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About Poincaré Inequalities for Functions Lacking Summability

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Abstract. We present in a unified context several Poincaré type inequalities involving median: the list includes also inequalities which hold true for functions lacking summability properties.

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Dedicated to our colleague and friend
Vincenzo Bruno Moscatelli

1 Introduction

Several kinds of L^p estimates of a function u in terms of L^q norms of its distributional derivatives are named Poincaré inequalities ([20], [26]). These inequalities are very useful in both existence and regularity theory for variational problems in Sobolev spaces and BV spaces. Our objective is to give an overview of analogous estimates in functional settings whose elements lack summability. Such inequalities take the form of L^p estimates of suitable truncations of u in terms of L^q norm of the absolutely continuous part of the derivatives.
These finer inequalities can be applied to several free discontinuity problems, mainly in image segmentation ([2], [19], [10], [11], [14], [16]) and in continuum mechanics ([8], [25]).
In Section 2 we introduce the notation.
In Section 3 we recall some classical results.

The main results related to Poincaré inequalities in *SBV*, *GSBV* and $GSBV^2$ (Theorems 7, 10, 11) are given in Sections 4 and 5, together with some comments on their consequences.

2 Notation

Given a set $E \subseteq \mathbb{R}^n$, we denote its k-dimensional Hausdorff measure by $\mathcal{H}^k(E)$ $(0 \le k \le n)$, its Lebesgue outer measure by $|E|$, its topological closure by $\overline{E}$ and its topological boundary by ∂E. We denote the ball $\{y \in \mathbb{R}^n; |y-x| < \varrho\}$ by $B_\varrho(x)$, and we set $B_\varrho = B_\varrho(0)$, $\omega_n = |B_1|$.

Definition 1. For a given measurable set E in $\mathbb{R}^n$ with $|E| > 0$ and u in $L^1(E)$, we define the *mean value* u_E of u as follows:

$$u_E := \frac{1}{|E|} \int_E u \, dx \, .$$

Definition 2. For a given measurable set E in $\mathbb{R}^n$ with $|E| > 0$ and u a.e. finite measurable real function in E, we say that a real number $m = m(u,E)$ is a *median* of u in E (here we do not require $u \in L^1(E)$) if

$$|\{u < m\} \cap E| \le \frac{1}{2}|E| \, , \qquad |\{u > m\} \cap E| \le \frac{1}{2}|E| \, .$$

We denote the set of medians of u in E by $M = M(u,E)$.

Existence of medians follows by a simple continuity argument. Evaluating medians $m(\cdot, E)$ is a multivalued non linear operator which has no simple relationship with the mean value u_E. Anyway the medians fulfil the following properties:

- The set of medians $M = M(u,E)$ is always a non empty compact interval.
- We set $m_* = m_*(u,E) =^{def} \min\{m \in M(u,E)\}$, $m^* = m^*(u,E) =^{def} \max\{m \in M(u,E)\}$ and we call respectively m_*, m^*, the least and the greatest median of u in E. Notice that m_* has the same value of med (Sect.4 of [10]), where:

$$m_* = \inf\{\, t \in \mathbb{R};\ |\{u < t\} \cap E| \ge \frac{1}{2}|E|\}.$$

- $|\{x \in E \, : \, m_* < u(x) < m^*\}| = 0$
- For any $\varepsilon > 0$, any increasing map $\varphi : \mathbb{R} \to \mathbb{R}$, which is strictly increasing in $[m_* - \varepsilon, m^* + \varepsilon]$, we have

$$\varphi(M(u,E)) = M(\varphi(u), E) \, . \tag{1}$$

- For any measurable set $F \subset E$ and any median $m(u,E)$ we have

$$M\big(u\chi_{E \setminus F} + m(u,E)\chi_F, E\big) = M(u,E) \, . \tag{2}$$

- $|m(u,E)| \le \frac{2}{|E|} \int_E |u| \, dx \quad \forall m(u,B)$ (see [26], (5.12.10)).

In some case median, least median and greatest median prove more useful than mean value, because (in contrast with the mean value u_E) they are defined even if u does not belong to $L^1(E)$ and they commute also with monotone nonlinear maps, as stated in (1): a choice of φ may be the usual truncation operator, provided the truncation operates on value outside the interval $[m_*, m^*]$. Useful examples of commutative truncation operators are given by the choices $\varphi(u) = \overline{u}$ and $\varphi(u) = T(u, a, \eta)$ referring to (13) and (21) (see Theorems 7 and 10).
In this Section Ω denotes a non empty open set in $\mathbb{R}^n$. We denote the Sobolev space of functions

$u \in L^p(\Omega)$ such that $Du \in L^p(\Omega;\mathbb{R}^n)$ by $W^{1,p}(\Omega)$ $(p \geq 1)$.
For every $u \in L^1_{\mathrm{loc}}(\Omega)$ we define the total variation of u as follows:

$$\int_\Omega |Du| = \sup\left\{\int_\Omega u \operatorname{div}\phi\, dx;\ \phi \in C^1_0(\Omega;\mathbb{R}^n), |\phi(x)| \leq 1\right\}.$$

We denote by $BV(\Omega)$ the Banach space of all functions u of $L^1(\Omega)$ with $\int_\Omega |Du| < +\infty$; by $BV_{\mathrm{loc}}(\Omega)$ we denote the space of functions which belong to $BV(\Omega')$ for every open set $\Omega' \subset\subset \Omega$ (*i.e.* $\overline{\Omega}'$ is compact and $\overline{\Omega}' \subset \Omega$).
If $E \subseteq \mathbb{R}^n$ is a Borel set, we define the perimeter of E in Ω as $P(E,\Omega) = \int_\Omega |D\chi_E|$ where χ_E is the characteristic function of E.
For the main properties of functions with bounded variation we refer *e.g.* to [2], [20], [21], [22], [26], in particular we recall the following statements.

Theorem 1. *(see [6], 18.1.3) For any ball $B \subset \mathbb{R}^n$, $n \geq 2$ we label*

$$\gamma_n = \frac{1}{\omega_{n-1}}\left(\frac{\omega_n}{2}\right)^{(n-1)/n} \tag{3}$$

the relative isoperimetric constant in a ball of $\mathbb{R}^n$: then for every Borel set E

$$\min\left\{|E \cap B|^{(n-1)/n},\ |B \setminus E|^{(n-1)/n}\right\} \leq \gamma_n P(E,B). \tag{4}$$

Theorem 2. *For any $u \in BV(\Omega)$ the following coarea formula holds true (see e.g. [26], 5.4.4):*

$$\int_\Omega |Du| = \int_{-\infty}^{+\infty} P(\{u > t\},\Omega)\, dt\,. \tag{5}$$

3 Classical Poincaré Inequality for Sobolev and Bounded Variation Functions

In this Section we list the Poincaré inequalities for a function up to the correction with its integral mean value or median, when the function belongs to a Sobolev or BV space.

Theorem 3. *Assume B is a ball in $\mathbb{R}^n$, $n \geq 2$, $1 \leq p < n$, $p^* = np/(n-p)$ and $u \in W^{1,p}(B)$. Then there exists a constant $C = C(n,p)$ such that:*

$$\left(\int_B |u - u_B|^{p^*}\, dx\right)^{1/p^*} \leq C\left(\int_B |Du|^p\, dx\right)^{1/p}.$$

Theorem 4. *Assume B is a ball in $\mathbb{R}^n$, $n \geq 2$, $u \in W^{1,p}(B)$, $m \in M(u,B)$. If in addition $1 \leq p < n$, and $p^* = np/(n-p)$, then*

$$\left(\int_B |u - m|^{p^*}, dx\right)^{1/p^*} \leq \gamma_n \frac{p(n-1)}{n-p}\left(\int_B |Du|^p dx\right)^{1/p}$$

Otherwise, if $p \geq n$, then

$$\left(\int_B |u - m|^q dx\right)^{1/q} \leq \gamma_n \frac{q(n-1)}{n}|B|^{\frac{1}{n}+\frac{1}{q}-\frac{1}{p}}\left(\int_B |Du|^p\, dx\right)^{1/p} \qquad \forall q \geq 1\,.$$

Theorem 5. *Assume B is a ball in $\mathbb{R}^n$, $n \geq 2$, $1^* = n/(n-1)$ and u belongs to $BV(B)$. Then there exists a constant $k = k(n)$ such that:*

$$\left(\int_B |u - u_B|^{1^*} dx\right)^{1/1^*} \leq k \int_B |Du| .$$

Theorem 6. *Assume B is a ball in $\mathbb{R}^n$, $n \geq 2$, $1^* = n/(n-1)$ and u belongs to $BV(B)$. Then*

$$\left(\int_B \mid u - m \mid^{1^*} dx\right)^{1/1^*} \leq \gamma_n \int_B |Du| \qquad \forall m \in M(u, B) ,$$

where the constant γ_n is optimal.

Proof. *(Theorem 6)* For any $r > 0$ and any measurable set E, we have

$$\{ \operatorname{sign}(m)|m|^r \, ; \, m \in M(u, E) \} = M(\operatorname{sign}(u) \, |u|^r , \, E).$$

Up to addition of a constant, it is not restrictive to assume $m = 0$. Then, by Lemma 3.48 of [2], isoperimetric inequality and coarea formula (see (3), (4), (5)), we get

$$\int_B |u|^{1^*} dx = \int_0^\infty |\{|u|^{1^*} > t\}| \, dt = \int_0^\infty |\{|u| > t^{1/1^*}\}| \, dt =$$

$$\int_0^\infty 1^* |\{|u| > s\}| \, s^{1^*-1} ds \leq \left(\int_0^\infty |\{|u| > s\}|^{1/1^*} ds\right)^{1^*} \leq$$

$$\left(\gamma_n \int_0^\infty P\left(\{|u| > s\}\right) ds\right)^{1^*} = \left(\gamma_n \int_B |Du|\right)^{1^*} .$$

Optimality of the constant follows by substituting the characteristic function of the half-ball to u in the inequality. *QED*

Proof. *(Theorem 4)* If $p = 1$, then the first inequality is a straightforward consequence of Theorem 6.
If $1 < p < n$, we may apply in a standard way Theorem 6 to the function $v = \operatorname{sign}(u) \, |u|^q$ where $q = p(n-1)/(n-p)$, say $M(v, B) = M(\operatorname{sign}(u)|u|^q, B)$.
If $q \geq n$, by the preceding case and by the Hölder inequality, we achieve the second inequality. *QED*

For the proof of Theorems 3 and 5 see Theorem 2 in Section 4.5.2 and Theorem 1.ii in Section 5.6.1 of [20].

4 Poincaré Inequality in *SBV*

We recall the main definitions and properties of functions whose derivatives are special measures in the sense of De Giorgi.
In this Section $u : \Omega \to \mathbb{R}$ always denotes a Borel function.
For any $x \in \Omega$, $z \in \widetilde{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ we set, according to [18], $z = \operatorname{aplim}_{y \to x} u(y)$, (the approximate limit of u at x, denoted also by $\widetilde{u}(x)$) if

$$g(z) = \lim_{\varrho \to 0} \frac{1}{|B_\varrho|} \int_{B_\varrho} g(u(x+y)) \, dy$$

for every $g \in C^0(\widetilde{\mathbb{R}})$; if $z \in \mathbb{R}$ this definition is equivalent to 2.9.12 in [21].

The set

$$S_u = \left\{ x \in \Omega;\ \operatorname*{aplim}_{y \to x} u(y) \text{ does not exist} \right\}$$

is a Borel set of negligible Lebesgue measure. Let $x \in \Omega \setminus S_u$ be such that $\widetilde{u}(x) \in \mathbb{R}$; we say that u is approximately differentiable at x if there exists a vector $\nabla u(x) \in \mathbb{R}^n$ (the approximate gradient of u at x) such that

$$\operatorname*{aplim}_{y \to x} \frac{|u(y) - \widetilde{u}(x) - \nabla u(x) \cdot (y - x)|}{|y - x|} = 0.$$

Here we recall only that for every $u \in BV(\Omega)$ the following properties hold:
S_u is countably $\mathcal{H}^{n-1}$-rectifiable, say there exist countably many Lipschitz functions $f_h : \mathbb{R}^{n-1} \to \mathbb{R}^n$ such that

$$\mathcal{H}^{n-1} \left(S_u \setminus \bigcup_{h=0}^{\infty} f_h(\mathbb{R}^{n-1}) \right) = 0\,;$$

$\mathcal{H}^{n-1}(\{x \in \Omega\,; \widetilde{u}(x) = \infty\}) = 0$; ∇u exists a.e. on Ω and coincides with a Radon-Nikodym derivative of Du with respect to the Lebesgue measure; moreover, for $\mathcal{H}^{n-1}$ almost all $x \in S_u$ there exist $\nu(x) \in \partial B_1$, $u_+(x) \in \mathbb{R}$, $u_-(x) \in \mathbb{R}$ with $u_+(x) > u_-(x)$ such that (see [21], 4.5.9(17), (22), (15))

$$\lim_{\varrho \to 0} \varrho^{-n} \int_{\{y \in B_\varrho; y \cdot \nu(x) > 0\}} |u(x+y) - u_+(x)|\, dy = 0,$$

$$\lim_{\varrho \to 0} \varrho^{-n} \int_{\{y \in B_\varrho; y \cdot \nu(x) < 0\}} |u(x+y) - u_-(x)|\, dy = 0,$$

$$\int_\Omega |Du| \geq \int_\Omega |\nabla u|\, dx + \int_{S_u \cap \Omega} (u_+ - u_-)\, d\mathcal{H}^{n-1}\,. \tag{6}$$

According to [18], we recall the definition of a class of special functions of bounded variation which are characterized by a property stronger than (6).

Definition 3. We define $SBV(\Omega)$ as the class of all functions $u \in BV(\Omega)$ such that

$$\int_\Omega |Du| = \int_\Omega |\nabla u|\, dx + \int_{S_u \cap \Omega} (u_+ - u_-)\, d\mathcal{H}^{n-1}. \tag{7}$$

By $SBV_{\mathrm{loc}}(\Omega)$ we denote the class of all functions which belong to $SBV(\Omega')$ for every open set $\Omega' \subset\subset \Omega$.

We recall that the well-known Cantor-Vitali function has bounded variation but does not satisfy (7).

Remark 1. Let $u \in BV(\Omega)$ and set $u_a = (u \wedge a) \vee (-a)$ for $0 < a < +\infty$. The following properties hold:

$$|\nabla u_a| \leq |\nabla u| \text{a.e. on } \Omega; \qquad \mathcal{H}^{n-1}\left((S_{u_a} \setminus S_u) \cap \Omega\right) = 0;$$

$$\int_{\Omega \setminus K} |Du_a| \leq \int_{\Omega \setminus K} |Du|; \qquad \int_{\Omega \setminus K} |\nabla u|\, dx = \lim_{a \to +\infty} \int_{\Omega \setminus K} |\nabla u_a| dx;$$

$$\int_{\Omega \setminus K} |Du| = \lim_{a \to +\infty} \int_{\Omega \setminus K} |Du_a|; \qquad \mathcal{H}^{n-1}(S_u \cap \Omega) = \lim_{a \to +\infty} \mathcal{H}^{n-1}(S_{u_a} \cap \Omega).$$

For any $u \in BV(\Omega)$ we have that $u \in SBV(\Omega)$ if and only if $\phi(u) \in SBV(\Omega)$ for every $\phi : \mathbb{R} \to \mathbb{R}$ uniformly Lipschitz continuous with $\phi(0) = 0$.

We point out some properties of the functions in $SBV(\Omega)$; for further results we refer to [1], [2], [18].

Lemma 1. *Let $\Omega \subseteq \mathbb{R}^n$ be open and $u \in L^\infty(\Omega) \cap L^1(\Omega)$. Let $K \subset \mathbb{R}^n$ be closed and assume*

$$u \in C^1(\Omega \setminus K), \qquad \int_{\Omega\setminus K} |\nabla u|\, dx < +\infty, \qquad \mathcal{H}^{n-1}(K \cap \Omega) < +\infty.$$

Then

$$u \in SBV(\Omega) \text{ and } S_u \cap \Omega \subseteq K.$$

We remark that, for $u \in SBV(\Omega)$ (see *e.g.* [21], 4.5.9(30)):

$$u \in W^{1,p}(\Omega) \text{ iff } \mathcal{H}^{n-1}(S_u \cap \Omega) = 0 \text{ and } \int_{\Omega\setminus K} (|\nabla u|^p + |u|^p)\, dx < \infty\,. \tag{8}$$

In order to prove a Poincaré inequality in the space $SBV(B)$, where $B \subset \mathbb{R}^n$ is an open ball, we introduce some notations.
For every measurable function $u : B \to \mathbb{R}$ we label u_* the non-decreasing rearrangement of u:

$$u_*(s, B) = \inf\{t \in \mathbb{R};\ |\{u < t\} \cap B| \geq s\} \text{ for } 0 \leq s \leq |B|. \tag{9}$$

In particular, the σ l least median of u in B $m_* = m_*(u, B)$ fulfils:

$$m_*(u, B) = u_*\left(\frac{1}{2}|B|, B\right); \tag{10}$$

moreover for every $u \in SBV(B)$ such that $(2\gamma_n \mathcal{H}^{n-1}(S_u \cap B))^{n/(n-1)} < \frac{1}{2}|B|$, where γ_n is the relative isoperimetric constant (3), we select the lower and upper levels τ', τ'' of truncation and $\overline{u}$ as follows

$$\tau' = \tau'(u,B) = u_*\Big((2\gamma_n \mathcal{H}^{n-1}(S_u \cap B))^{n/(n-1)}, B\Big), \tag{11}$$

$$\tau'' = \tau''(u,B) = u_*\Big(|B| - (2\gamma_n \mathcal{H}^{n-1}(S_u \cap B))^{n/(n-1)}, B\Big), \tag{12}$$

$$\overline{u} = \big(u \wedge \tau''(u,B)\big) \vee \tau'(u,B). \tag{13}$$

In the next theorem we show some estimates which were first proved in [19] for $m = m_*$.

Theorem 7. *Assume B is an open ball in $\mathbb{R}^n$, $n \geq 2$, $1 \leq p < n$, $p^* = np/(n-p)$, $u \in SBV(B)$ and $\mathcal{H}^{n-1}(S_u \cap B) < \dfrac{1}{2\gamma_n}(\frac{1}{2}|B|)^{(n-1)/n}$. Referring to* (9)-(13)*,we have*

$$\int_B |D\overline{u}| \leq 2 \int_B |\nabla \overline{u}|\, dx \tag{14}$$

and

$$\left(\int_B |\overline{u} - m|^{p^*}\, dx\right)^{1/p^*} \leq \frac{2\gamma_n p\,(n-1)}{n-p} \left(\int_B |\nabla u|^p\, dx\right)^{1/p} \tag{15}$$

for any median $m \in M(u, B)$.
If $p \geq n$, for every $q \geq 1$, we have

$$|\ \overline{u} - m\ |_{L^q(B)} \leq \frac{2\gamma_n q(n-1)}{n} |B|^{\frac{1}{n} + \frac{1}{q} - \frac{1}{p}}\ |\ \nabla u\ |_{L^p(B)}\ . \tag{16}$$

Proof. By assumption $\tau' \leq m_* \leq m \leq m^* \leq \tau''$. We may assume that $m = 0$ and moreover that $\mathcal{H}^{n-1}(S_u \cap B) > 0$ (because, if $\mathcal{H}^{n-1}(S_u \cap B) = 0$, then $\overline{u} = u \in W^{1,1}(B)$ and the assertion is well known). Since $\overline{u} \in SBV(B)$ we have, by (7),

$$\int_B |D\overline{u}| = \int_B |\nabla \overline{u}|\, dx + \int_{S_{\overline{u}} \cap B} (\overline{u}_+ - \overline{u}_-)\, d\mathcal{H}^{n-1},$$

and also, since $\mathcal{H}^{n-1}((S_{\overline{u}} \setminus S_u) \cap B) = 0$,

$$\int_B |D\overline{u}| \leq \int_B |\nabla \overline{u}|\, dx + \left(\tau'' - \tau'\right) \mathcal{H}^{n-1}(S_u \cap B). \tag{17}$$

By using the coarea formula (5) and the isoperimetric inequality (4), we have

$$\begin{aligned} \int_B |D\overline{u}| &= \int_{\tau'}^{\tau''} P(\{\overline{u} < t\}, B)\, dt \\ &\geq \frac{1}{\gamma_n} \int_{\tau'}^{0} |\{\overline{u} < t\} \cap B|^{(n-1)/n}\, dt + \frac{1}{\gamma_n} \int_0^{\tau''} |B \setminus \{\overline{u} < t\}|^{(n-1)/n}\, dt. \end{aligned}$$

Since for $\tau'(u, B) < t < 0$,

$$\frac{1}{\gamma_n} |\{\overline{u} < t\} \cap B|^{(n-1)/n} \geq 2\mathcal{H}^{n-1}(S_u \cap B)$$

and for $0 < t < \tau''(u, B)$,

$$\frac{1}{\gamma_n} |B \setminus \{\overline{u} < t\}|^{(n-1)/n} \geq 2\mathcal{H}^{n-1}(S_u \cap B),$$

we have

$$\int_B |D\overline{u}| \geq 2\left(\tau'' - \tau'\right) \mathcal{H}^{n-1}(S_u \cap B).$$

Hence, by comparison with (17),

$$\left(\tau''(u, B) - \tau'(u, B)\right) \mathcal{H}^{n-1}(S_u \cap B) \leq \int_B |\nabla \overline{u}|\, dx;$$

therefore, again by (17)

$$\int_B |D\overline{u}| \leq 2 \int_B |\nabla \overline{u}|\, dx,$$

which proves (14). Eventually by Theorem 6 we obtain

$$\left(\int_B |\,\overline{u}\,|^{n/(n-1)}\, dx\right)^{(n-1)/n} \leq 2\gamma_n \int_B |\,\nabla \overline{u}\,|\, dx.$$

Thus the proof is complete for $p = 1$. For $1 < p < n$, since $\overline{u}$ is bounded, we may apply in a standard way the previous inequality to the function $v = |\overline{u}|^{q-1}\overline{u}$ where $q = p(n-1)/(n-p)$. If $p \geq n$, by the preceding case and by the Hölder inequality, we achieve the thesis. QED

Remark 2. With the previous notation and under the same assumptions as in Theorem 7, we have, by the definition of $\overline{u}$ and by (14), that

$$|\{\overline{u} \neq u\} \cap B| \leq 2\left(2\gamma_n \mathcal{H}^{n-1}(S_u \cap B)\right)^{n/(n-1)}.$$

If, in addition $\nabla u \in L^p(B)$, then

$$\int_B |D\overline{u}| \leq 2 \int_B |\nabla u|\, dx \leq 2|B|^{(p-1)/p} \left(\int_B |\nabla u|^p\, dx\right)^{1/p}.$$

Theorem 7 may be extended to any class of bounded open sets Ω fulfilling an isoperimetric inequality of the following type:

$$\exists\,\gamma_\Omega > 0\ : \qquad \min\left\{|E\cap\Omega|^{(n-1)/n},\ |\Omega\setminus E|^{(n-1)/n}\right\} \le \gamma_\Omega P(E,\Omega).$$

Moreover the following statement holds true.

Corollary 1. *Assume Ω is a bounded open set with Lipschitz boundary. Then the estimates* (14)*,* (15) *and* (16) *hold true if B and γ_n are substituted by Ω and γ_Ω.*

PROOF. The proof is the same as the one given above, starting from the existence of a relative isoperimetric inequality for a Lipschitz domain (see [21], [26]). QED

We show a relevant example arising in regularization of weak minimizers, in case where blown-up sequences do not even belong to SBV. Starting from Theorem 7, we deduce a compactness and lower semicontinuity result in $SBV(B)$, which is useful in studying the regularity of weak minimizers for variational problems with free discontinuity ([18], [24]).
When dealing with sequences, in order to avoid further selections, we always choose the least median of each element in the sequence.

Theorem 8. ***(Compactness and lower semicontinuity)***
Assume $B\subset\mathbb{R}^n$ is an open ball, $(u_h)\subset SBV(B)$, $p>1$ and

$$\sup_{h\in\mathbb{N}}\int_B |\nabla u_h|^p\,dx < +\infty, \qquad \lim_h \mathcal{H}^{n-1}(S_{u_h}\cap B) = 0.$$

Then

(i) *the functions $\overline{u}_h - m_*(u_h,B)$, defined as in Theorem 7, are uniformly bounded in $BV(B)$;*

(ii) *there exist a subsequence (u_{h_i}) and a function $u_\infty\in W^{1,p}(B)$ such that*

$$\lim_i\,[u_{h_i} - m_*(u_{h_i},B)] = u_\infty \ \textit{a.e. on } B.$$

Moreover

$$\mathcal{H}^{n-1}(S_{u_\infty}\cap B) \le \liminf_i \mathcal{H}^{n-1}(S_{\overline{u}_{h_i}}\cap B) = 0,$$

and

$$\int_B |\nabla u_\infty|^p\,dx \le \liminf_i \int_B |\nabla u_{h_i}|^p\,dx < +\infty.$$

PROOF. (i) In the proof of the assertion we may assume $1<p<n$. For h large enough, by Theorem 7, we have

$$\int_B |\overline{u}_h - m_*(u_h,B)|^{p^*}\,dx \le \text{const}\,.; \tag{18}$$

moreover, by Remark 2,

$$\int_B |D\overline{u}_h| \le \text{const}\,.$$

Therefore the functions $\overline{u}_h - m_*(u_h,B)$ are uniformly bounded in $BV(B)$.

(ii) From (18) in the case $1<p<n$ or from (16) in the case $p\ge n$, by virtue of the compactness theorem in $BV(B)$ (see *e.g.* [22], Theorem 1.19) there exist a subsequence (u_{h_i}) and $u_\infty\in BV(B)$ such that

$$\lim_i\,[\overline{u}_{h_i} - m_*(u_{h_i},B)] = u_\infty \tag{19}$$

in $L^r(B)$ for every $1\le r< np/(n-p)$ if $1<p<n$, and in $L^r(B)$ for every $r\ge 1$ if $p\ge n$. Now, by the semicontinuity results in [1], u_∞ belongs to $SBV(B)$; moreover

$$\mathcal{H}^{n-1}(S_{u_\infty}\cap B) \le \liminf_i \mathcal{H}^{n-1}(S_{\overline{u}_{h_i}}\cap B) = 0,$$

and

$$\int_B |\nabla u_\infty|^p \, dx \leq \liminf_i \int_B |\nabla u_{h_i}|^p \, dx < +\infty,$$

thus, by (8), u_∞ is in $W^{1,p}(B)$. Finally, since from Remark 2

$$|\{\overline{u}_{h_i} \neq u_{h_i}\} \cap B| \leq 2 \left(2\gamma_n \mathcal{H}^{n-1}(S_{u_{h_i}} \cap B)\right)^{n/(n-1)},$$

from (19) it follows that, possibly by restriction again to a subsequence,

$$\lim_i \left[u_{h_i} - m_*(u_{h_i}, B)\right] = u_\infty \text{ a.e. on } B.$$

QED

A significant application of the above theory is the regularization of SBV minimizers for the Mumford-Shah functional, arising in image segmentation. The following result has been proved in [19].

Theorem 9. *Let $n \in \mathbb{N}$, $n \geq 2$, let $\Omega \subseteq \mathbb{R}^n$ be an open set, $q \geq 1$, $\alpha > 0$, $\mu > 0$, $g \in L^q(\Omega) \cap L^\infty(\Omega)$; then there exists at least one pair (K, u) minimizing the functional G defined for every closed set $K \subset \mathbb{R}^n$ and for every $u \in C^1(\Omega \setminus K)$ by*

$$G(K,u) := \int_{\Omega\setminus K} |Du|^2 dx + \mu \int_{\Omega\setminus K} |u-g|^q dx + \alpha \mathcal{H}^{n-1}(K \cap \Omega), \tag{20}$$

If the datum g in the Mumford and Shah functional (20) is unbounded, the maximum principle fails, therefore the natural class of competing functions must include possibly unbounded functions: this class is given by $GSBV(\Omega)$ (see [23] or [2] for other applications).

Definition 4. The class of generalized functions with special bounded variation is defined as follows:

$$GSBV(\Omega) \;:=\; \left\{v : \Omega \to \mathbb{R} \text{ Borel function}; -k \vee v \wedge k \in SBV_{loc}(\Omega)\ \forall k \in \mathbb{N}\right\}.$$

For $v \in GSBV(\Omega)$ we have that S_v is countably $(\mathcal{H}^{n-1}, n-1)$ rectifiable and ∇v exists a.e. in Ω. If we assume that

$$\int_\Omega |\nabla v| \, dy < +\infty, \quad \mathcal{H}^{n-1}(S_v) < +\infty,$$

then for every $a, b \in \mathbb{R}, a < b$ we have $(a \vee v \wedge b) \in SBV(\Omega)$,

$$\int_\Omega |D(a \vee v \wedge b)| = \int_a^b P(\{v > \sigma\}, \Omega) \, d\sigma,$$

and $(a \vee v \wedge b)\chi_E \in SBV(\Omega)$ for every set E with finite perimeter in Ω.

Theorem 7 has been proved for scalar valued functions of the class SBV in a ball. Now we use a technical refinement of Theorem 7, which is useful in the class $GSBV(\Omega)$.

5 Poincaré Inequality in $GSBV^2(\Omega)$

In order to study second order variational problems in image segmentation ([3]) and elasto-plastic plates ([25]) we introduced Poincaré inequalities ([8], [9], [12], [13]) for non integrable functions whose gradient has a non integrable absolutely continuous part. In this context the truncation of competing functions v has to be smoothed so that it operates with

a controlled increase of $\nabla^2 v$ and avoids the introduction of discontinuity in the gradient itself: in this way an interpolation inequality (see [9] or Theorem 2.8 of [10]) provides estimates of first gradient and existence of $GSBV^2(\Omega)$ minimizers.
In the regularization process, there is no way of truncating gradient ∇v and preserving the $\operatorname{curl} \nabla v = 0$ property at the same time. Area and perimeter of the set where v and ∇v are too high can be *a priori* estimated by $\mathcal{H}^{n-1}(S_v \cup S_{\nabla v})$, hence instead of truncating v in this set we replace v by a suitable affine polynomial, in such a way that least median of modified function and of its gradient are preserved.

Definition 5.

$$GSBV^2(\Omega) \ := \ \{ v \in GSBV(\Omega), \ \nabla v \in \big(GSBV(\Omega)\big)^n \}.$$

We emphasize that $v \in GSBV^2(\Omega)$ does not even entail that either v or ∇v belongs to $L^1_{loc}(\Omega)$.
We state a Poincaré inequality in the class $GSBV(\Omega)$, which is used also for estimating derivatives involved in the study of the Blake and Zisserman functional (see Theorem 13).
For every $v \in GSBV(B)$ and $a \in \mathbb{R}$ with $(2\gamma_n \mathcal{H}^{n-1}(S_v))^{\frac{n}{n-1}} \le a \le \frac{1}{2}|B|$, we set

$$\tau'(v,a,B) = \inf\{t \in \mathbb{R};\ |\{v < t\} \cap B| \ \ge \ a\},$$

$$\tau''(v,a,B) = \inf\{t \in \mathbb{R};\ |\{v \ge t\} \cap B| \ \le \ a\},$$

where γ_n is the isoperimetric constant relative to the balls of $\mathbb{R}^n$ (see (3) and (4)).
For every $\eta \ge 0$ and a as above we define the truncation operator

$$T(v,a,\eta) = (\tau'(v,a,B) - \eta) \vee v \wedge (\tau''(v,a,B) + \eta). \tag{21}$$

We get easily $T(T(v,a,\eta),a,\eta) = T(v,a,\eta)$, $m_*(T(v,a,\eta),B) = m_*(v,B)$ and $T(\lambda v, a, \lambda\eta) = \lambda T(v,a,\eta)$ for every $\lambda > 0$. Moreover $|\nabla T(v,a,\eta)| \le |\nabla v|$ a.e. on B and

$$|\{v \ne T(v,a,\eta)\}| \le 2a. \tag{22}$$

In case v is vector-valued the operators med and T are defined componentwise.
In this Section we write explicitly the dependence of medians, τ', τ'' on function, level set and the domain, since they are not fixed and we must operate "truncations" on both the competing functions and their gradients.

Theorem 10. *Assume $B \subset \mathbb{R}^n$ is an open ball, $n \ge 2$, $p \ge 1$, $v \in GSBV(B)$ and $a \in \mathbb{R}$ with*

$$\big(2\gamma_n \mathcal{H}^{n-1}(S_v)\big)^{\frac{n}{n-1}} \le a \le \frac{1}{2}|B|. \tag{23}$$

Let $\eta \ge 0$ and $T(v,a,\eta)$ as in (21). Then

$$\int_B |D\,T(v,a,\eta)| \le 2|B|^{\frac{p-1}{p}} \left(\int_B |\nabla T(v,a,\eta)|^p\,dy \right)^{\frac{1}{p}} + 2\eta \mathcal{H}^{n-1}(S_v). \tag{24}$$

If in addition $p < n$, setting $p^* = \frac{np}{n-p}$, then for any median $m(v,B)$

$$\begin{aligned} &\int_B |T(v,a,\eta) - m(v,B)|^{p^*} dy \\ &\le \frac{1}{2} \left(\frac{4\gamma_n p(n-1)}{n-p} \right)^{p^*} \left(\int_B |\nabla T(v,a,0)|^p\,dy \right)^{\frac{p^*}{p}} + (2\eta)^{p^*} a. \end{aligned} \tag{25}$$

If in addition $p \geq n$, then for every $s \geq \frac{n}{n-1}$ and for any median $m(v,B)$

$$\begin{aligned}&\int_B |T(v,a,\eta)v - m(v,B)|^s dy \\ &\leq \frac{1}{2}\left(\frac{4\gamma_n s(n-1)}{n}\right)^s \left(\int_B |\nabla T(v,a,0)|^p \, dy\right)^{\frac{s}{p}} |B|^{1+\frac{s}{n}-\frac{s}{p}} + (2\eta)^s a.\end{aligned} \tag{26}$$

PROOF. We may assume that $m(v,B) = 0$ and the right hand sides are finite. If $\mathcal{H}^{n-1}(S_v) = 0$ and $a = 0$ then $v \in W^{1,p}(B)$, $T(v,0,\eta) = v$ and the inequalities are well-known. By Theorem 11, $T(v,a,\eta) \in SBV(B)$ and we obtain

$$\begin{aligned}&\int_B |DT(v,a,\eta)| \\ &= \int_B |\nabla T(v,a,\eta)| \, dy + \int_{S_{T(v,a,\eta)}} |(T(v,a,\eta))^+ - (T(v,a,\eta))^-| d\mathcal{H}^{n-1} \\ &\leq \int_B |\nabla T(v,a,\eta)| \, dy + (\tau''(v,a,B) - \tau'(v,a,B) + 2\eta)\mathcal{H}^{n-1}(S_v).\end{aligned} \tag{27}$$

By the coarea formula and the isoperimetric inequality we obtain

$$\begin{aligned}&\int_B |DT(v,a,\eta)| = \int_{-\infty}^{+\infty} P(\{T(v,a,\eta) < \sigma\}, B) \, d\sigma \\ &\geq \frac{1}{\gamma_n}\int_0^{\tau''} |\{T(v,a,\eta) \geq \sigma\}|^{\frac{n-1}{n}} d\sigma + \frac{1}{\gamma_n}\int_{\tau'}^0 |\{T(v,a,\eta) < \sigma\}|^{\frac{n-1}{n}} d\sigma \\ &\geq \frac{1}{\gamma_n}(\tau''(v,a,B) - \tau'(v,a,B))a^{\frac{n-1}{n}}.\end{aligned} \tag{28}$$

By the assumption on a and by comparison with (27) in the case $\eta = 0$ we have

$$\begin{aligned}&2(\tau''(v,a,B) - \tau'(v,a,B))\mathcal{H}^{n-1}(S_v) \leq \frac{1}{\gamma_n}(\tau''(v,a,B) - \tau'(v,a,B))a^{\frac{n-1}{n}} \\ &\leq \int_B |\nabla T(v,a,0)| \, dy + (\tau''(v,a,B) - \tau'(v,a,B))\mathcal{H}^{n-1}(S_v),\end{aligned}$$

hence

$$(\tau''(v,a,B) - \tau'(v,a,B))\mathcal{H}^{n-1}(S_v) \leq \int_B |\nabla T(v,a,0)| \, dy. \tag{29}$$

By substitution in (27), we obtain for every $\eta \geq 0$

$$\int_B |DT(v,a,\eta)| \leq 2\int_B |\nabla T(v,a,\eta)| \, dy + 2\eta\mathcal{H}^{n-1}(S_v), \tag{30}$$

so (24) follows by Hölder inequality. By (28) and (30) we get also

$$\begin{aligned}&(\tau''(v,a,B) - \tau'(v,a,B))a^{\frac{n-1}{n}} \\ \leq \;& 2\gamma_n\left(|B|^{\frac{p-1}{p}}\left(\int_B |\nabla T(v,a,\eta)|^p \, dy\right)^{\frac{1}{p}} + \eta\mathcal{H}^{n-1}(S_v)\right).\end{aligned} \tag{31}$$

By Theorem 6 applied to $T(v,a,\eta)$ we get

$$\int_B |T(v,a,\eta)|^{1^*} dy \leq \left(\gamma_n \int_B |DT(v,a,\eta)|\right)^{1^*}. \tag{32}$$

We define

$$E = \{y \in B; \tau'(v,a,B) \leq v(y) \leq \tau''(v,a,B)\ \},$$

$$E' = \{y \in B; v(y) < \tau'(v,a,B)\ \}\ , \qquad E'' = \{y \in B; v(y) > \tau''(v,a,B)\ \}.$$

Then, by taking into account (21), (22) and (24), we have for every $s \geq 1$

$$\begin{aligned} &\int_B |T(v,a,\eta)|^s dy = \int_E |T(v,a,0)|^s dy + \int_{E'\cup E''} |T(v,a,\eta)|^s dy \\ &\leq \int_E |T(v,a,0)|^s dy + \int_{E'} |\tau'(v,a,B)-\eta|^s dy + \int_{E''} |\tau''(v,a,B)+\eta|^s dy \\ &\leq \int_E |T(v,a,0)|^s dy + 2^{s-1}\left(\int_{E'\cup E''} |T(v,a,0)|^s dy + \eta^s |E'\cup E''|\right) \\ &\leq 2^{s-1}\int_B |T(v,a,0)|^s dy + (2\eta)^s a. \end{aligned} \tag{33}$$

Hence if $p = 1$, by (33) with $s = 1^*$, (32) and (30) for $T(v,a,0)$ we get

$$\int_B |T(v,a,\eta)|^{1^*} dy \leq 2^{1^*-1}\left(2\gamma_n \int_B |\nabla T(v,a,0)|\, dy\right)^{1^*} + (2\eta)^{1^*} a,$$

so that (25) is proved in the case $p = 1$. We focus now the case $1 < p < n$. We set

$$w = T(v,a,0)|T(v,a,0)|^{\frac{p^*}{1^*}-1}$$

and we notice that $w \in SBV(B)$, $m_*(w,B) = m_*(v,B)$ and

$$|\nabla w| = \frac{p^*}{1^*}|T(v,a,0)|^{\frac{p^*}{1^*}-1}|\nabla T(v,a,0)| \quad \text{a.e. on } B.$$

By plugging w in (32) and (30) with $\eta = 0$ and by Hölder inequality

$$\begin{aligned} &\left(\int_B |T(v,a,0)|^{p^*} dy\right)^{\frac{1}{1^*}} = \left(\int_B |T(w,a,0)|^{1^*} dy\right)^{\frac{1}{1^*}} \\ &\leq 2\gamma_n \frac{p^*}{1^*}\int_B |\nabla T(v,a,0)||T(v,a,0)|^{\frac{p^*}{1^*}-1} dy \\ &\leq 2\gamma_n \frac{p^*}{1^*}\left(\int_B |\nabla T(v,a,0)|^p dy\right)^{\frac{1}{p}}\left(\int_B |T(v,a,0)|^{p^*} dy\right)^{1-\frac{1}{p}}. \end{aligned}$$

Dividing by $\mid T(v,a,0)\mid_{L^{p^*}(B)}^{p^*(1-\frac{1}{p})}$, which is finite since $T(v,a,0)$ is bounded, we get

$$\textstyle\int_B |T(v,a,0)|^{p^*} dy \leq \left(\frac{2\gamma_n p(n-1)}{n-p}\right)^{p^*}\left(\int_B |\nabla T(v,a,0)|^p dy\right)^{\frac{p^*}{p}}. \tag{34}$$

By (33) with $s = p^*$ and (34) we obtain

$$\begin{aligned} &\int_B |T(v,a,\eta)|^{p^*} dy \\ &\leq \frac{1}{2}\left(\frac{4\gamma_n p(n-1)}{n-p}\right)^{p^*}\left(\int_B |\nabla T(v,a,0)|^p dy\right)^{\frac{p^*}{p}} + (2\eta)^{p^*} a. \end{aligned}$$

If $p \geq n$, fix any $s \geq \frac{n}{n-1}$ and set $r = \frac{ns}{n+s}$. Then $r < n$ and $r^* = s$, hence by (25) and Hölder inequality, still assuming $m_*(v, B) = 0$ and setting $c = (4\gamma_n \frac{r(n-1)}{n-r})^s = (4\gamma_n \frac{s(n-1)}{n})^s$, we have

$$\begin{aligned}\int_B |T(v,a,\eta)|^s \, dy \ &\leq \frac{c}{2}\left(\int_B |\nabla T(v,a,0)|^r \, dy\right)^{\frac{s}{r}} + (2\eta)^s a \\ &\leq \frac{c}{2}|B|^{\left(1-\frac{r}{p}\right)\frac{s}{r}} \left(\int_B |\nabla T(v,a,0)|^p \, dy\right)^{\frac{s}{p}} + (2\eta)^s a \\ &\leq \frac{c}{2}|B|^{1+\frac{s}{n}-\frac{s}{p}} \left(\int_B |\nabla T(v,a,0)|^p \, dy\right)^{\frac{s}{p}} + (2\eta)^s a\end{aligned}$$

and the proof is completed. QED

We show that, besides the volume estimate (22), also the perimeter of the set $\{v \neq T(v,a,\eta)\}$ can be estimated for sufficiently many $\eta \in (0,1)$.

Theorem 11. *l Let $B \subset \mathbb{R}^n$ be an open ball, $n \geq 2$, $s \geq 1$. Let $v \in GSBV(B)$ and $a \in \mathbb{R}$ with $\left(2\gamma_n \mathcal{H}^{n-1}(S_v)\right)^{\frac{n}{n-1}} \leq a \leq \frac{1}{2}|B|$. Then there exists $\eta \in (0,1)$ such that*

$$\begin{aligned}&P(\{v > \tau''(v,a,B) + \eta\}, B) \\ &\leq 3a^{1-\frac{1}{s}}\left(\int_{\{v > \tau''(v,a,B)\}} |\nabla T(v,a,1)|^s dy\right)^{\frac{1}{s}} + 3\mathcal{H}^{n-1}(S_v),\end{aligned} \tag{35}$$

$$\begin{aligned}&P(\{v < \tau'(v,a,B) - \eta\}, B) \\ &\leq 3a^{1-\frac{1}{s}}\left(\int_{\{v < \tau'(v,a,B)\}} |\nabla T(v,a,1)|^s dy\right)^{\frac{1}{s}} + 3\mathcal{H}^{n-1}(S_v).\end{aligned} \tag{36}$$

Actually $|\{\eta \in (0,1) : \text{both } (35) \text{ and } (36) \text{ hold }\}| \geq \frac{1}{3}$.

Proof. By coarea formula and by the definition of $SBV(B)$

$$\int_{\tau''}^{\tau''+1} P(\{v > \sigma\}, B) \, d\sigma = \int_B |D(\tau'' \vee v \wedge (\tau''+1))|$$

$$\leq \int_{\{\tau'' < v < \tau''+1\}} |\nabla T(v,a,1)| \, dy + \mathcal{H}^{n-1}(S_v) \leq \int_{\{\tau'' < v\}} |\nabla T(v,a,1)| \, dy + \mathcal{H}^{n-1}(S_v),$$

and analogously

$$\int_{\tau'-1}^{\tau'} P(\{v < \sigma\}, B) \, d\sigma = \int_B |D((\tau'-1) \vee v \wedge \tau')|$$

$$\leq \int_{\{\tau'-1 < v < \tau'\}} |\nabla T(v,a,1)| \, dy + \mathcal{H}^{n-1}(S_v) \leq \int_{\{v < \tau'\}} |\nabla T(v,a,1)| \, dy + \mathcal{H}^{n-1}(S_v).$$

We get the thesis by Chebyshev and Hölder inequalities and by (22). QED

Now we show how the previous Theorem 10 can be used to obtain a compactness and lower semicontinuity theorem useful in regularity theory.

For any given function in $GSBV^2$, we define an affine polynomial correction such that both median and gradient median vanish.

Let $B_r(x) \subset \Omega$ and $v \in GSBV^2(B_r(x))$; for every $y \in \mathbb{R}^n$ we set

$$(M_{x,r}v)(y) = m_*(\nabla v, B_r(x)) \cdot (y - x)$$

$$(\mathcal{P}_{x,r}v)(y) = (M_{x,r}v)(y) + m_*(v - M_{x,r}v, B_r(x)).$$

Since $m_*(v-c, B_r(x)) = m_*(v, B_r(x)) - c$ for every $c \in \mathbb{R}$ and $\nabla(\mathcal{P}_{x,r}v) = \nabla(M_{x,r}v) = m_*(\nabla v, B_r(x))$ then we have $\mathcal{P}_{x,r}(v - \mathcal{P}_{x,r}v) = 0$, say

$$m_*(v - \mathcal{P}_{x,r}v, B_r(x)) = 0, \qquad m_*(\nabla(v - \mathcal{P}_{x,r}v), B_r(x)) = \mathbf{0}.$$

We notice that there are v such that $m_*(v, B_r(x)) \neq m_*(\mathcal{P}_{x,r}v, B_r(x))$, take e.g. $v(x) = (x_1^2 - x_1)H(-x_1) - \frac{x_1}{2}H(x_1)$, where H is the Heaviside function.

In the following we denote by s' the conjugate exponent of s in the Hölder inequality.

Theorem 12. ***(Compactness and lower semicontinuity)*** *l Let $p \geq n \geq 2$, $B_r(x) \subset \mathbb{R}^n$ be an open ball, $(v_h) \subset GSBV^2(B_r(x))$.*
Set $L_h := \mathcal{H}^{n-1}(S_{v_h} \cup S_{\nabla v_h})$. Assume

$$\sup_h \int_{B_r(x)} |\nabla^2 v_h|^p dy < +\infty, \tag{37}$$

$$\lim_h L_h = 0. \tag{38}$$

Then there exist $z \in W^{2,p}(B_r(x))$, a sequence $(z_h) \subset GSBV^2(B_r(x))$ and a positive constant c (depending on the left hand side of (37)) such that up to a finite number of indices,

$$|\{z_h \neq (v_h - \mathcal{P}_{x,r}v_h)\}| \leq c{L_h}^{n'}, \tag{39}$$

$$P(\{z_h \neq (v_h - \mathcal{P}_{x,r}v_h)\}, B_r(x)) \leq cL_h. \tag{40}$$

Moreover there is a subsequence (z_{h_k}) such that for every $\vartheta \geq 1$

$$\lim_k z_{h_k} = z \quad \text{strongly in } L^\vartheta(B_r(x)), \tag{41}$$

$$\lim_k \nabla z_{h_k} = Dz \quad \text{strongly in } L^\vartheta(B_r(x), \mathbb{R}^n), \tag{42}$$

$$\int_{B_r(x)} |D^2 z|^p dy \leq \liminf_k \int_{B_r(x)} |\nabla^2 z_{h_k}|^p dy \leq \liminf_k \int_{B_r(x)} |\nabla^2 v_{h_k}|^p dy, \tag{43}$$

$$\lim_k \left(v_{h_k} - \mathcal{P}_{x,r}v_{h_k}\right) = z \quad \text{a.e. on } B_r(x), \tag{44}$$

$$\lim_k \nabla(v_{h_k} - \mathcal{P}_{x,r}v_{h_k}) = Dz \quad \text{a.e. on } B_r(x). \tag{45}$$

Proof. We can assume $x = 0$, $\mathcal{P}_{0,r}v_h = 0$ and we extract subsequences without relabeling. We use properties of functions that hold true only up to a finite number of indices, hence by (38) we can assume that

$$a_h := \left(2\gamma_n L_h\right)^{n'} \leq \frac{1}{2}|B_r|.$$

Arguing separately on each component of ∇v_h we deduce that $\forall h \in \mathbb{N}$, $\forall k \in \{1, \ldots, n\}$, there exist $\eta_h^k \in (0,1)$ and c depending on the l.h.s. of (37) such that

$$|\{T(\nabla_k v_h, a_h, \eta_h^k) \neq \nabla_k v_h\}| \leq c{L_h}^{n'}, \tag{46}$$

$$P\left(\{T(\nabla_k v_h, a_h, \eta_h^k) \neq \nabla_k v_h\}, B_r\right) \leq c\left(L_h^{\frac{n'}{p'}} + \mathcal{H}^{n-1}(S_{\nabla_k v_h})\right). \tag{47}$$

Inequality (46) follows by (22), while (47) follows by Theorem 11 with $s = p$ applied to $\nabla_k v_h$ and estimating $\nabla T(\nabla_k v_h, a_h, 1)$ through (37).

We define small subsets $E_h \subset B_r$ where we have to modify v_h in order to force boundedness of ∇v_h, and we perform a former tuning of v_h. To this aim set

$$E_h = \bigcup_{k=1}^{n} \{y \in B_r; T(\nabla_k v_h, a_h, \eta_h^k) \neq \nabla_k v_h\}, \tag{48}$$

$$w_h \;=\; v_h \chi_{B_r \setminus E_h}. \tag{49}$$

Since $m_*(v_h, B_r) = 0$ then $m_*(w_h, B_r) = 0$ by (2); moreover by the definition $m_*(\nabla w_h, B_r) = m_*(\nabla v_h, B_r) = 0$. By (47) E_h has finite perimeter, hence by Theorem 11 we have $w_h \in GSBV^2(B_r)$ and $\nabla w_h \in SBV(B_r, \mathbb{R}^n) \cap L^\infty(B_r, \mathbb{R}^n)$. Then summarizing

$$\begin{cases} \mathcal{P}_{0,r} w_h = 0, \\ |\nabla_k w_h| \leq |T(\nabla_k v_h, a_h, \eta_h^k)|, \quad |\nabla^2 w_h| \leq |\nabla^2 v_h| \;\text{ a.e. on }\; B_r, \\ \mathcal{H}^{n-1}(S_{w_h} \cup S_{\nabla w_h}) \leq c \left(L_h^{\frac{n'}{p'}} + L_h \right), \end{cases} \tag{50}$$

and by (38), (46) and (47)

$$\lim_h \Big(|E_h| + P(E_h, B_r) \Big) = 0.$$

Fix $\vartheta \geq p$. Since $0 < \eta_h^k < 1$, by (50), (26), (37) and (38) we get

$$\begin{aligned} \int_{B_r} |\nabla_k w_h|^\vartheta dy &\leq \int_{B_r} |T(\nabla_k v_h, a_h, \eta_h^k)|^\vartheta dy \\ &\leq c \left(\int_{B_r} |\nabla T(\nabla_k v_h, a_h, 0)|^p \, dy \right)^{\frac{t}{p}} + 2^\vartheta a_h \leq c < +\infty. \end{aligned} \tag{51}$$

By (50), (29) and by the assumptions (37), (38) and $p \geq n$, we get

$$\begin{aligned} \int_{B_r} |D\nabla_k w_h| \leq & \int_{B_r} |\nabla^2 w_h| \, dy \\ & + (\tau''(\nabla_k w_h, a_h, B_r) - \tau'(\nabla_k w_h, a_h, B_r) + 2)\mathcal{H}^{n-1}(S_{\nabla_k w_h}) \\ \leq & 2|B_r|^{\frac{1}{p'}} \left(\int_{B_r} |\nabla^2 w_h|^p \, dy \right)^{\frac{1}{p}} + 2\mathcal{H}^{n-1}(S_{\nabla_k w_h}) \leq c < +\infty. \end{aligned} \tag{52}$$

Now we want to force boundedness of (w_h). By (50) we can assume

$$b_h := \left(2\gamma_n \mathcal{H}^{n-1}(S_{w_h} \cup S_{\nabla w_h}) \right)^{n'} \leq \frac{1}{2} |B_r|$$

and there are $\eta_h \in (0,1)$ and a constant, depending on the l.h.s. of (37), such that

$$|\{T(w_h, b_h, \eta_h) \neq w_h\}| \leq c \left(L_h^{\frac{n'}{p'}} + L_h \right)^{n'}, \tag{53}$$

$$P(\{T(w_h, b_h, \eta_h) \neq w_h\}, B_r) \leq c \left(L_h^{\frac{n'^2}{p'^2}} + L_h^{\frac{n'}{p'}} + L_h \right). \tag{54}$$

Inequality (53) follows by (22), while (54) follows by Theorem 11 with $s = p$, applied to w_h, taking into account (50) and (51). Then we define

$$U_h = \{y \in B_r; T(w_h, b_h, \eta_h) \neq w_h\}$$

and we perform the following tuning of the sequence (w_h):

$$z_h \ = T(w_h, b_h, \eta_h).$$

We notice that $z_h \in GSBV^2(B_r)$ and

$$\begin{cases} m_*(z_h, B_r) = 0,\ T(z_h, b_h, \eta_h) = z_h,\ \{z_h \neq v_h\} = E_h \cup U_h, \\ |\nabla_k z_h| \le |\nabla_k w_h|, \quad |\nabla^2 z_h| \le |\nabla^2 w_h| \text{ a.e. on } B_r, \\ \mathcal{H}^{n-1}(S_{z_h} \cup S_{\nabla z_h}) \le c\left(L_h^{\frac{n'2}{p'2}} + L_h^{\frac{n'}{p'}} + L_h \right) . \end{cases} \tag{55}$$

Hence (39) and (40) follow by (46), (53), and (47), (54) respectively, taking into account the assumption $p \ge n$. By (27), (29) and (51) we have

$$\begin{aligned} &\int_{B_r} |Dz_h| \\ &\quad \le \int_{B_r} |\nabla w_h|\, dy + (\tau''(w_h, b_h, B_r) - \tau'(w_h, b_h, B_r) + 2)\mathcal{H}^{n-1}(S_{w_h}) \\ &\quad \le 2|B_r|^{\frac{1}{p'}} \left(\int_{B_r} |\nabla w_h|^p dy \right)^{\frac{1}{p}} + 2\mathcal{H}^{n-1}(S_{w_h}) \le c < +\infty. \end{aligned} \tag{56}$$

Moreover, by (55) and (51),

$$\int_{B_r} |\nabla_k z_h|^{\vartheta} dy \le \int_{B_r} |\nabla_k w_h|^{\vartheta} dy \le c < +\infty, \tag{57}$$

and by (50), (55) we get

$$\begin{aligned} &\int_{B_r} |D\nabla_k z_h| \le \int_{B_r} |\nabla^2 z_h|\, dy \\ &\quad + (\tau''(\nabla_k v_h, a_h, B_r) - \tau'(\nabla_k v_h, a_h, B_r) + 2)\mathcal{H}^{n-1}(S_{\nabla_k z_h}) \\ &\le c\int_{B_r} |\nabla^2 v_h|\, dy + c(\tau''(\nabla_k v_h, a_h, B_r) - \tau'(\nabla_k v_h, a_h, B_r) + 2)\Big(L_h^{\frac{n'2}{p'2}} + L_h^{\frac{n'}{p'}} + L_h \Big), \end{aligned}$$

hence by (31) and the assumption $p \ge n$ we have

$$\int_{B_r} |D\nabla_k z_h| \le c < +\infty. \tag{58}$$

Since $0 < \eta_h < 1$, by (26) and (57)

$$\begin{aligned} \int_{B_r} |z_h|^{\vartheta} dy \le \frac{1}{2} \left(\frac{4\gamma_n \vartheta(n-1)}{n} \right)^{\vartheta} \left(\int_{B_r} |\nabla T(z_h, b_h, 0)|^p\, dy \right)^{\frac{t}{p}} |B_r|^{1+\frac{\vartheta}{n}-\frac{\vartheta}{p}} \\ + 2^{\vartheta} \left(2\gamma_n \mathcal{H}^{n-1}(S_{z_h}) \right)^{n'} \le c < +\infty. \end{aligned} \tag{59}$$

By (57), (58) and the compactness theorem in $BV(B_r)$ applied to (∇z_h) there exists $f \in BV(B_r, \mathbb{R}^n)$ such that

$$\nabla z_h \to f \qquad \text{strongly in } L^s(B_r, \mathbb{R}^n) \quad \forall s \in [1, 1^*). \tag{60}$$

By (55), (37), (38) and Theorem 2.1 in [1], $f \in SBV(B_r, \mathbb{R}^n)$ and also

$$\int_{B_r} |\nabla f|^p dy \leq \liminf_h \int_{B_r} |\nabla^2 z_h|^p dy,$$

$$\mathcal{H}^{n-1}(S_f) \leq \lim_h \mathcal{H}^{n-1}(S_{\nabla z_h}) = 0,$$

hence $f \in W^{1,p}(B_r, \mathbb{R}^n)$. By (57) and by (60) we get

$$\nabla z_h \to f \qquad \text{strongly in } L^s(B_r, \mathbb{R}^n) \quad \forall s \in [1, \vartheta). \tag{61}$$

By (56), (59) and the compactness theorem in $BV(B_r)$ applied to (z_h) , there exists $z \in BV(B_r)$ such that

$$z_h \to z \quad \text{strongly in } L^s(B_r),\ \forall s \in [1, 1^*). \tag{62}$$

By (55), (57) and Theorem 2.1 in [1] we have $z \in SBV(B_r)$ and also

$$\nabla z_h \to \nabla z \qquad \text{weakly in } L^\vartheta(B_r, \mathbb{R}^n), \tag{63}$$

$$\mathcal{H}^{n-1}(S_z) \leq \lim_h \mathcal{H}^{n-1}(S_{z_h}) = 0,$$

hence $z \in W^{1,p}(B_r)$ and also by (59) and (62)

$$z_h \to z \qquad \text{strongly in } L^s(B_r) \quad \forall s \in [1, \vartheta).$$

By (61) and (63) $f = \nabla z = Dz$ so that $z \in W^{2,p}(B_r)$. By the arbitrariness of ϑ, (41), (42), (43) hold true and (44), (45) follow by (39). QED

The Blake & Zisserman functional was proposed for the study of a 2-dimensional monochromatic image of brightness intensity g in [3] and [9]. For the most recent results and updated bibliography on this functional we refer to [4], [5], [13] and [15].
Precisely the strong formulation of the Blake & Zisserman functional is

$$\begin{aligned} F(K_0, K_1, u) \ :=\ & \int_{\Omega\setminus(K_0\cup K_1)} \left(|D^2u|^2 + \mu|u-g|^q\right) dx \\ & + \alpha\,\mathcal{H}^1(K_0\cap\Omega) \ +\ \beta\,\mathcal{H}^1((K_1\setminus K_0)\cap\Omega)\,, \end{aligned} \tag{64}$$

to be minimized over triplets (K_0, K_1, u).

Theorem 13. *(see [10]) Under assumptions $n = 2$ and*

$$q \geq 1\ ,\ \mu > 0\ ,\ 0 < \beta \leq \alpha \leq 2\beta\ ,\ g \in L^{2q}_{loc}(\Omega) \cap L^q(\Omega)\,,$$

there is at least one triplet among $K_0, K_1 \subset \mathbb{R}^2$ Borel sets with $K_0 \cup K_1$ closed and $u \in C^2(\Omega \setminus (K_0 \cup K_1))$ approximately continuous on $\Omega \setminus K_0$ minimizing the functional (64) and having finite energy. Moreover the sets $K_0 \cap \Omega$ and $K_1 \cap \Omega$ are countably $\mathcal{H}^1$-rectifiable.

We remark that if (K_0, K_1, u) is a minimizing triplet of F, then K_0 and K_1 can be interpreted respectively as the jump set and the crease set of the image g, and u as a smoothing of g.

An application to image inpainting, which has been announced in [17], follows from [14] and is performed in [7].

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