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Dynkin diagrams, support spaces and representation type

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Abstract. This survey article is an expanded version of a series of lectures given at the conference on Advances in Group Theory and its Applications which was held in Porto Cesareo in June of 2011. We are concerned with representations of finite group schemes, a class of objects that generalizes the more familiar finite groups. In the last 30 years, this discipline has enjoyed considerable attention. One reason is the application of geometric techniques that originate in Quillen's fundamental work concerning the spectrum of the cohomology ring [25, 26]. The subsequent developments pertaining to cohomological support varieties and representation-theoretic support spaces have resulted in many interesting applications. Here we will focus on those aspects of the theory that are motivated by the problem of classifying indecomposable modules. Since the determination of the simple modules is often already difficult enough, one can in general not hope to solve this problem in a naive sense. However, the classification problem has resulted in an important subdivision of the category of algebras, which will be our general theme.

The algebras we shall be interested in are the so-called cocommutative Hopf algebras, which are natural generalizations of group algebras of finite groups. The module categories of these algebras are richer than those of arbitrary algebras:

- They afford tensor products which occasionally allow the transfer of information between various blocks of the algebra.
- Their cohomology rings are finitely generated, making geometric methods amenable to application.

The purpose of these notes is to illustrate how a combination of these features with methods from the abstract representation theory of algebras and quivers provides insight into classical questions.

Keywords: Dynkin diagram, support variety, representation type, small quantum groups

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1 Motivation and basic examples

1.1 Motivation

We fix the following notation once and for all:

- k denotes an algebraically closed field.

- Unless mentioned otherwise all k-vector spaces are assumed to be finite-dimensional.

- Λ denotes an associative k-algebra.

Given $d \in \mathbb{N}$, we let mod_Λ^d be the affine variety of d-dimensional Λ-modules. More precisely, mod_Λ^d is the variety of Λ-module structures on a fixed d-dimensional k-vector space V. If $\{ x_1, \ldots, x_n \} \subseteq \Lambda$ is a basis of Λ such that $x_1 = 1$ and $x_i x_j = \sum_{\ell=1}^n \alpha_{ij\ell} x_\ell$, then a representation of Λ on V is given by an n-tuple $(A_1, \ldots, A_n)$ of $(d \times d)$-matrices such that $A_1 = I_d$ and $A_i A_j = \sum_{\ell=1}^n \alpha_{ij\ell} A_\ell$. In this fashion, mod_Λ^d is a Zariski closed subspace of k^{nd^2}.

The algebraic group $\mathrm{GL}_d(k)$ acts on mod_Λ^d via conjugation. Thus, the orbits correspond to the isoclasses of Λ-modules. Note that the set ind_Λ^d of indecomposable modules of mod_Λ^d is $\mathrm{GL}_d(k)$-invariant. (The set ind_Λ^d is a constructible subset of mod_Λ^d.)

Definition 1. Given $d \in \mathbb{N}$, we let $C_d \subseteq \mathrm{mod}_\Lambda^d$ be a closed subset of minimal dimension subject to $\mathrm{ind}_\Lambda^d \subseteq \mathrm{GL}_d(k).C_d$. The algebra Λ is

(a) *representation-finite*, provided $\dim C_d = 0$ for every $d \in \mathbb{N}$,

(b) *tame*, provided Λ is not representation-finite and $\dim C_d \leq 1$ for all $d \in \mathbb{N}$,

(c) *wild*, otherwise.

Remark 1. (1) An algebra is representation-finite if and only if there are only finitely many isoclasses of indecomposable Λ-modules. This follows from the so-called second Brauer-Thrall conjecture for Artin algebras, which is known to hold in our context.

(2) If an algebra is wild, then its module category is at least as complicated as that of any other algebra. For such algebras the classification of its indecomposable modules is deemed hopeless [3].

Example 1. (1) Every semi-simple algebra is representation-finite.

(2) The algebra $k[X]/(X^n)$ is representation-finite.

(3) More generally, Nakayama algebras are representation-finite. By definition, the projective indecomposable and injective indecomposable modules of such algebras are uniserial.

(4) The Kronecker algebra $k[X,Y]/(X^2,Y^2)$ is tame.

One may ask what this subdivision looks like for certain classes of algebras. As the representation type of an algebra is an invariant of its Morita equivalence class, the criteria one is looking for are often given in terms of the associated basic algebras. Such algebras can be described by finite directed graphs.

Definition 2. Let Q be a quiver (a directed graph). The *path algebra* kQ of Q has an underlying vector space, whose basis consists of all paths. We multiply paths by concatenation if possible and postulate that their product be zero otherwise.

The above definition is meant to include paths e_i of length 0, labelled by the vertices of Q. They form a system of primitive orthogonal idempotents of kQ. If the quiver is finite, then $\sum_i e_i$ is the identity element of kQ. We shall only be concerned with finite quivers (i.e., Q has finitely many vertices and arrows). In that case, kQ is finite-dimensional if and only if Q does not afford any oriented cycles. The following basic result concerning Morita equivalence $\sim_M$ of k-algebras indicates an interesting connection between representations of quivers and Lie theory:

Theorem 1. *Let Λ be an associative k-algebra.*

(1) There exists a finite quiver Q_Λ and a certain ideal $I \trianglelefteq kQ_\Lambda$ such that $\Lambda \sim_M kQ_\Lambda/I$ [14].

(2) If Λ is hereditary (i.e., submodules of projectives are projective) and Q_Λ is connected, then $\Lambda \sim_M kQ_\Lambda$ and

  (a) Λ is representation-finite if and only if Q_Λ is a Dynkin diagram of type A, D, E [14].

  (b) Λ is tame if and only if Q_Λ is an extended Dynkin diagram of type $\tilde{A}, \tilde{D}, \tilde{E}$ [2, 24].

In either case, the indecomposable modules can be classified via the associated root system.

The quiver Q_Λ is the so-called Ext-*quiver* of Λ. Its vertices are formed by a complete set of representatives for the simple Λ-modules. There are $\dim_k \mathrm{Ext}^1_\Lambda(S,T)$ arrows from S to T. There is no general rule for the computation of the relations generating the non-unique ideal I.

While the above results are very satisfactory from the point of view of abstract representation theory, they do rely on the knowledge of the quiver and the relations of the given algebra. However, even if an algebra is basic to begin with (that is, if all simple modules are one-dimensional), the given presentation may not be suitable for our purposes. Let me illustrate this point by considering an easy example.

Example 2. Let $\mathrm{char}(k) = p > 0$, and consider the algebra given by

$$\Lambda = k\langle t, x\rangle/(tx - xt - x, t^p - t, x^p).$$

This is the natural presentation of the restricted enveloping algebra of the two-dimensional, non-abelian Lie algebra. The bound quiver presentation we are looking for is

$$\Lambda \cong k\tilde{A}_{p-1}/(k\tilde{A}_{p-1})_{\geq p},$$

where the quiver $\tilde{A}_{p-1}$ is the clockwise oriented circle with p vertices and $(k\tilde{A}_{p-1})_{\geq p}$ is the subspace with basis the set of all paths of length $\geq p$.

The more complicated quiver presentation contains more information. One readily sees that Λ is a Nakayama algebra, which is not apparent in the natural presentation.

In these notes we will show how a combination of geometric and representation theoretic methods affords the transition to such a more complicated presentation for certain Hopf algebras of positive characteristic. The classical examples of Hopf algebras are of course the group algebras of finite groups. Here we have the following situation:

Theorem 2. *Suppose that* $\operatorname{char}(k) = p > 0$. *Let kG be the group algebra of a finite group G, $P \subseteq G$ be a Sylow-p-subgroup.*

(1) kG is representation-finite $\Leftrightarrow$ P is cyclic [16].

(2) kG is tame $\Leftrightarrow$ $p = 2$, and P is dihedral, semidihedral, or generalized quaternion [1].

Like any algebra, the group algebra kG is the direct sum of indecomposable two-sided ideals of kG, the so-called *blocks* of kG. Each block is an algebra in its own right and the module category of kG is the direct sum of the module categories of the blocks. (The block decomposition corresponds to the connected components of the Ext-quiver.) The basic algebras of the representation-finite and tame blocks of kG are completely understood. The representation-finite blocks were determined in the late sixties. Almost 20 years later, Karin Erdmann classified blocks of tame representation type via the stable Auslander-Reiten quiver [5].

1.2 Finite algebraic groups and their Hopf algebras

We let M_k and Gr be the categories of not necessarily finite-dimensional commutative k-algebras and groups, respectively. A representable functor

$$\mathcal{G} : M_k \longrightarrow \mathrm{Gr} \;\;;\;\; R \mapsto \mathcal{G}(R)$$

is called an *affine group scheme*. By definition, there exists a commutative k-algebra $k[\mathcal{G}]$ such that $\mathcal{G}(R)$ is the set of algebra homomorphisms $k[\mathcal{G}] \longrightarrow R$ for every $R \in M_k$. By Yoneda's Lemma, the group functor structure of

$\mathcal{G}$ corresponds to a Hopf algebra structure of the coordinate ring $k[\mathcal{G}]$, which renders $k[\mathcal{G}]$ a commutative Hopf algebra.

We say that $\mathcal{G}$ is an *algebraic group* if the representing object $k[\mathcal{G}]$ is finitely generated. If $k[\mathcal{G}]$ is finite-dimensional, then $\mathcal{G}$ is referred to as a *finite algebraic group*. In this case,

$$k\mathcal{G} := k[\mathcal{G}]^*$$

is a finite-dimensional, cocommutative Hopf algebra, the so-called *algebra of measures* on $\mathcal{G}$. In fact, the correspondence

$$\mathcal{G} \mapsto k\mathcal{G}$$

provides an equivalence between the categories of finite algebraic groups and finite-dimensional cocommutative Hopf algebras. In this equivalence, group algebras of finite groups correspond to reduced finite algebraic groups. An algebraic group $\mathcal{G}$ is called *reduced* or *smooth*, provided its coordinate ring $k[\mathcal{G}]$ does not possess any non-trivial nilpotent elements. If $\mathrm{char}(k) = 0$, then Cartier's Theorem asserts that any algebraic group is reduced, thus all cocommutative Hopf algebras are semisimple in this case. We shall therefore henceforth assume that $\mathrm{char}(k) = p > 0$.

Definition 3. A finite group scheme $\mathcal{G}$ is called *infinitesimal*, provided $\mathcal{G}(k) = \{ 1 \}$.

Let $\mathcal{G}$ be a finite algebraic group. General theory shows that

$$k\mathcal{G} = \Lambda * G$$

is a skew group algebra, where $G = \mathcal{G}(k)$ is the finite group of k-rational points of $\mathcal{G}$ and $\Lambda = k\mathcal{G}^0$ is the Hopf algebra of a certain infinitesimal normal subgroup $\mathcal{G}^0$ of $\mathcal{G}$.

Example 3. Let $r \in \mathbb{N}$.

(1) For $n \in \mathbb{N}$, let $\mathrm{GL}(n)_r : M_k \longrightarrow \mathrm{Gr}$ be given by

$$\mathrm{GL}(n)_r(R) := \{ (\zeta_{ij}) \in \mathrm{GL}(n)(R) \mid \zeta_{ij}^{p^r} = \delta_{ij} \}.$$

By general theory, every infinitesimal group $\mathcal{G}$ is a subgroup of a suitable $\mathrm{GL}(n)_r$.

(2) Consider $\mathbb{G}_{m(r)} := \mathrm{GL}(1)_r$, that is,

$$\mathbb{G}_{m(r)}(R) := \{ x \in R^\times \mid x^{p^r} = 1 \} \subseteq R^\times.$$

Then we have

$$k\mathbb{G}_{m(r)} \cong k^{p^r}.$$

(3) Let $\mathbb{G}_{a(r)} : M_k \longrightarrow \mathrm{Gr}$ be given by

$$\mathbb{G}_{a(r)}(R) := \{ x \in R \mid x^{p^r} = 0 \} \subseteq (R, +).$$

Then we have

$$k\mathbb{G}_{a(r)} \cong k[X_1, \ldots, X_r]/(X_1^p, \ldots, X_r^p).$$

As an algebra, $k\mathbb{G}_{a(r)}$ is the group algebra of an elementary abelian p-group of rank r. In particular, we have

$$k\mathbb{G}_{a(r)} \text{ is representation-finite } \Leftrightarrow \ r = 1 \ \ ;$$
$$k\mathbb{G}_{a(r)} \text{ is tame } \Leftrightarrow \ p = 2 \text{ and } r = 2.$$

(4) For $m = np^r$ with $(n, p) = 1$ we consider

$$\mathcal{Q}_{(m)}(R) := \{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2)(R) \mid a^m = 1 = d^m \ , \ b^p = 0 = c^p \}.$$

Then we have $\mathcal{Q}_{(m)}(k) = \{ \left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right) \mid a^n = 1 \}$. Thus, $\mathcal{Q}_{(m)}$ is a finite algebraic group, which is infinitesimal if and only if $n = 1$. The infinitesimal group $\mathcal{Q}_{(p^r)} = \mathrm{SL}(2)_1 T_r$ is the product of the first Frobenius kernel of $\mathrm{SL}(2)$ with the r-th Frobenius kernel of its standard maximal torus T.

Let $\mathcal{G} \subseteq \mathrm{GL}(n)$ be an algebraic group, $r \in \mathbb{N}$. Then

$$\mathcal{G}_r := \mathcal{G} \cap \mathrm{GL}(n)_r$$

is the *r-th Frobenius kernel* of $\mathcal{G}$. Thus, $\mathcal{G}_r$ is an infinitesimal group. One can show that the definition does not depend on the choice of the inclusion $\mathcal{G} \subseteq \mathrm{GL}(n)$.

If $\mathcal{G}$ is infinitesimal, then there exists $r \in \mathbb{N}$ with $\mathcal{G} = \mathcal{G}_r$ and

$$\mathrm{ht}(\mathcal{G}) := \min\{ r \mid \mathcal{G}_r = \mathcal{G} \}$$

is called the *height* of $\mathcal{G}$. The Hopf algebra $k\mathcal{G}$ possesses a co-unit $\varepsilon : k\mathcal{G} \longrightarrow k$. The unique block $\mathcal{B}_0(\mathcal{G}) \subseteq k\mathcal{G}$ with $\varepsilon(\mathcal{B}_0(\mathcal{G})) \neq (0)$ is called the *principal block* of $k\mathcal{G}$. **Problem**. Let $\mathcal{G}$ be a finite algebraic group. When is $\mathcal{B}_0(\mathcal{G})$ representation-finite or tame?

Roughly speaking, we shall pursue the following strategy. Using geometric tools we reduce the problem to the consideration of small examples that are amenable to the methods from abstract representation theory. The latter will enable us to see which of the examples have the desired representation type and what their quivers and relations are.

We conclude this section by stating the analogue of Maschke's Theorem in the context finite algebraic groups. Since the tensor product of a module with a projective module is projective, a Hopf algebra $k\mathcal{G}$ is semi-simple if and only if its principal block is simple.

Theorem 3 (Nagata). *Let $\mathcal{G}$ be a finite algebraic group. Then $k\mathcal{G}$ is semisimple if and only if $p \nmid \operatorname{ord}(\mathcal{G}(k))$ and $\mathcal{G}^0 \cong \prod_{i=1}^n \mathbb{G}_{m(r_i)}$ for some $n \in \mathbb{N}_0$ and $r_i \in \mathbb{N}$.*

2 Support varieties and rank varieties of restricted Lie algebras

2.1 Cohomological support varieties

Let $\mathcal{G}$ be a finite group scheme over an algebraically closed field k of characteristic $p > 0$. We shall study the category $\operatorname{mod} \mathcal{G}$ of finite-dimensional $k\mathcal{G}$-modules, whose objects will be referred to as $\mathcal{G}$-modules. Our tools will be geometric in nature; we begin by outlining the main features.

Let M be a $\mathcal{G}$-module. We denote by

$$\operatorname{Ext}^*_{\mathcal{G}}(M, M) := \bigoplus_{n \geq 0} \operatorname{Ext}^n_{\mathcal{G}}(M, M)$$

the *Yoneda algebra* of self-extensions of M. If $M = k$ is the trivial $\mathcal{G}$-module, then

$$\mathrm{H}^\bullet(\mathcal{G}, k) := \bigoplus_{n \geq 0} \operatorname{Ext}^{2n}_{\mathcal{G}}(k, k)$$

is the *even cohomology ring* of $\mathcal{G}$. This is a commutative k-algebra.

A classical result due to Evens [6] and Venkov [28] asserts that $\mathrm{H}^\bullet(G, k)$ is finitely generated whenever G is a finite group. The most general result of this type is the following:

Theorem 4 ([13]). *Let M be a $\mathcal{G}$-module.*

(1) The commutative k-algebra $\mathrm{H}^\bullet(\mathcal{G}, k)$ is finitely generated.

(2) The homomorphism

$$\Phi_M : \mathrm{H}^\bullet(\mathcal{G}, k) \longrightarrow \operatorname{Ext}^*_{\mathcal{G}}(M, M) \quad ; \quad [f] \mapsto [f \otimes \mathrm{id}_M]$$

is finite.

This fundamental result enables us to introduce geometric techniques by associating varieties to modules. We denote by $\operatorname{Maxspec}(\mathrm{H}^\bullet(\mathcal{G}, k)) := \{ \mathfrak{M} \trianglelefteq \mathrm{H}^\bullet(\mathcal{G}, k) \mid \mathfrak{M} \text{ maximal ideal} \}$ the *maximal ideal spectrum of* $\mathrm{H}^\bullet(\mathcal{G}, k)$. For an arbitrary ideal $I \trianglelefteq \mathrm{H}^\bullet(\mathcal{G}, k)$, we let $Z(I) := \{ \mathfrak{M} \in \operatorname{Maxspec}(\mathrm{H}^\bullet(\mathcal{G}, k)) \mid I \subseteq \mathfrak{M} \}$ be the *zero locus* of I. These sets form the closed sets of the Zariski topology of the affine variety $\operatorname{Maxspec}(\mathrm{H}^\bullet(\mathcal{G}, k))$.

Definition 4. Let M be a $\mathcal{G}$-module. The affine variety

$$\mathcal{V}_{\mathcal{G}}(M) := Z(\ker \Phi_M) \subseteq \mathrm{Maxspec}(\mathrm{H}^{\bullet}(\mathcal{G}, k))$$

is called the *cohomological support variety* of M.

Before looking at an example, let us see how varieties provide information about the representation type of the algebra $k\mathcal{G}$.

Theorem 5. *Let $\mathcal{B} \subseteq k\mathcal{G}$ be a block, $M \in \mathrm{mod}\,\mathcal{B}$.*

(1) If $\mathcal{B}$ is representation-finite, then $\dim \mathcal{V}_{\mathcal{G}}(M) \leq 1$ [15].

(2) If $\mathcal{B}$ is tame, then $\dim \mathcal{V}_{\mathcal{G}}(M) \leq 2$ [8].

Example 4. Let $k\mathcal{G} = k(\mathbb{Z}/(p))^r$ be the group algebra of a p-elementary abelian group of rank r. Then

$$\mathrm{H}^*(\mathcal{G}, k) := k[X_1, \ldots, X_r] \otimes_k \Lambda(Y_1, \ldots, Y_r) \qquad \deg(X_i) = 2,\ \deg(Y_i) = 1,$$

is the tensor product of a polynomial ring and an exterior algebra. We thus obtain:

- $\mathcal{V}_{\mathcal{G}}(k) = \mathrm{Maxspec}(\mathrm{H}^{\bullet}(\mathcal{G}, k)) \cong \mathbb{A}^r$.
- $k\mathcal{G}$ is representation-finite $\Rightarrow r = 1$.
- $k\mathcal{G}$ is tame $\Rightarrow r = 2$.

In view of Theorem 2 this tells us that homological methods alone can in general not be expected to give complete answers to the problem of finding blocks of a given representation type.

2.2 Lie algebras

We have seen that finite algebraic groups consist of two building blocks, reduced groups and infinitesimal groups. In this section we focus on infinitesimal groups of height 1. It turns out that this is equivalent to studying restricted Lie algebras. Given a finite group scheme $\mathcal{G}$, we let $\Delta : k\mathcal{G} \longrightarrow k\mathcal{G} \otimes_k k\mathcal{G}$ denote the comultiplication of $k\mathcal{G}$. Then

$$\mathrm{Lie}(\mathcal{G}) := \{\, x \in k\mathcal{G} \mid \Delta(x) = x \otimes 1 + 1 \otimes x \,\}$$

is called the *Lie algebra* of $\mathcal{G}$. Writing $[x, y] = xy - yx$, we have

(a) $[x, y] \in \mathrm{Lie}(\mathcal{G})$ for every $x, y \in \mathrm{Lie}(\mathcal{G})$, and

(b) $x^p \in \mathrm{Lie}(\mathcal{G})$ for every $x \in \mathrm{Lie}(\mathcal{G})$.

A subspace $\mathfrak{g} \subseteq \Lambda$ of an associative k-algebra Λ satisfying (a) and (b) is called a *restricted Lie algebra*. These algebras may also be defined axiomatically: A restricted Lie algebra is a pair $(\mathfrak{g}, [p])$ consisting of an abstract Lie algebra $\mathfrak{g}$ and an operator $\mathfrak{g} \longrightarrow \mathfrak{g}$; $x \mapsto x^{[p]}$ that satisfies the formal properties of an associative p-th power.

Given such a restricted Lie algebra $(\mathfrak{g}, [p])$ with universal enveloping algebra $U(\mathfrak{g})$, one defines the *restricted enveloping algebra* via

$$U_0(\mathfrak{g}) := U(\mathfrak{g})/(\{ x^p - x^{[p]} \mid x \in \mathfrak{g} \}).$$

The algebra $U_0(\mathfrak{g})$ inherits the Hopf algebra structure from $U(\mathfrak{g})$ and we have

$$\mathfrak{g} = \{ x \in U_0(\mathfrak{g}) \mid \Delta(x) = x \otimes 1 + 1 \otimes x \}.$$

The connection with infinitesimal groups of height 1 is given by:

Proposition 1. *Let $\mathcal{G}$ be an infinitesimal group of height* 1. *Then there exists an isomorphism*

$$k\mathcal{G} \cong U_0(\mathrm{Lie}(\mathcal{G}))$$

of Hopf algebras.

Many of our results to follow will depend on the following basic examples pertaining to solvable and simple restricted Lie algebras.

Example 5. (1) Let V be a k-vector space, $t : V \longrightarrow V$ be a non-zero linear transformation satisfying $t^p = t$. Then $\mathfrak{g}(t, V) := kt \oplus V$ obtains the structure of a restricted Lie algebra via

$$[(\alpha t, v), (\beta t, w)] := (0, \alpha t(w) - \beta t(v)) \quad ; \quad (\alpha t, v)^{[p]} = (\alpha^p t, \alpha^{p-1} t^{p-1}(v)).$$

For the corresponding restricted enveloping algebra one can compute the Ext-quiver and the relations. Abstract representation theory then shows:

- $U_0(\mathfrak{g}(t, V))$ is representation-finite $\Leftrightarrow \dim_k V \leq 1$.
- $U_0(\mathfrak{g}(t, V))$ is tame $\Leftrightarrow \dim_k V = 2$ and $p = 2$.

(2) Let $\mathfrak{g} := \mathfrak{sl}(2)$ be the restricted Lie algebra of trace zero (2×2)-matrices. The restricted enveloping algebra $U_0(\mathfrak{sl}(2))$ possesses exactly p simple modules $L(0), \ldots, L(p-1)$ with $\dim_k L(i) = i+1$. In the early 1980's Fischer [11], Drozd [4] and Rudakov [27] independently computed the quiver and the relations of $U_0(\mathfrak{sl}(2))$. For $p \geq 3$, the algebra $U_0(\mathfrak{sl}(2))$ has blocks $\mathcal{B}_0, \ldots, \mathcal{B}_{\frac{p-3}{2}}$, each $\mathcal{B}_i$ possessing two simple modules $L(i)$ and $L(p-2-i)$. There is one additional simple block $\mathcal{B}_{p-1}$ belonging to the Steinberg module $L(p-1)$. The non-simple blocks have bound quiver presentation given by the quiver Δ_1:

$$\begin{tikzcd} 0 \arrow[r, shift left=4.5, "\alpha_0"] \arrow[r, shift left=1.5, "\beta_0"] & 1, \arrow[l, shift left=1.5, "\alpha_1"'] \arrow[l, shift left=4.5, "\beta_1"'] \end{tikzcd}$$

and relations defining the ideal $J \trianglelefteq k\Delta_1$ generated by

$$\{\, \beta_{i+1}\alpha_i - \alpha_{i+1}\beta_i, \quad \alpha_{i+1}\alpha_i, \quad \beta_{i+1}\beta_i \;\mid\; i \in \mathbb{Z}/(2)\,\}.$$

These examples will turn out to be of major importance for our determination of the tame infinitesimal groups of odd characteristic. The first example is essentially the reason for the validity of the following result:

Proposition 2 ([9])**.** *Suppose that $p \geq 3$, and let $\mathcal{G}$ be a solvable infinitesimal group. Then $\mathcal{B}_0(\mathcal{G})$ is either representation-finite or wild.*

Turning to the second example, we observe that the algebra $k[\Delta_1]/J$ is tame. In fact, our algebra belongs to an important class of tame algebras, the so-called *special biserial algebras*. The uniformity of the presentation of these blocks is not accidental; it is a consequence of the so-called translation principle [18], which affords the passage between certain blocks. Roughly speaking, one proceeds as follows: Given two blocks $\mathcal{B}, \mathcal{C}$ of $U_0(\mathfrak{g})$ and a simple module S, one considers the functor

$$\mathrm{Tr}_S : \mathrm{mod}\,\mathcal{B} \longrightarrow \mathrm{mod}\,\mathcal{C} \quad ; \quad M \mapsto e_{\mathcal{C}} \cdot (S \otimes_k M).$$

Here $e_{\mathcal{C}} \in U_0(\mathfrak{g})$ is the central idempotent defining the block $\mathcal{C}$. Under certain compatibility conditions on $\mathcal{B}, \mathcal{C}$ and S, this functor is in fact a Morita equivalence. The easiest instance of the translation principle is given by one-dimensional modules. In particular, all blocks of basic cocommutative Hopf algebras (i.e., those corresponding to group schemes of upper triangular matrices) are isomorphic.

2.3 Rank varieties

Although being of theoretical importance, support varieties are inherently intractable. Quillen's early work on the spectrum of the cohomology ring of a finite group and Chouinard's result on projective modules suggested that elementary abelian groups could play an important rôle. Dade noticed a further reduction to cyclic shifted subgroups. These observations led Jon Carlson to his representation-theoretic notion of a rank variety. A few years later a similar theory for restricted Lie algebras was developed by Friedlander-Parshall and Jantzen.

About 7 years ago, Eric Friedlander and Julia Pevtsova introduced a theory of representation-theoretic support spaces that applies to all finite group schemes. Since this approach is a bit technical, we confine our attention to restricted Lie algebras.

Definition 5. Let $(\mathfrak{g}, [p])$ be a restricted Lie algebra. The conical variety

$$V(\mathfrak{g}) := \{ x \in \mathfrak{g} \mid x^{[p]} = 0 \}$$

is called the *nullcone* of $\mathfrak{g}$. Let M be a $U_0(\mathfrak{g})$-module. Then

$$V(\mathfrak{g})_M := \{ x \in V(\mathfrak{g}) \mid M|_{k[x]} \text{ is not free} \} \cup \{ 0 \}$$

is referred to as the *rank variety* of M.

The name derives from the following alternative description of $V(\mathfrak{g})_M$: Given $x \in V(\mathfrak{g})$, we denote by $x_M : M \longrightarrow M \; ; \; m \mapsto x.m$ the left multiplication by x on M. Then we have $x \in V(\mathfrak{g})_M$ if and only if $\operatorname{rk}(x_M) < \frac{p-1}{p} \dim_k M$.

Example 6. Let $\mathfrak{g} = \mathfrak{sl}(2)$.

- Note that $V(\mathfrak{sl}(2))$ is the set of nilpotent (2×2)-matrices, so that

  $$V(\mathfrak{sl}(2)) = \{ \left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right) \mid a^2 + bc = 0 \}.$$

  Thus, $V(\mathfrak{sl}(2))$ is a two-dimensional, irreducible variety.

- Recall that there are exactly p simple $U_0(\mathfrak{sl}(2))$-modules $L(i)$ $0 \le i \le p-1$ with $\dim_k L(i) = i+1$. If $x \in V(\mathfrak{sl}(2)) \smallsetminus V(\mathfrak{sl}(2))_{L(i)}$, then $L(i)$ is a free module for the p-dimensional algebra $k[x]$. Thus, $p| \dim_k L(i)$ and $i = p-1$. Hence $L(i) = L(p-1)$ is the Steinberg module, which is projective. We therefore have (see also Corollary 2 below)

  $$V(\mathfrak{sl}(2))_{L(i)} = \begin{cases} V(\mathfrak{sl}(2)) & i \ne p-1 \\ \{0\} & i = p-1. \end{cases}$$

- The rank varieties of the baby Verma modules $Z(i) := U_0(\mathfrak{sl}(2)) \otimes_{U_0(\mathfrak{b})} k_i$ are of dimension 0 or 1. Here $\mathfrak{b} \subseteq \mathfrak{sl}(2)$ is the Borel subalgebra of upper triangular matrices of trace zero, and k_i denotes the one-dimensional $U_0(\mathfrak{b})$-module with weight $i \in \{ 0, \dots, p-1 \}$.

Theorem 6 ([17, 12])**.** *Let $(\mathfrak{g}, [p])$ be a restricted Lie algebra. Then there exists a homeomorphism*

$$\Psi : \mathcal{V}_{\mathfrak{g}}(k) \longrightarrow V(\mathfrak{g})$$

such that $\Psi(\mathcal{V}_{\mathfrak{g}}(M)) = V(\mathfrak{g})_M$ for every $M \in \operatorname{mod} U_0(\mathfrak{g})$.

This result tells us that for our intents and purposes rank varieties are as good a cohomological support varieties. Theorem 5 now implies:

Corollary 1. *Let $\mathcal{G}$ be a finite algebraic group with Lie algebra $\mathfrak{g}$.*

(1) If $\mathcal{B}_0(\mathcal{G})$ is representation-finite, then $\dim V(\mathfrak{g}) \leq 1$.

(2) If $\mathcal{B}_0(\mathcal{G})$ is tame, then $\dim V(\mathfrak{g}) \leq 2$.

So why did we introduce support varieties to begin with? Let us look at the following result:

Corollary 2. *Let $(\mathfrak{g}, [p])$ be a restricted Lie algebra, M be a $U_0(\mathfrak{g})$-module. Then the following statements are equivalent:*

(1) M is projective.

(2) $V(\mathfrak{g})_M = \{0\}$.

Proof. (1) $\Rightarrow$ (2). Let $x \in V(\mathfrak{g})$. By the PBW-Theorem, $U_0(\mathfrak{g})$ is a free $k[x]$-module. Hence $M|_{k[x]}$ is projective, so that $x = 0$.

(2) $\Rightarrow$ (1). If $V(\mathfrak{g})_M = \{0\}$, then $\mathcal{V}_{\mathfrak{g}}(M)$ is finite. Recall that

$$\Phi_M : \mathrm{H}^\bullet(\mathfrak{g}, k) \longrightarrow \mathrm{Ext}^*_{U_0(\mathfrak{g})}(M, M)$$

is a finite morphism. Since the Krull dimension $\dim \mathrm{H}^\bullet(\mathfrak{g}, k)/\ker \Phi_M = \dim \mathcal{V}_{\mathfrak{g}}(M) = 0$, the algebra $\mathrm{Ext}^*(M, M)$ is finite-dimensional. It follows that there exists $n_0 \in \mathbb{N}$ such that $\mathrm{Ext}^n_{U_0(\mathfrak{g})}(M, -) = 0$ for all $n \geq n_0$. Hence M has finite projective dimension. But $U_0(\mathfrak{g})$ is a Hopf algebra and hence self-injective. This implies that M is projective. $\boxed{QED}$

The foregoing result suggests that $\dim V(\mathfrak{g})_M$ has a representation-theoretic interpretation. Indeed,

$$\dim V(\mathfrak{g})_M = \mathrm{cx}_{U_0(\mathfrak{g})}(M)$$

is the *complexity* of M, that is, the polynomial rate of growth of a minimal projective resolution of M.

3 Binary polyhedral groups, McKay quivers, and tame blocks

Throughout, $\mathcal{G}$ denotes a finite group scheme over an algebraically closed field k of characteristic $\mathrm{char}(k) = p > 0$. We want to know when the principal block $\mathcal{B}_0(\mathcal{G})$ has tame representation type. If G is a finite group, this happens precisely, when $p = 2$, and the Sylow-2-subgroups of G are dihedral, semidihedral, or generalized quaternion.

Recall that

$$k\mathcal{G} = \Lambda {*} G$$

is a skew group algebra, where $G = \mathcal{G}(k)$ is the finite group of k-rational points of $\mathcal{G}$, and $\Lambda = k\mathcal{G}^0$ is the Hopf algebra of an infinitesimal group scheme.

3.1 A basic reduction

In the sequel, we write $\mathfrak{g} := \mathrm{Lie}(\mathcal{G})$ and assume that $p \geq 3$.

Definition 6. The group scheme $\mathcal{G}$ is *linearly reductive* if the associative algebra $k\mathcal{G}$ is semi-simple.

Recall that Nagata's Theorem 3 describes the structure of the linearly reductive groups. Using rank varieties, we obtain the following result:

Theorem 7 ([7, 9])**.** *If $\mathcal{B}_0(\mathcal{G})$ is tame, then*

(a) $p \nmid |\mathcal{G}(k)|$, and

(b) $\mathfrak{g}/C(\mathfrak{g}) \cong \mathfrak{sl}(2)$, where $C(\mathfrak{g})$ denotes the center of $\mathfrak{g}$.

In particular, $\mathfrak{g}$ is a central extension of $\mathfrak{sl}(2)$. Since the Chevalley-Eilenberg cohomology group $\mathrm{H}^2(\mathfrak{sl}(2), k)$ vanishes, such an extension splits, when considered as one of ordinary Lie algebras. General theory then shows that the structure of $\mathfrak{g} = \mathfrak{sl}(2) \oplus V$ is given as follows:

$$[(x, v), (y, w)] := ([x, y], 0) \quad \text{and} \quad (x, v)^{[p]} = (x^{[p]}, \psi(x) + v^{[p]}),$$

where $\psi : \mathfrak{sl}(2) \longrightarrow V$ is p-semilinear. One can say when exactly $U_0(\mathfrak{g})$ is tame. Instead of going into the technical details, let us look at one particular example, that reveals fundamental differences between finite groups and restricted Lie algebras.

Example 7. Let $\{ e, h, f \}$ be the standard basis of $\mathfrak{sl}(2)$ and suppose that $V = kv$ is one-dimensional. We define the Lie algebra $\mathfrak{sl}(2)_s := \mathfrak{sl}(2) \oplus kv$ via

$$e^{[p]} = 0 = f^{[p]} \quad ; \quad h^{[p]} = h + v \quad ; \quad v^{[p]} = 0.$$

(This amounts to choosing the p-semilinear map $\psi_s : \mathfrak{sl}(2) \longrightarrow kv$; $\psi_s(\left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right)) = a^p v$.) The algebra $U_0(\mathfrak{sl}(2)_s)$ turns out to be tame. However, the subalgebra $U_0(ke \oplus kv) \cong k(\mathbb{Z}/(p) \times \mathbb{Z}/(p))$ is wild. By contrast, Brauer's Third Main Theorem implies that subgroups of tame finite groups are always tame.

We are going to simplify matters a little and assume from now on that $C(\mathfrak{g}) = (0)$. In the context of finite groups this amounts to assuming that the Sylow-2-subgroup is a Klein four group. We are thus studying a Hopf algebra

$$\Lambda = U_0(\mathfrak{sl}(2)) {*} G,$$

where G is a linearly reductive finite group that acts on $U_0(\mathfrak{sl}(2))$ via automorphisms of Hopf algebras. Hence G acts on $\mathfrak{sl}(2)$ via automorphisms. If N denotes the kernel of this action, then the principal block of Λ is isomorphic to that of $U_0(\mathfrak{sl}(2)) {*} (G/N)$. Since $\mathrm{Aut}(\mathfrak{sl}(2)) \cong \mathrm{PSL}(2)(k)$, we may thus assume $G \subseteq \mathrm{PSL}(2)(k)$. Passage to the double cover does not change the principal block and thus yields $G \subseteq \mathrm{SL}(2)(k)$. In other words, G is a *binary polyhedral group*. These groups were classified by Klein around 1884.

3.2 Extended Dynkin diagrams and finite groups

Extended Dynkin diagrams are perhaps best known from Lie theory, where they appear in the structure theory of affine Kac-Moody algebras. We have seen in Section 1 another occurrence in the representation theory of hereditary algebras. In this case, these diagrams describe the Ext-quivers of hereditary algebras of tame representation type.

Extended Dynkin diagrams also appear in the representation theory of finite groups. In his seminal work, J. McKay [22, 23] associated to a finite group G and a complex G-module V a quiver $\Psi_V(G)$ that has since played a rôle in a number of contexts. Let's generalize this a little to cover our setting.

- Let $H = k\mathcal{G}$ be the Hopf algebra of a linearly reductive finite group scheme.
- $\{ S_1, \ldots, S_n \}$ denotes a complete set of representatives for the isoclasses of the simple H-modules.
- Fix an H-module V. Then V defines an $(n \times n)$-matrix $(a_{ij}) \in \mathrm{Mat}_n(\mathbb{Z})$ such that
$$V \otimes_k S_j \cong \bigoplus_{i=1}^{n} a_{ij} S_i \qquad 1 \leq j \leq n.$$

In other words, the integral $(n \times n)$-matrix (a_{ij}) describes the left multiplication by V in the Grothendieck ring $K_0(H)$ of H relative to its standard basis of simple modules.

Definition 7. Let $\mathcal{G}$ be a linearly reductive finite group scheme, V be a $\mathcal{G}$-module. The *McKay quiver* $\Psi_V(\mathcal{G})$ *of* $\mathcal{G}$ *relative to* V is given by the following data:

- Vertices: $\{1, \dots, n\}$
- Arrows: $i \xrightarrow{a_{ij}} j$.

Example 8. (1) Let G be an abelian group with $p \nmid \mathrm{ord}(G)$. If V is a faithful G-module with simple constituents $k_{\lambda_1}, \dots, k_{\lambda_r}$, then the character group $X(G)$ is generated by $S := \{\lambda_1, \dots, \lambda_r\}$ and the McKay quiver of G relative to V is the Cayley graph of $X(G)$ relative to S.

(2) Let G be a finite group with $p \nmid \mathrm{ord}(G)$, V be a faithful G-module. By Burnside's classical theorem, every simple G-module is a direct summand of some tensor power $V^{\otimes n}$. This implies that the quiver $\Psi_V(G)$ is connected. There is a version of Burnside's result for finite group schemes.

Let us return to our simplified context. We thus have

$$\Lambda = U_0(\mathfrak{sl}(2)) * G,$$

with $G \subseteq \mathrm{SL}(2)(k)$ acting on $\mathfrak{sl}(2)$ via automorphisms, and p not dividing the order of G. This implies that the McKay quiver $\Psi_{L(1)}(G)$ of G relative to the two-dimensional standard representation $L(1) = k^2$ is connected.

It turns out that a binary polyhedral group is uniquely determined by its McKay graph $\Psi_{L(1)}(G)$. Here is the list of groups up to conjugation in $\mathrm{SL}(2)(k)$:

G	$\overline{\Psi_{L(1)}(G)}$
$\mathbb{Z}/(n)$	$\tilde{A}_{n-1}$
Q_n	$\tilde{D}_{n+2}$
T	$\tilde{E}_6$
O	$\tilde{E}_7$
I	$\tilde{E}_8$.

The left-hand column gives the isomorphism types of the finite groups. Here Q_n denotes the quaternion group of order $4n$, and T, O, and I refer to the binary tetrahedral group (of order 24), the binary octahedral group (of order 48) and the binary icosahedral group (of order 120), respectively. The quivers corresponding to the graphs in the right-hand column are obtained by replacing each bond by $\bullet \leftrightarrows \bullet$.

The above list will be sufficient for our simplified context. In general, one needs to deal with linearly reductive group schemes $\mathcal{G} \subseteq \mathrm{SL}(2)$.

Thus, modulo our simplifications, we know the groups that can occur, i.e., we understand the Hopf algebra structure. Moreover, the affine quivers describing the tame hereditary algebras also appear. How can we get the Ext-quiver of Λ? The first step consists of finding the simple modules.

Lemma 1. *Let $\mathcal{N} \trianglelefteq \mathcal{G}$ be a normal subgroup. Suppose that $L_1, \dots, L_n$ are simple $\mathcal{G}$-modules such that $\{ L_1|_{\mathcal{N}}, \dots, L_n|_{\mathcal{N}} \}$ is a complete set of representatives for the simple $\mathcal{N}$-modules.*

(1) Every simple $\mathcal{G}$-module S is of the form

$$S \cong L_i \otimes_k M$$

for a unique $i \in \{ 1, \dots, n \}$ and a unique simple $\mathcal{G}/\mathcal{N}$-module M.

(2) Suppose that $\mathrm{Ext}^1_{\mathcal{N}}(V, V) = (0)$ for every simple $\mathcal{N}$-module V. If M, N are simple $\mathcal{G}/\mathcal{N}$-modules, then

$$\mathrm{Ext}^1_{\mathcal{G}}(L_i \otimes_k M, L_j \otimes_k N) \cong \begin{cases} (0) & i = j \\ \mathrm{Hom}_{\mathcal{G}/\mathcal{N}}(M, \mathrm{Ext}^1_{\mathcal{N}}(L_i, L_j) \otimes_k N) & i \neq j. \end{cases}$$

If $\mathcal{G}/\mathcal{N}$ is linearly reductive, then the dimension of our Ext-group describes the multiplicity of M in the $\mathcal{G}/\mathcal{N}$-module $\mathrm{Ext}^1_{\mathcal{N}}(L_i, L_j) \otimes_k N$. We thus obtain a connection between the Ext-quiver of $k\mathcal{G}$ and the McKay quiver of $\mathcal{G}/\mathcal{N}$ relative to $\mathrm{Ext}^1_{\mathcal{N}}(L_i, L_j)$.

The technical conditions of the Lemma may seem somewhat contrived, but they do hold in classical contexts such as ours: There exist simple Λ-modules $L(0), \dots, L(p-1)$, whose restrictions to $U_0(\mathfrak{sl}(2))$ give all simple $U_0(\mathfrak{sl}(2))$-modules. Moreover, there are isomorphisms of G-modules

$$\mathrm{Ext}^1_{\mathfrak{sl}(2)}(L(i), L(j)) \cong \begin{cases} (0) & i+j \neq p-2 \\ L(1) & \text{otherwise.} \end{cases}$$

The Lemma now shows that the Ext-graph of Λ consists of the extended Dynkin diagrams that appear in the classification of the tame hereditary algebras.

Group algebras, or Hopf algebras in general, are self-injective and thus are hereditary only in case they are semi-simple (no arrows). The passage from hereditary algebras to self-injective algebras is given by the notion of *trivial extension.*

Given an algebra Λ, the *trivial extension* of Λ is the semidirect product $T(\Lambda) := \Lambda \ltimes \Lambda^*$ of Λ with its bimodule Λ^*:

$$(a, f) \cdot (b, g) := (ab, a.g + f.b) \quad \forall\, a, b \in \Lambda,\ f, g \in \Lambda^*.$$

The algebra $T(\Lambda)$ is symmetric, and one can often compute the quiver and the relations of $T(\Lambda)$. For instance, if $\Delta_n = \tilde{A}_{2n-1}$ is the quiver without paths of length 2, then $T(k\Delta_n) = kQ/I$, where Q is given by

$$
\begin{tikzcd}
 & 1 \arrow[r, shift left, "\alpha_1"] \arrow[dl, shift left, "\beta_1"] & 2 \arrow[l, shift left, "\beta_2"] \arrow[r, shift left] & \cdots \arrow[l, shift left] \arrow[r, shift left] & n-2 \arrow[l, shift left] \arrow[r, shift left, "\alpha_{n-2}"] & n-1 \arrow[l, shift left, "\beta_{n-1}"] \arrow[dr, shift left, "\alpha_{n-1}"] & \\
0 \arrow[ur, shift left, "\alpha_0"] \arrow[dr, shift left, "\beta_0"] & & & & & & n \arrow[ul, shift left, "\beta_n"] \arrow[dl, shift left, "\alpha_n"] \\
 & 2n-1 \arrow[ul, shift left, "\alpha_{2n-1}"] \arrow[r, shift left, "\beta_{2n-1}"] & 2n-2 \arrow[l, shift left, "\alpha_{2n-2}"] \arrow[r, shift left] & \cdots \arrow[l, shift left] \arrow[r, shift left] & n+2 \arrow[l, shift left] \arrow[r, shift left, "\beta_{n+2}"] & n+1 \arrow[l, shift left, "\alpha_{n+1}"] \arrow[ur, shift left, "\beta_{n+1}"] &
\end{tikzcd}
$$

and $I \subseteq kQ$ is the ideal generated by

$$\{\, \beta_{i+1}\alpha_i - \alpha_{i-1}\beta_i, \quad \alpha_{i+1}\alpha_i, \quad \beta_i\beta_{i+1} \;\mid\; i \in \mathbb{Z}/(2n) \,\}.$$

Thus, the effect of passing to the trivial extension is the familiar doubling process. For $n = 1$ we obtain the algebra of Example 5(2).

It turns out that the tame principal blocks of finite group schemes are algebras of this type:

Theorem 8 ([7]). *Let $\mathcal{G}$ be a finite group scheme of characteristic $p \geq 3$ such that $\mathcal{B}_0(\mathcal{G})$ tame.*

(1) There exists a linearly reductive group scheme $\tilde{\mathcal{G}} \subseteq \mathrm{SL}(2)$ such that the Ext*-quiver of $\mathcal{B}_0(\mathcal{G})$ is isomorphic to the McKay quiver $\Psi_{L(1)}(\tilde{\mathcal{G}})$.*

(2) The block $\mathcal{B}_0(\mathcal{G})$ is Morita equivalent to a generalized trivial extension of a tame hereditary algebra.

Let us return to our example and consider $G = T_{(2n)}$, the cyclic group of order $2n$ contained in the standard maximal torus $T \subseteq \mathrm{SL}(2)$ of diagonal matrices. In that case, $\tilde{\mathcal{G}}$ is the reduced group with $\tilde{\mathcal{G}}(k) = T_{(2n)}$, and our Theorem says that

$$\mathcal{B}_0(\Lambda) \sim_M T(k\tilde{A}_{2n-1})$$

is Morita equivalent to the trivial extension, which we have considered above. The other binary polyhedral groups give rise to the trivial extensions of the corresponding affine quivers.

4 Small quantum groups

Let $\mathfrak{g}$ be a finite-dimensional complex semi-simple Lie algebra. Given a complex number $\zeta \in \mathbb{C} \smallsetminus \{0\}$, Drinfeld and Jimbo defined the quantum group $U_\zeta(\mathfrak{g})$ of $\mathfrak{g}$. Roughly speaking, this Hopf algebra is a deformation of the ordinary enveloping algebra $U(\mathfrak{g})$. Technically, it is defined via a Chevalley basis of $\mathfrak{g}$ and the so-called quantum Serre relations. If ζ is not a root of unity, all finite-dimensional $U_\zeta(\mathfrak{g})$-modules are completely reducible. Alternatively, its representation theory resembles that of Lie algebras in positive characteristic. Lusztig defined a finite-dimensional Hopf subalgebra $u_\zeta(\mathfrak{g})$ of $U_\zeta(\mathfrak{g})$, which can be thought of as an analogue of the restricted enveloping algebra of a restricted Lie algebra. If ζ is a primitive ℓ-th root of unity, then $\dim_k u_\zeta(\mathfrak{g}) = \ell^{\dim_k \mathfrak{g}}$. To cut down on subtle technicalities, we shall henceforth assume that $6 \nmid \ell$.

In order to develop a theory of supports for $u_\zeta(\mathfrak{g})$, one needs an analogue of the Friedlander-Suslin Theorem. Since there are other cases of Hopf algebras, where such a result is available, it is expedient to formulate the relevant properties in broader context. A rather detailed summary of the current state of the art can be found in [19].

We consider a (finite-dimensional) Hopf algebra Λ over a algebraically closed field k (of arbitrary characteristic). It is well-known that the cohomology ring $\mathrm{H}^*(\Lambda, k)$ is graded commutative, so that the even cohomology ring $\mathrm{H}^\bullet(\Lambda, k)$ is a commutative k-algebra.

Definition 8. Let Λ be a Hopf algebra. We say that Λ is an *fg-Hopf algebra*, provided

(a) the algebra $\mathrm{H}^\bullet(\Lambda, k)$ is finitely generated, and

(b) for every $M \in \operatorname{mod}\Lambda$, the algebra homomorphism $\Phi_M : \mathrm{H}^\bullet(\Lambda, k) \longrightarrow \mathrm{Ext}^*_\Lambda(M, M)$ is finite.

In this case, $\operatorname{Maxspec}(\mathrm{H}^\bullet(\Lambda, k))$ carries the structure of an affine variety and one defines the support variety

$$\mathcal{V}_\Lambda(M) := Z(\ker \Phi_M)$$

for every $M \in \operatorname{mod}\Lambda$. One can show that $M \mapsto \mathcal{V}_\Lambda(M)$ enjoys properties analogous to those known for finite group schemes. In particular, Feldvoss and Witherspoon [10] have generalized Theorem 5 to the present context. Using these techniques one obtains the following result:

Theorem 9 ([20])**.** *Let $\mathfrak{g}$ be simple and suppose that ℓ is good for the root system of $\mathfrak{g}$. If $\mathcal{B} \subseteq u_\zeta(\mathfrak{g})$ is a block, then the following statements hold:*

(1) $\mathcal{B}$ is representation-finite if and only and if $\mathcal{B}$ is the simple block belonging to the Steinberg module.

(2) If $\mathcal{B}$ has tame representation type, then $\mathfrak{g} \cong \mathfrak{sl}(2)$ and $\mathcal{B}$ is Morita equivalent to $T(k(\bullet \rightrightarrows \bullet))$.

The representation theory of the trivial extension of the Kronecker quiver $\bullet \rightrightarrows \bullet$ is completely understood.

Support spaces and Dynkin diagrams also appear in the context of Auslander-Reiten theory. Given a self-injective Λ, one defines a quiver $\Gamma_s(\Lambda)$, which is an important invariant of its Morita equivalence class. The vertices of the so-called *stable Auslander-Reiten quiver* are the isoclasses of the non-projective indecomposable Λ-modules. Arrows are given by irreducible morphisms. Roughly, speaking such a non-isomorphism does not factor non-trivially through any indecomposable Λ-module. A third ingredient is the Auslander-Reiten translation $\tau : \Gamma_s(\Lambda) \longrightarrow \Gamma_s(\Lambda)$, which reflects homological properties. A fundamental result by Riedtmann states that the isomorphism class of a connected component $\Theta \subseteq \Gamma_s(\Lambda)$ is essentially determined by an undirected tree T_Θ, the *tree class* of Θ. For fg-Hopf algebras, the possible tree classes are finite Dynkin diagrams, Euclidean diagrams or infinite Dynkin diagrams of type A_∞, D_∞, A_∞^∞. In concrete cases, support varieties can be used to decide, which tree class a given component has.

Recall that $u_\zeta(\mathfrak{g}) \subseteq U_\zeta(\mathfrak{g})$ is a Hopf subalgebra. We let h denote the Coxeter number of $\mathfrak{g}$.

Theorem 10 ([21])**.** *Let $\ell \geq h$. Suppose that $\Theta \subseteq \Gamma_s(u_\zeta(\mathfrak{g}))$ is a component containing the restriction of a $U_\zeta(\mathfrak{g})$-module. If $\mathfrak{g} \neq \mathfrak{sl}(2)$ is simple, then $T_\Theta = A_\infty$.*

Proof. Given $M, N \in \Theta$ one can show that $\mathcal{V}_{u_\zeta(\mathfrak{g})}(M) = \mathcal{V}_{u_\zeta(\mathfrak{g})}(N)$, so that we have the support variety $\mathcal{V}_{u_\zeta(\mathfrak{g})}(\Theta)$. This variety corresponds to a Zariski closed subspace X_Θ of the nilpotent cone $\mathcal{N} \subseteq \mathfrak{g}$. Since Θ contains the restriction of a $U_\zeta(\mathfrak{g})$-module, X_Θ is invariant under the adjoint action of the algebraic group G of $\mathfrak{g}$. As $\mathfrak{g} \neq \mathfrak{sl}(2)$, a little more structure theory implies that $\dim \mathcal{V}_{u_\zeta(\mathfrak{g})}(\Theta) \geq 3$. Such components are known to have tree class A_∞. $\boxed{QED}$

Using this result, one can for instance locate the simple $u_\zeta(\mathfrak{g})$-modules within the AR-quiver and show that they have precisely one predecessor. This in turn yields information concerning the structure of certain subfactors of principal indecomposable $u_\zeta(\mathfrak{g})$-modules.

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