Note di Matematica ISSN 1123-2536, e-ISSN 1590-0932
Note Mat. **31** (2011) n. 1, 139–148. doi:10.1285/i15900932v31n1p139

On the domain of a Fleming–Viot-type operator on an L^p-space with invariant measure

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Received: 08/09/2010; accepted: 29/09/2010.

Abstract. We characterize the domain of a Fleming-Viot type operator of the form $L\varphi(x) := \sum_{i=1}^N x_i(1-x_i)D_{ii}\varphi(x) + \sum_{i=1}^N (\alpha_i(1-x_i) - \alpha_{i+1}x_i)D_i\varphi(x)$ on $L^p([0,1]^N, \mu)$ for $1 < p < \infty$, where μ is the corresponding invariant measure. Our approach relies on the characterization of the domain of the one-dimensional Fleming-Viot operator and the Dore-Venni operator sum method.

Keywords: Fleming–Viot process, degenerate elliptic problems, analytic C_0-semigroups

MSC 2000 classification: primary 47D07, secondary 60J35

Dedicated to the memory of V.B. Moscatelli

1 Introduction

In this paper we are dealing with the following Fleming-Viot type operator

$$L\varphi(x) := \sum_{i=1}^N x_i(1-x_i)D_{ii}\varphi(x) + \sum_{i=1}^N (\alpha_i(1-x_i) - \alpha_{i+1}x_i)D_i\varphi(x), \quad x \in [0,1]^N,$$

where the constants $\alpha_i > 0$ for all $i = 1, \ldots, N+1$. Note that in the one-dimensional case the above given operator is the classical Fleming-Viot operator arising in population genetics, whereas the usual N-dimensional formulation of the Fleming–Viot model takes place on a

[i]This work was done while A. Rhandi visited the university of Ulm, Germany. He wishes to express his gratitude to the University of Ulm for the financial support.

simplex instead of a cube (see however also [2]). We refer to [10] for the original derivation of the model and to [8,9] for surveys on the theory of Fleming–Viot processes and its applications to the genetic evolution of a population.

Let

$$\beta_i := \frac{1}{\int_0^1 x^{\alpha_i - 1}(1-x)^{\alpha_{i+1}-1}\,dx}.$$

Since $\alpha_i > 0$ for all $i = 1, \dots, N+1$, it is not difficult to see that $0 < \beta_i < \infty$ for all $i = 1, \dots, N+1$ and the probability measure

$$d\mu(x) = \prod_{i=1}^N \beta_i x_i^{\alpha_i - 1}(1-x_i)^{\alpha_{i+1}-1}\,dx, \quad x = (x_1, \dots, x_N) \in [0,1]^N$$

is an invariant measure for the operator L, i.e.

$$\int_{[0,1]^N} L\varphi(x)\,d\mu(x) = 0, \qquad \text{for all } \varphi \in C^2([0,1]^N).$$

We refer e.g. to [5, Chapter 11] or [6] for an introduction to this theory.

It is known that if $N = 1$ then

$$\begin{aligned} &L\varphi(x) = x(1-x)\varphi''(x) + (\alpha_1(1-x) - \alpha_2 x)\varphi'(x), \quad x \in [0,1] \text{ with domain} \\ &D(L) = \{\varphi \in C^1[0,1] \cap C^2(0,1) : \lim_{x \to 0^+,\, 1^-} x(1-x)\varphi''(x) = 0\} \end{aligned}$$

generates a C_0-semigroup $T(\cdot)$ of contractions on $C[0,1]$ which is positive and analytic, and $C^2[0,1]$ is a core (see [11] and [1, § 3]). Hence in particular the invariance of the measure μ_1 is equivalent to saying that

$$\int_{[0,1]} T(t)\varphi(x)\,d\mu_1(x) = \int_{[0,1]} \varphi(x)\,d\mu_1(x), \qquad \text{for all } \varphi \in C[0,1] \text{ and all } t \geq 0.$$

Since the probability measure $d\mu_1(x) = \beta_1 x^{\alpha_1 - 1}(1-x)^{\alpha_2 - 1}dx$, $x \in [0,1]$, is an invariant measure for L, it is known (see e.g. [6, Thm. 3.7]) that the semigroup $T(\cdot)$ can be extended to an analytic C_0-semigroup $T_p(\cdot)$ of contractions on $L^p(0,1;\mu_1)$, $1 \leq p < \infty$. However, to the best of our knowledge an explicit form for the domain of the generators of such L^p-semigroups has not yet been obtained – not even in the one-dimensional case. Aim of the present article is to solve this problem. We remark that a related result has been obtained for $p = 2$ and under certain technical assumptions in [1, §4].

In $[0,1]^N$ the operator L, defined on $C^2([0,1]^N)$, can be written as

$$L = L^{(1)} + \dots L^{(N)} \text{ with } L^{(i)}\varphi = x_i(1-x_i)D_{ii}\varphi + (\alpha_i(1-x_i) - \alpha_{i+1}x_i)D_i\varphi.$$

Since the operators $L^{(i)}$ are commuting in the resolvent sense, it follows that the realization L_p of L in $L^p([0,1]^N, \mu)$ generates an analytic C_0-semigroup $T_p(\cdot)$ of contractions. Moreover L_p is the closure of the sum $L_p^{(1)} + \dots L_p^{(N)}$ defined on $D(L_p^{(1)}) \cap \dots \cap D(L_p^{(N)})$.

The aim of this paper is to give an explicit characterization of $D(L_p)$ by mean of some weighted Sobolev spaces. To get such a characterization we have to compute the domain in the one dimensional case and to apply the Dore-Venni theorem which gives us the closedness of the operator sum $L_p^{(1)} + \dots L_p^{(N)}$ defined on $D(L_p^{(1)}) \cap \dots \cap D(L_p^{(N)})$.

2 Main results

We first investigate the domain of the one-dimensional Fleming-Viot operator

$$L\varphi(x) := x(1-x)\varphi''(x) + (\alpha_1(1-x) - \alpha_2 x)\varphi'(x), \quad \varphi \in C^2([0,1]),$$

on the space $L^p(\mu_1) := L^p(0,1;\mu_1)$ for $1 < p < \infty$, where

$$d\mu_1(x) := \beta_1 x^{\alpha_1 - 1}(1-x)^{\alpha_2 - 1}dx.$$

To this purpose let us introduce the weighted Sobolev spaces

$$W_c^{1,p}(\mu_1) \text{ and } W_c^{2,p}(\mu_1)$$

as the completion of $C^1[0,1]$ and $C^2[0,1]$ respectively with respect to the norm

$$\|\varphi\|^p_{W_c^{1,p}(\mu_1)} := \|\varphi\|^p_{L^p(\mu_1)} + \left\|\sqrt{c}\varphi'\right\|^p_{L^p(\mu_1)} \quad \text{and}$$
$$\|\varphi\|^p_{W_c^{2,p}(\mu_1)} := \|\varphi\|^p_{W_c^{1,p}(\mu_1)} + \left\|c\varphi''\right\|^p_{L^p(\mu_1)},$$

where $c(x) := x(1-x)$, $x \in [0,1]$, $1 < p < \infty$.

Lemma 1. *If $\varphi \in W_c^{1,p}(\mu_1)$, $1 < p < \infty$, then there is a constant $M = M(p, \alpha_1, \alpha_2) > 0$ such that*

$$\|(\alpha_1(1-x) - \alpha_2 x)\varphi\|^p_{L^p(\mu_1)} \le M(\left\|\sqrt{c}\varphi\right\|^p_{L^p(\mu_1)} + \left\|c\varphi'\right\|^p_{L^p(\mu_1)}). \tag{1}$$

Hence, if $\varphi \in W_c^{2,p}(\mu_1)$, $1 < p < \infty$, then

$$\left\|(\alpha_1(1-x) - \alpha_2 x)\varphi'\right\|_{L^p(\mu_1)} \le M \|\varphi\|_{W_c^{2,p}(\mu_1)}. \tag{2}$$

Proof. To prove (1) it suffices to consider $\varphi \in C^1[0,1]$. Then,

$$\begin{aligned}
&\int_0^1 |(\alpha_1(1-x) - \alpha_2 x)\varphi(x)|^p d\mu_1(x) \\
&= \beta_1 \int_0^1 |(\alpha_1(1-x) - \alpha_2 x)|^{p-1} sign(\alpha_1(1-x) - \alpha_2 x)|\varphi(x)|^p \frac{d}{dx}(x^{\alpha_1}(1-x)^{\alpha_2})dx \\
&= -\beta_1 \int_0^1 \frac{d}{dx}[|(\alpha_1(1-x) - \alpha_2 x)|^{p-1} sign(\alpha_1(1-x) - \alpha_2 x)|\varphi(x)|^p] x^{\alpha_1}(1-x)^{\alpha_2}dx \\
&= (p-1)(\alpha_1+\alpha_2) \int_0^1 |(\alpha_1(1-x) - \alpha_2 x)|^{p-2} sign(\alpha_1(1-x) - \alpha_2 x)|\varphi(x)|^p c(x) d\mu_1 \\
&- p \int_0^1 |(\alpha_1(1-x) - \alpha_2 x)|^{p-1} sign((\alpha_1(1-x) - \alpha_2 x)\varphi(x))\varphi'(x)|\varphi(x)|^{p-1} c(x) d\mu_1 \\
&=: (p-1)(\alpha_1+\alpha_2) I_1 - p I_2.
\end{aligned}$$

Step 1: $2 \le p < \infty$.
Applying Hölder and Young inequalities we get

$$\begin{aligned}
|I_1| &\le \left(\int_0^1 |(\alpha_1(1-x) - \alpha_2 x)\varphi(x)|^p d\mu_1(x)\right)^{\frac{p-2}{p}} \left(\int_0^1 |\varphi(x)|^p c(x)^{\frac{p}{2}} d\mu_1(x)\right)^{\frac{2}{p}} \\
&\le \varepsilon^{\frac{p}{p-2}} \frac{p-2}{p} \int_0^1 |(\alpha_1(1-x) - \alpha_2 x)\varphi(x)|^p d\mu_1(x) + \frac{2}{p\varepsilon^{\frac{p}{2}}} \int_0^1 |\varphi(x)|^p c(x)^{\frac{p}{2}} d\mu_1(x)
\end{aligned}$$

and

$$|I_2| \le \Big(\int_0^1 |(\alpha_1(1-x)-\alpha_2 x)\varphi(x)|^p d\mu_1(x)\Big)^{\frac{p-1}{p}} \Big(\int_0^1 |c(x)\varphi'(x)|^p d\mu_1(x)\Big)^{\frac{1}{p}}$$
$$\le \varepsilon^{\frac{p}{p-1}} \frac{p-1}{p} \int_0^1 |(\alpha_1(1-x)-\alpha_2 x)\varphi(x)|^p d\mu_1(x) + \frac{1}{p\varepsilon^p}\int_0^1 |c(x)\varphi'(x)|^p d\mu_1(x)$$

for any $\varepsilon > 0$. Hence,

$$\begin{aligned}
&\int_0^1 |(\alpha_1(1-x)-\alpha_2 x)\varphi(x)|^p d\mu_1(x)\\
&\le (p-1)(\alpha_1+\alpha_2)\varepsilon^{\frac{p}{p-2}} \frac{p-2}{p}\int_0^1 |(\alpha_1(1-x)-\alpha_2 x)\varphi(x)|^p d\mu_1(x)\\
&\quad + \frac{2(p-1)(\alpha_1+\alpha_2)}{p\varepsilon^{\frac{p}{2}}}\int_0^1 |\varphi(x)|^p c(x)^{\frac{p}{2}} d\mu_1(x)\\
&\quad + (p-1)\varepsilon^{\frac{p}{p-1}}\int_0^1 |(\alpha_1(1-x)-\alpha_2 x)\varphi(x)|^p d\mu_1(x)\\
&\quad + \varepsilon^{-p}\int_0^1 |c(x)\varphi'(x)|^p d\mu_1(x).
\end{aligned}$$

Thus,

$$\Big[1-(p-1)(\alpha+\alpha_2)\varepsilon^{\frac{p}{p-2}}\frac{p-2}{p} - (p-1)\varepsilon^{\frac{p}{p-1}}\Big]\int_0^1 |(\alpha_1(1-x)-\alpha_2 x)\varphi(x)|^p d\mu_1$$
$$\le \frac{2(p-1)(\alpha_1+\alpha_2)}{p\varepsilon^{\frac{p}{2}}}\int_0^1 |\varphi(x)|^p c(x)^{\frac{p}{2}} d\mu_1(x) + \varepsilon^{-p}\int_0^1 |c(x)\varphi'(x)|^p d\mu_1(x).$$

So, one gets (1) by taking a sufficiently small ε and (2) follows from (1).
Step 2: $1 < p < 2$.
We have only to estimate

$$\int_0^1 |(\alpha_1(1-x)-\alpha_2 x)|^{p-2}|\varphi(x)|^p c(x) d\mu_1(x).$$

Set $\gamma := \frac{\alpha_1}{\alpha_1+\alpha_2}$ and consider $\varepsilon < \min(\gamma, 1-\gamma)$. Then

$$\begin{aligned}
&\int_0^1 |(\alpha_1(1-x)-\alpha_2 x)|^{p-2}|\varphi(x)|^p c(x) d\mu_1(x)\\
&= \int_0^{\gamma-\varepsilon} |(\alpha_1(1-x)-\alpha_2 x)|^{p-2}|\varphi(x)|^p c(x)^{p/2} c(x)^{1-p/2} d\mu_1(x)\\
&+ \int_{\gamma+\varepsilon}^1 |(\alpha_1(1-x)-\alpha_2 x)|^{p-2}|\varphi(x)|^p c(x)^{p/2} c(x)^{1-p/2} d\mu_1(x)\\
&+ \int_{\gamma-\varepsilon}^{\gamma+\varepsilon} |(\alpha_1(1-x)-\alpha_2 x)|^{p-2}|\varphi(x)|^p c(x) d\mu_1(x)\\
&\le C_1\int_0^1 |\varphi(x)|^p c(x)^{p/2} d\mu_1(x) \int_{\gamma-\varepsilon}^{\gamma+\varepsilon} |(\alpha_1(1-x)-\alpha_2 x)|^{p-2}|\varphi(x)|^p c(x) d\mu_1(x).
\end{aligned}$$

Using the Sobolev embedding $W^{1,p}(\gamma-\varepsilon,\gamma+\varepsilon) \hookrightarrow L^\infty(\gamma-\varepsilon,\gamma+\varepsilon)$ we get

$$\begin{aligned}
&\int_{\gamma-\varepsilon}^{\gamma+\varepsilon} |(\alpha_1(1-x)-\alpha_2 x)|^{p-2}|\varphi(x)|^p c(x)d\mu_1(x)\\
&\quad\le C_2\int_{\gamma-\varepsilon}^{\gamma+\varepsilon} |(\alpha_1(1-x)-\alpha_2 x)|^{p-2}|\varphi(x)|^p\,dx\\
&\quad\le C_2\left(\sup_{|x-\gamma|<\varepsilon}|\varphi(x)|\right)^p\int_{\gamma-\varepsilon}^{\gamma+\varepsilon} |(\alpha_1(1-x)-\alpha_2 x)|^{p-2}dx\\
&\quad\le C_3\left(\sup_{|x-\gamma|<\varepsilon}|\varphi(x)|\right)^p\\
&\quad\le C_3\left(\int_{\gamma-\varepsilon}^{\gamma+\varepsilon}|\varphi(x)|^p\,dx+\int_{\gamma-\varepsilon}^{\gamma+\varepsilon}|\varphi'(x)|^p\,dx\right)\\
&\quad\le C_4\left(\int_{\gamma-\varepsilon}^{\gamma+\varepsilon}|\varphi(x)|^p c(x)^{p/2}d\mu_1(x)+\int_{\gamma-\varepsilon}^{\gamma+\varepsilon}|\varphi'(x)|^p c(x)^p d\mu_1(x)\right)\\
&\quad\le C_4\left(\int_0^1|\varphi(x)|^p c(x)^{p/2}d\mu_1(x)+\int_0^1|\varphi'(x)|^p c(x)^p d\mu_1(x)\right),
\end{aligned}$$

since the functions $c(\cdot)^{-1}$ and $x\mapsto x^{1-\alpha_1}(1-x)^{1-\alpha_2}$ are bounded on $[\gamma-\varepsilon,\gamma+\varepsilon]$. Thus, there is a constant $M>0$ such that

$$\int_0^1 |(\alpha_1(1-x)-\alpha_2 x)|^{p-2}|\varphi(x)|^p c(x)d\mu_1(x) \le M\left(\left\|\sqrt{c}\varphi\right\|^p_{L^p(\mu_1)}+\left\|c\varphi'\right\|^p_{L^p(\mu_1)}\right). \quad (3)$$

Now, (1) for the case $1<p<2$ follows from the estimate of $|I_2|$ and (3). QED

As a consequence we obtain a first characterization of the domain of the realization L_p of L in $L^p(\mu_1)$.

Proposition 1. *For $1<p<\infty$ the realization L_p of L in $L^p(\mu_1)$ is the closure of the differential operator L defined in $W^{2,p}_c(\mu_1)$.*

Proof. It is known that L_p is the closure of L defined on $C^2([0,1])$ (see [1, Theorem 4.3]). So, let $\varphi\in W^{2,p}_c(\mu_1)$ and $(\varphi_n)\subset C^2([0,1])$ converges to φ in the norm of $W^{2,p}_c(\mu_1)$. Then $\varphi_n\in D(L_p)$ and using (2) one obtains that $L_p\varphi_n=L\varphi_n$ converges to $L\varphi$. Since L_p is closed, it follows that $\varphi\in D(L_p)$ and $L_p\varphi=L\varphi$. QED

Our purpose is now to prove that the operator L with domain $W^{2,p}_c(\mu_1)$ is closed in $L^p(\mu_1)$. We start with the following lemma.

Lemma 2. *If $\varphi\in D(L_p)$ and $1\le p<\infty$ then $\varphi\in W^{1,p}_c(\mu_1)$ and the following holds*

$$\|\varphi\|_{W^{1,p}_c(\mu_1)}\le M(\|L_p\varphi\|_{L^p(\mu_1)}+\|\varphi\|_{L^p(\mu_1)}) \quad (4)$$

for some constant $M>0$.

Proof. Take $\varphi\in C^2([0,1])$ and set $f:=\varphi-L\varphi$ and $\psi:=\varphi'$. Then,

$$f(y)-\varphi(y)=-y(1-y)\psi'(y)-(\alpha_1(1-y)-\alpha_2 y)\psi(y),\quad y\in[0,1].$$

Integrating we get (assuming for simplicity $\beta_1=1$)

$$x^{\alpha_1}(1-x)^{\alpha_2}\psi(x)=\int_x^1(f(y)-\varphi(y))d\mu_1(y) \quad (5)$$

and

$$x^{\alpha_1}(1-x)^{\alpha_2}\psi(x) = -\int_0^x (f(y)-\varphi(y))d\mu_1(y). \tag{6}$$

Setting $v(x) := x^{\frac{\alpha_1-1}{p}}(1-x)^{\frac{\alpha_2-1}{p}}\psi(x)$ and $g(x) := x^{\frac{\alpha_1-1}{p}}(1-x)^{\frac{\alpha_2-1}{p}}(f(x)-\varphi(x))$, we obtain, by (5) and (6) respectively,

$$v(x)\sqrt{c(x)} = \int_x^1 g(y)\left(\frac{y}{x}\right)^{\frac{\alpha_1-1}{p'}}\left(\frac{1-y}{1-x}\right)^{\frac{\alpha_2-1}{p'}}\frac{1}{\sqrt{c(x)}}dy \tag{7}$$

and

$$v(x)\sqrt{c(x)} = -\int_0^x g(y)\left(\frac{y}{x}\right)^{\frac{\alpha_1-1}{p'}}\left(\frac{1-y}{1-x}\right)^{\frac{\alpha_2-1}{p'}}\frac{1}{\sqrt{c(x)}}dy, \tag{8}$$

where $\frac{1}{p}+\frac{1}{p'}=1$. Applying first (7) and Hölder's inequality we deduce

$$\begin{aligned}
\int_{\frac{1}{2}}^1 |\sqrt{c(x)}v(x)|^p\,dx &= \int_{\frac{1}{2}}^1\left|\int_x^1 g(y)\left(\frac{y}{x}\right)^{\frac{\alpha_1-1}{p'}}\left(\frac{1-y}{1-x}\right)^{\frac{\alpha_2-1}{p'}}\frac{1}{\sqrt{c(x)}}dy\right|^p dx\\
&\leq \int_{\frac{1}{2}}^1\left(\int_x^1\frac{1}{\sqrt{c(x)}}|g(y)|^p\,dy\right)\left(\int_x^1\frac{1}{\sqrt{c(x)}}\left(\frac{y}{x}\right)^{\alpha_1-1}\left(\frac{1-y}{1-x}\right)^{\alpha_2-1}dy\right)^{p-1}dx\\
&\leq M_1\int_{\frac{1}{2}}^1\int_x^1\frac{1}{\sqrt{c(x)}}|g(y)|^p\,dy\,dx\\
&= M_1\int_{\frac{1}{2}}^1|g(y)|^p\left(\int_{\frac{1}{2}}^y\frac{1}{\sqrt{c(x)}}dx\right)dy \leq M\int_{\frac{1}{2}}^1|g(y)|^p\,dy,
\end{aligned}$$

since

$$\int_x^1\frac{(y/x)^{\alpha_1-1}}{\sqrt{c(x)}}\left(\frac{1-y}{1-x}\right)^{\alpha_2-1}dy = x^{\frac{1}{2}-\alpha_1}\sqrt{1-x}\int_0^1(1-t(1-x))^{\alpha_1-1}t^{\alpha_2-1}dt \leq M_1$$

for any $x\in[\frac{1}{2},1]$.
Now, using (8) and by the same arguments we have

$$\int_0^{\frac{1}{2}}|\sqrt{c(x)}v(x)|^p\,dx \leq M\int_0^{\frac{1}{2}}|g(y)|^p\,dy.$$

Therefore,

$$\left\|\sqrt{c}\varphi'\right\|^p_{L^p(\mu_1)} \leq M\left\|f-\varphi\right\|^p_{L^p(\mu_1)}.$$

So, by Proposition 1, the above estimate holds for any $\varphi\in D(L_p)$ and this ends the proof of the lemma. QED

The first main result of this section is the following characterization of the domain of the operator L_p in dimension one.

Theorem 1. *The operator* L_p *defined by*

$$L_p\varphi = x(1-x)\varphi'' + (\alpha_1(1-x)-\alpha_2 x)\varphi'$$

with domain

$$D(L_p) = W^{2,p}_c(\mu_1)$$

generates an analytic C_0*-semigroup on* $L^p(\mu_1)$ *for all* $1<p<\infty$.

Proof. By (2) we know that

$$\|L_p\varphi\|_{L^p(\mu_1)} \le M_1 \|\varphi\|_{W_c^{2,p}(\mu_1)}.$$

Hence it suffices to prove

$$\|\varphi\|_{W_c^{2,p}(\mu_1)} \le M_2(\|L_p\varphi\|_{L^p(\mu_1)} + \|\varphi\|_{L^p(\mu_1)}). \tag{9}$$

To this purpose let us recall the first step of the proof of Lemma 1. For $\varphi \in C^2([0,1])$ we have

$$\int_0^1 |\xi_{\alpha_1,\alpha_2}(x)\varphi'(x)|^p d\mu_1(x) = (p-1)(\alpha_1+\alpha_2)I_1 - pI_2,$$

where

$$\begin{aligned}
\xi_{\alpha_1,\alpha_2}(x) &:= \alpha_1(1-x) - \alpha_2 x, \;\; x \in [0,1],\\
I_1 &:= \int_0^1 |\xi_{\alpha_1,\alpha_2}(x)|^{p-2} sign(\xi_{\alpha_1,\alpha_2}(x))|\varphi'(x)|^p c(x) d\mu_1(x) \;\text{ and}\\
I_2 &:= \int_0^1 |\xi_{\alpha_1,\alpha_2}(x)|^{p-1} sign\left(\xi_{\alpha_1,\alpha_2}(x)\varphi'(x)\right) \varphi''(x)|\varphi'(x)|^{p-1} c(x) d\mu_1(x)\\
&= \int_0^1 |\xi_{\alpha_1,\alpha_2}(x)\varphi'(x)|^{p-1} sign\left(\xi_{\alpha_1,\alpha_2}(x)\varphi'(x)\right) L_p\varphi(x)\, d\mu_1(x) -\\
&\int_0^1 |\xi_{\alpha_1,\alpha_2}(x)\varphi'(x)|^p \, d\mu_1(x).
\end{aligned}$$

Thus,

$$\begin{aligned}
(1-p)&\int_0^1 |\xi_{\alpha_1,\alpha_2}(x)\varphi'(x)|^p \, d\mu_1(x)\\
&= (p-1)(\alpha_1+\alpha_2)I_1 -\\
&\quad p\int_0^1 |\xi_{\alpha_1,\alpha_2}(x)\varphi'(x)|^{p-1} sign\left(\xi_{\alpha_1,\alpha_2}(x)\varphi'(x)\right) L_p\varphi(x)\, d\mu_1(x).
\end{aligned}$$

So, using Hölder's and Young's inequality we deduce that

$$\begin{aligned}
\left\|\xi_{\alpha_1,\alpha_2}\varphi'\right\|^p_{L^p(\mu_1)} \le &(\alpha_1+\alpha_2)\left\|\xi_{\alpha_1,\alpha_2}\varphi'\right\|^{p-2}_{L^p(\mu_1)} \left\|\sqrt{c}\varphi'\right\|^2_{L^p(\mu_1)}\\
&+ \frac{p}{p-1}\left\|\xi_{\alpha_1,\alpha_2}\varphi'\right\|^{p-1}_{L^p(\mu_1)} \|L_p\varphi\|_{L^p(\mu_1)}
\end{aligned}$$

for $2 \le p < \infty$. Therefore,

$$\begin{aligned}
&\left\|\xi_{\alpha_1,\alpha_2}\varphi'\right\|^2_{L^p(\mu_1)} \le (\alpha_1+\alpha_2)\left\|\sqrt{c}\varphi'\right\|^2_{L^p(\mu_1)} + \frac{p}{p-1}\left\|\xi_{\alpha_1,\alpha_2}\varphi'\right\|_{L^p(\mu_1)} \|L_p\varphi\|_{L^p(\mu_1)}\\
&\le (\alpha_1+\alpha_2)\left\|\sqrt{c}\varphi'\right\|^2_{L^p(\mu_1)} + \frac{p\varepsilon}{2(p-1)}\left\|\xi_{\alpha_1,\alpha_2}\varphi'\right\|^2_{L^p(\mu_1)} + \frac{p}{2\varepsilon(p-1)}\|L_p\varphi\|^2_{L^p(\mu_1)}
\end{aligned}$$

for $2 \le p < \infty$ and any $\varepsilon > 0$. Hence, using Lemma 2 and taking ε sufficiently small we obtain (9) for $2 \le p < \infty$.

Let us now consider the case $1 < p < 2$.

Take $\varphi \in C^2([0,1])$ and set $f := \varphi - L\varphi$, $v(x) := x^{\frac{\alpha_1-1}{p}}(1-x)^{\frac{\alpha_2-1}{p}}\varphi'(x)$ and $g(x) := x^{\frac{\alpha_1-1}{p}}(1-x)^{\frac{\alpha_2-1}{p}}(f(x)-\varphi(x))$. Using (7) and (8) we get

$$v(x)\xi_{\alpha_1,\alpha_2}(x) = \int_x^1 g(y)\left(\frac{y}{x}\right)^{\frac{\alpha_1-1}{p'}}\left(\frac{1-y}{1-x}\right)^{\frac{\alpha_2-1}{p'}}\left(\frac{\alpha_1}{x} - \frac{\alpha_2}{1-x}\right) dy \tag{10}$$

and

$$v(x)\xi_{\alpha_1,\alpha_2}(x) = -\int_0^x g(y)\left(\frac{y}{x}\right)^{\frac{\alpha_1-1}{p'}}\left(\frac{1-y}{1-x}\right)^{\frac{\alpha_2-1}{p'}}\left(\frac{\alpha_1}{x}-\frac{\alpha_2}{1-x}\right)dy. \tag{11}$$

By the same arguments as in the proof of Lemma 2 we have, applying (10) and Hölder's inequality

$$\begin{aligned}
&\int_{\frac{1}{2}}^1 |\xi_{\alpha_1,\alpha_2}(x)v(x)|^p\,dx\\
&\quad=\int_{\frac{1}{2}}^1\left|\int_x^1 g(y)\left(\frac{y}{x}\right)^{\frac{\alpha_1-1}{p'}}\left(\frac{1-y}{1-x}\right)^{\frac{\alpha_2-1}{p'}}\left(\frac{\alpha_1}{x}-\frac{\alpha_2}{1-x}\right)dy\right|^p dx\\
&\quad\le\int_{\frac{1}{2}}^1\left(\int_x^1\left|\frac{\alpha_1}{x}-\frac{\alpha_2}{1-x}\right|^{p-1}|g(y)|^p\,dy\right)\cdot\\
&\qquad\cdot\left(\int_x^1\left|\frac{\alpha_1}{x}-\frac{\alpha_2}{1-x}\right|\left(\frac{y}{x}\right)^{\alpha_1-1}\left(\frac{1-y}{1-x}\right)^{\alpha_2-1}dy\right)^{p-1}dx\\
&\quad\le M_1\int_{\frac{1}{2}}^1\int_x^1\left|\frac{\alpha_1}{x}-\frac{\alpha_2}{1-x}\right|^{p-1}|g(y)|^p\,dy\,dx\\
&\quad=M_1\int_{\frac{1}{2}}^1|g(y)|^p\left(\int_{\frac{1}{2}}^y\left|\frac{\alpha_1}{x}-\frac{\alpha_2}{1-x}\right|^{p-1}dx\right)dy\le M\int_{\frac{1}{2}}^1|g(y)|^p\,dy,
\end{aligned}$$

since $2-p>0$ and

$$\begin{aligned}
&\int_x^1\left|\frac{\alpha_1}{x}-\frac{\alpha_2}{1-x}\right|\left(\frac{y}{x}\right)^{\alpha_1-1}\left(\frac{1-y}{1-x}\right)^{\alpha_2-1}dy\\
&\quad=x^{1-\alpha_1}\left|\frac{\alpha_1(1-x)}{x}-\alpha_2\right|\int_0^1(1-t(1-x))^{\alpha_1-1}t^{\alpha_2-1}\,dt\le M_1
\end{aligned}$$

for any $x\in[\frac{1}{2},1]$. We repeat the same argument and use (11), we obtain

$$\int_0^{\frac{1}{2}}|\xi_{\alpha_1,\alpha_2}(x)v(x)|^p\,dx\le M\int_0^{\frac{1}{2}}|g(y)|^p\,dy.$$

Thus,

$$\left\|\xi_{\alpha_1,\alpha_2}\varphi'\right\|_{L^p(\mu_1)}\le M\left\|L\varphi\right\|_{L^p(\mu_1)}.$$

This and Lemma 2 imply (9). QED

We now treat the N-dimensional case. To this purpose let us denote by C the diagonal matrix

$$C(x):=\operatorname{diag}(c(x_1),\dots,c(x_N)),\quad x=(x_1,\dots,x_N)\in[0,1]^N,$$

and consider the N-dimensional weighted Sobolev spaces

$$W_C^{k,p}(\mu):=\bigcap_{i=1}^N W_{c_i}^{k,p}(\mu),\quad k=1,\,2,$$

endowed respectively with the norm

$$\|\varphi\|^p_{W^{1,p}_C(\mu)} := \|\varphi\|^p_{L^p(\mu)} + \sum_{i=1}^N \|\sqrt{c_i} D_i\varphi\|^p_{L^p(\mu)} \,,$$
$$\|\varphi\|^p_{W^{2,p}_C(\mu)} := \|\varphi\|^p_{W^{1,p}_C(\mu)} + \sum_{i=1}^N \|c_i D_{ii}\varphi\|^p_{L^p(\mu)} \,,$$

where $c_i(x) := x_i(1 - x_i)$ and $L^p(\mu) := L^p([0, 1]^N, \mu(dx))$.

We come now to the main result of this paper.

Theorem 2. *Let $1 < p < \infty$. Then the realization L_p of L in $L^p(\mu)$ with domain $W^{2,p}_C(\mu)$ generates a C_0-semigroup of contractions which is positive and analytic.*

Proof. By Theorem 1 we know that the operator $L^{(i)}_p := (L^{(i)}, W^{2,p}_{c_i}(\mu))$ generates a positive C_0-semigroup of contractions on $L^p(\mu)$ which is analytic. Thanks to the transference principle [4, Section 4] (see [3, Theorem 5.8]) the operator $I - L^{(i)}_p$ admits bounded imaginary powers on $L^p(\mu)$ with power angle

$$\theta(L^{(i)}_p) := \lim_{|s|\to\infty} \frac{1}{|s|} \log \left\| (I - L^{(i)}_p)^{is} \right\| \le \frac{\pi}{2}.$$

Moreover, $L^{(i)}_2$ is self adjoint on $L^2(\mu)$ and thus has power angle 0 on $L^2(\mu)$. So, by the Riesz-Thorin interpolation theorem, we get

$$\theta(L^{(i)}_p) < \frac{\pi}{2}.$$

Therefore one can apply the Dore-Venni theorem [7] in the version of [12, Corollary 4], since the resolvents of $L^{(i)}_p$ commute. Thus, $L^{(1)}_p + \ldots + L^{(N)}_p$ is closed on the intersection of $D(L^{(i)}_p)$, $1 \le i \le N$. Hence, $D(L_p) = W^{2,p}_C(\mu)$ and the theorem is proved. QED

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