S_2 A_{\mu_2}$. Let $\mu = (\mu_1 + \mu_2)/|\mu_1 + \mu_2|$. Then $A_{\mu_i} = A_{\mu\mu_i} A_\mu$ and because $(B^*|_P)^* = B$ we get $f = A_\mu^* A_{\mu\mu_2}^* S_2^* S_1 A_{\mu\mu_1} A_\mu$. Putting $f_\mu = A_{\mu\mu_2}^* S_2^* S_1 A_{\mu\mu_1}$ we get the desired factorization.

4) The triplet $(f,\mu,f_\mu) \in \mathscr{L} \times M^+ \times \mathscr{L}(H_\mu, H_\mu^*)$ will be called suitable if $f = A_\mu^* f_\mu A_\mu$ where $f_\mu : H_\mu \to H_\mu^*$. The set of all suitable triplets will be denoted by V. Let us choose in each Hilbert space H_μ some orthonormal basis with the corresponding system of projections $\{P_n^\mu\}$, $P_n^\mu x \to x$ in the norm, $\|P_n^\mu\| \leqq 1$ and let us define for every suitable pair $(f,\mu,f_\mu) \in$ $\in V$

$$f_n^{\mu,f_\mu} = A_\mu^* P_n f_\mu A_\mu \in \mathscr{K}(P,P^*) = \mathscr{K}.$$

5) The following is evident:

a) If $(f,\mu,f_\mu) \in V, (g,\mu,g_\mu) \in V$ then $(f+g,\mu,f_\mu+g_\mu) \in V$ and $(f+g)_n^{\mu,f_\mu+g_\mu} =$ $= f_n^{\mu+f_\mu} + g_n^{\mu,g_\mu}$.

b) If $(f,\mu,f_\mu) \in V$ then $(f,\nu,f_\nu) \in V$ for all $k\nu \geqq \mu$ and for some f_ν.

6) For all $p \in P$ all $p^{**} \in P^{**}$ and for (every suitable (f,μ,f_μ) we have

$$\lim_{n\to\infty} \|f_n^{\mu,f_\mu}(p) - f(p)\| = 0 \qquad \text{and}$$

$$\lim_{n\to\infty} \|(f_n^{\mu,f_\mu})^*(p^{**}) - f^*(p^{**})\| = 0.$$

This together with the identification of $\mathscr{K}$ as a subspace of $C(K)$ and the Lebesgue dominated convergence theorem implies: If $f \in \mathscr{K}$ then $f_n^{\mu, f_\mu} \underset{n}{\to} f$ weakly.

7) Let (f, μ, f_μ) and (f, ν, f_ν) be suitable triplets and let $\{n_k\}$ and $\{m_k\}$ be two sequences of natural numbers tending to infinity. Then $\lim\limits_{k\to\infty} f_{n_k}^{\mu, f_\mu} - f_{m_k}^{\nu, f_\nu} = 0$ in the weak topology of the space $\mathscr{K}(P, P^*) \subset \mathscr{L}(P, P^*)$.

Indeed, from 6 follows

$$\lim_{k\to 0}((f_{n_k}^{\mu, f_\mu})^*(p^{**}) - (f_{m_k}^{\nu, f_\nu})^*(p^{**})) = f^*(p^{**}) - f^*(p^{**}) = 0$$

and 1) together with Lebesgue dominated konvergence yield the statement.

8) For any $\Phi \in \mathscr{K}^*$ and any suitable triplet (f, μ, f_μ) we define $J(\Phi, f, \mu, f_\mu) = \lim\limits_{k\to\infty} \Phi(f_{n_k}^{\mu, f_\mu})$ for some subsequence $\{n_k\}$ such that this limit exists. (Notice that f_n^{μ, f_μ} is bounded in $\mathscr{K}$).

Now we claim that $J(\Phi, f, \mu, f_\mu)$ depends neither on the choice of $\{n_k\}$ nor on μ and f_μ (such that $(f, \mu, f_\mu) \in V$). Indeed,

$$\lim_k \Phi(f_{n_k}^{\mu, f_\mu}) - \Phi(f_{m_k}^{\nu, f_\nu}) = \lim_k \Phi(f_{n_k}^{\mu, f_\mu} - f_{m_k}^{\nu, f_\nu}) = 0.$$

Here we used 7 of course.

Thus we may define

$$J(\Phi, f) = J(\Phi, f, \mu, f_\mu) \quad \text{for arbitrary} \quad (f, \mu, f_\mu) \in V.$$

The statement contained in 3 implies that J is defined on the whole $\mathscr{K}^* \times \mathscr{L}$. Furthermore J is bilinear form on $\mathscr{K}^* \times \mathscr{L}$ (use 5): If $(f, \mu, f_\mu) \in V, (g, \nu, g_\nu) \in V$ we may suppose that

$$J(\Phi, f) = \lim \Phi(f_{n_k}^{\mu+\nu, f_{\mu+\nu}}) \quad \text{and} \quad J(\Phi, g) = \lim \Phi(g_{n_k}^{\mu+\nu, g_{\mu+\nu}}).$$

Then $J(\Phi, f) + J(\Phi, g) = \lim \Phi(f+g)_{n_k}^{\mu+\nu, f_{\mu+\nu}+g_{\mu+\nu}} = J(\Phi, f+g)$.

9) For $f \in \mathscr{L}(P, P^*)$ we define

$$p(f) = \inf\{\|A_\mu\|^2 \|f_\mu\|; (f, \mu, f_\mu) \in V\}$$

and

$$|||f||| = \inf\left\{\sum_{i=1}^n p(f_i); \qquad f = \sum_{i=1}^n f_i\right\}.$$

Then $|||\cdot|||$ is an equivalent norm on $\mathscr{L}$ and

$$\|f\| \leqq |||f||| \leqq c\|f\|.$$

Indeed, easily $\|f\| \leqq p(f) \leqq c\|f\|$ by 3.

10) We have

$$|J(\Phi, f)| \leqq \|\Phi\| \cdot |||f|||.$$

Indeed $J(\Phi, f) = \lim \Phi(f_{n_k}^{\mu, f_\mu}) = \lim \Phi(A_\mu^* P_n^\mu f_\mu A_\mu) \leqq \|\Phi\| \cdot \|A_\mu\|^2 \|f_\mu\|$ for every $(f, \mu, f_\mu) \in V$. Thus

$$|J(\Phi, f)| \leqq \|\Phi\| p(f).$$

11) $J(\Phi, f) = \Phi(f)$ for all $(\Phi, f) \in \mathscr{K}^* \times \mathscr{K}$. Indeed by 6) $f_n^{\mu, f_\mu} \to f$ weakly.

12) The bilinear form J on $\mathscr{K}^* \times \mathscr{L}$ gives rise to two canonically defined operators J_K and J_L:

$$J_K : \mathscr{K}^* \to \mathscr{L}^*;\ J_K \Phi(f) = J(\Phi, f)$$

$$J_L : \mathscr{L} \to \mathscr{K}^{**};\ J_L f(\Phi) = J(\Phi, f).$$

13) J_K is c-isomorphism and $\|\Phi\| \leqq J_K \Phi\| \leqq c\|\Phi\|$ for all $\Phi \in \mathscr{K}^*$.

Indeed, the equality 11) implies

$$\|\Phi\| = \sum\{|\Phi(f)|; f \in \mathscr{K}, \|f\| \leqq 1\} \leqq \sum\{|J(\Phi, f); f \in \mathscr{L}, \|f\| \leqq 1\} = \|J_K(\Phi)\|.$$

On the other hand 10) and 9) yield

$$\|J_K \Phi\| = \sum\{|J(\Phi, f); \|f\| \leqq 1\} \leqq \|\Phi\| \sup\{|||f|||; \|f\| \leqq 1\} \leqq c\|\Phi\|.$$

14) Let Re be the restriction map $\mathrm{Re} : \mathscr{L}^* \to \mathscr{K}^*$. Then $\mathrm{Re}\, J_K = \mathrm{Id}\, \mathscr{K}^*$ and $P = J_K \mathrm{Re}$ is the projection in $\mathscr{L}^*$ and $\ker P = \mathscr{K}^0$. Indeed, by 11) we have for any $f \in \mathscr{K}$

$$\mathrm{Re}\, J_K \Phi(f) = J_K \Phi(f) = J(\Phi, f) = \Phi(f).$$

Then $P^2 = J_K \mathrm{Re}\, J_K \mathrm{Re} = J_K \mathrm{Re} = J$.

Finally we have

$$P\Phi = J_K \mathrm{Re}\, \Phi = 0 \Leftrightarrow \mathrm{Re}\, \Phi = 0 \Leftrightarrow \Phi \in \mathscr{K}^0.$$

The proof of Proposition 2 is finished.

Our last result is again inspired by [5, Lemma 2].

Proposition 4. *There is an isomorphism of* $\mathscr{L}(P, P^*)$ *into* $\mathscr{K}(P, P^*)^{**}$ *whose restriction to* $\mathscr{K}(E, F)$ *is the canonical embedding.*

Proof. The isomorphism $J_L : \mathscr{L}(P, P^*) \to \mathscr{K}(P, P^*)^{**}$ is defined by

$$J_L(f)(\Phi) = J(\Phi, f) \quad \text{for each} \quad f \in \mathscr{L}(P, P^*) \quad \text{and each} \quad \Phi \in \mathscr{K}(P, P^*)^*.$$

Here J is the bilinear form on $\mathscr{K}^* \times \mathscr{L}$ defined in the point 8) of the preceding proof. From 10) and 9) we see that

$$||J_L(f)|| = \sup\{|J_L(f)(\phi)|; ||\phi|| \leqq 1\} \leqq |||f||| \leqq c||f||.$$

Thus $||J_L|| \leqq c$. For each $f \in \mathscr{K}(P, P^*)$ we see 11) that $J_L(f)(\Phi) = \Phi(f)$ showing the last assertion. Now let $\varepsilon > 0$ and $f : P \to P^*$ are given and let $x, y \in P_1$ be such that $||f|| - \varepsilon \leqq f(x)y$ then by 6)

$$\begin{aligned} ||f|| - \varepsilon &\leqq \lim f_n^{\mu, f_\mu}(x)(y) = \lim(x \otimes y)(f_n^{\mu, f_\mu}) = J(x \otimes y, f) = \\ &= J_L(f)(x \otimes y) \leqq \sup\{|J_L(f)|; ||\phi|| \leqq 1\} = ||J_L f||. \end{aligned}$$

Added in proof. For substantially generalized version of the Proposition 3 see the forthcoming paper of the author: on a result of J. Johnson, Crechoslovak Math. Journal.

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Received April 23, 1991 and in revised form August 23, 1991
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