Note di Matematica ISSN 1123-2536, e-ISSN 1590-0932
Note Mat. **31** (2011) n. 1, 15–27. doi:10.1285/i15900932v31n1p15

On a class of positive C_0-semigroups of operators on weighted continuous function spaces

Francesco Altomare[i]
Dipartimento di Matematica,
Universitá degli Studi di Bari "A. Moro"
Via E. Orabona, 4 - 70125 Bari - Italia
altomare@dm.uniba.it

Sabina Milella
Dipartimento di Matematica,
Universitá degli Studi di Bari "A. Moro"
Via E. Orabona, 4 - 70125 Bari - Italia
smilella@dm.uniba.it

Graziana Musceo
Dipartimento di Matematica,
Universitá degli Studi di Bari "A. Moro"
Via E. Orabona, 4 - 70125 Bari - Italia
musceo@dm.uniba.it

Received: 24/06/2010; accepted: 07/10/2010.

Abstract. This paper is mainly concerned with the study of the generators of those positive C_0-semigroups on weighted continuous function spaces that leave invariant a given closed sublattice of bounded continuous functions and whose relevant restrictions are Feller semigroups. Additive and multiplicative perturbation results for this class of generators are also established. Finally, some applications concerning multiplicative perturbations of the Laplacian on $\mathbb{R}^n$, $n \geq 1$, and degenerate second-order differential operators on unbounded real intervals are showed.

Keywords: Positive semigroup, Feller property, weighted continuous function space, evolution equation.

MSC 2000 classification: 47D06, 47D07, 35K15

Dedicated to the memory of V.B. Moscatelli

[i]Corresponding author. This work has been partially supported by the Research Project "Real Analysis and Functional Analytic Methods for Differential Problems and Approximation Problems", University of Bari, 2010.

Introduction

Positive C_0-semigroups have proved to be a large and important subject of investigation with a vast range of applications from evolution equations to probability theory and to quantum statistical mechanics, just to quote a few.

The main aspects of the subject are documented in several monographs (see, e.g., [9], [10], [12], [13], [14], [16], [17]). In particular, the theory of positive C_0-semigroups has reached a satisfactory level of completeness in the setting of spaces of bounded continuous functions where, among other things, it is strongly connected with probability theory (see, e.g., [11], [12], [20]).i

Rather recently ([1], [3], [6]) positive C_0-semigroups have been investigated in some Banach lattices of unbounded continuous functions, namely, in weighted continuous function spaces on a locally compact Hausdorff space X.

In this setting, the positive C_0-semigroups satisfying the so-called Feller property, i.e., they leave invariant the space $C_0(X)$ and the restrictions to $C_0(X)$ are contractive and strongly continuous, play an important role because they are the only possible positive C_0-semigroups on the bigger weighted continuous function spaces that are associated with probability transition functions and hence with right-continuous Markov processes.

This special class of positive semigroup was object of investigation in [1], [3] and [6]. Here we deepen their study by reformulating the previous results in a more general form and by highlighting some new additional aspects.

In Section 2 we first discuss a simple counterexample showing that, in general, the Feller property can fail also in the framework of weighted continuous function spaces.

In the above mentioned setting we then characterize the generators of those positive C_0-semigroups that leave invariant a given closed sublattice of bounded continuous functions and whose relevant restrictions are Feller semigroups.

The given characterization involves, in particular, a generalized positive maximum principle. We also establish some additive and multiplicative perturbation results for this class of generators and we show some applications concerning multiplicative perturbations of the Laplacian on $\mathbb{R}^n$, $n \geq 1$, (see also [5]). Finally, Section 3 is devoted to applications to second-order differential operators on unbounded real intervals.

1 Positive semigroups on weighted continuous function spaces

In this section we shall present a characterization of the generators of a particular class of positive C_0-semigroups on weighted continuous function spaces, namely those positive C_0-semigroups that leave invariant a given closed sublattice of bounded continuous functions and whose relevant restrictions are Feller semigroups. As we explained in the Introduction, the main interest for these classes of positive C_0-semigroups rests on the fact that they are the only possible C_0-semigroups associated with probability transition functions (and hence with right-continuous Markov processes) (for more details on the theory of C_0-semigroups of operators we refer, e.g., to [12], [16]).

Let X be a locally compact noncompact Hausdorff space. As usual we shall denote by $C(X)$ the space of all real valued continuous functions on X and by $C_b(X)$ and $C_0(X)$ the Banach lattices of all bounded continuous functions and of all continuous functions vanishing at infinity, respectively, endowed with the natural pointwise order and the uniform norm $\|\cdot\|_\infty$.

Given a bounded weight w on X, i.e., $w \in C_b(X)$ and $w(x) > 0$ for every $x \in X$, we denote by $C_0^w(X)$ the Banach lattice of all functions $f \in C(X)$ such that $wf \in C_0(X)$, endowed with the natural pointwise order and the weighted norm $\|\cdot\|_w$ defined by

$$\|f\|_w := \|wf\|_\infty \qquad (f \in C_0^w(X)). \tag{1}$$

Note that the space $C_0(X)$ is contained in $C_0^w(X)$ and it is dense in it.
Moreover,

$$\|f\|_w \leq \|w\|_\infty \|f\|_\infty \qquad (f \in C_0(X)). \tag{2}$$

If, in addition, $w \in C_0(X)$, then $C_b(X) \subset C_0^w(X)$ and (2) holds true for every $f \in C_b(X)$.

Throughout this section we shall fix a bounded weight w on X and we shall denote by E the space $C_0(X)$ or, if $w \in C_0(X)$, an arbitrary closed sublattice of $C_b(X)$ that is dense in $C_0^w(X)$. Therefore, E, endowed with the uniform norm and the natural pointwise order, is a Banach lattice.

Typical examples of such Banach lattices are, other than $C_0(X)$ and $C_b(X)$, the space

$$C_*(X) := \{f \in C(X) : f \text{ is convergent at infinity}\} \tag{3}$$

or, more generally, the space

$$E_Y(X) := \{f \in C(X) : \text{there exists a (unique) } \widetilde{f} \in C(Y) \text{ such that } \widetilde{f}\,|_X = f\} \tag{4}$$

where Y is a compact metric space such that X is a dense open subset of Y and the topology of X is inherited by Y.

Our main aim is to characterize those linear operators $A : D(A) \subset C_0^w(X) \to C_0^w(X)$ that are the generators of a C_0-semigroup $(T(t))_{t\geq 0}$ of positive linear operators on $C_0^w(X)$ satisfying the following additional properties:

(F_1) $T(t)(E) \subset E$ for every $t \geq 0$,

(F_2) $(T(t)\,|_E)_{t\geq 0}$ is strongly continuous on $(E, \|\cdot\|_\infty)$

and

(F_3) $0 \leq T(t)f \leq 1$ for every $t \geq 0$ and $f \in E$, $0 \leq f \leq 1$ (i.e., each $T(t)$ is positive and contractive on $(E, \|\cdot\|_\infty)$).

It is customary to refer to properties (F_1)-(F_3) by saying that $(T(t)\,|_E)_{t\geq 0}$ is a Feller semigroup on E. Accordingly, we shall also say that $(T(t))_{t\geq 0}$ satisfies the Feller property with respect to E.

The main interest for these particular classes of positive semigroups rests on the fact that, as it was shown in [1, Theorem 2.10], when X has a countable base and $E = C_0(X)$, then properties (F_1)-(F_3) hold true if and only if there exists a uniformly stochastically continuous normal C_0-transition function $(P_t)_{t\geq 0}$ on X, such that

(1) $\lim\limits_{x\to\infty} \sup\limits_{0\leq t\leq s} P_t(x, K) = 0$ for every $s > 0$ and for every compact subset K of X;

(2) each function $f \in C_0^w(X)$ is $P_t(x, \cdot)$-integrable and

$$T(t)(f)(x) = \int_X f\,dP_t(x, \cdot) \tag{5}$$

for every $t \geq 0$ and $x \in X$.

Moreover, there exists a right-continuous Markov process

$$(\Omega, \Sigma, (P^x)_{x\in X_\infty}, (Z_t)_{0\leq t\leq +\infty})$$

with state space X_∞ and whose paths have left-hand limits almost surely, such that

(i) $P^x_{Z_t}(B) = P_t(x, B)$ for every $t \geq 0$, $x \in X$ and $B \in \mathcal{B}(X)$,

(ii) w^{-1} as well as each function $f \in C_0^w(X)$ is $P^x_{Z_t}$-integrable,

and

(iii) $T(t)(f)(x) = \int_X f dP^x_{Z_t} = E_x(f^*(Z_t))$,

where f^* is the extension of f to X_∞ vanishing at ∞ and $\mathcal{B}$(X) denotes the σ-algebra of Borel subsets of X. Here $X_\infty := X \cup \{\infty\}$ denotes the one-point compactification of X (for more details on C_0-transition functions and Markov processes we refer, e.g., to [1, Section 1], [12], [15], [20]).

We also point out that, if $E = E_Y(X)$, then properties (F_1)-(F_3) hold true if and only if there exists a uniformly stochastically continuous normal Feller function $(P_t)_{t\geq 0}$ on Y such that

(1) $P_t(x, Y \backslash X) = 0$ for every $t \geq 0$ and $x \in X$;

(2) each function $f \in C_0^w(X)$ is $P_t(x, \cdot)$-integrable and

$$T(t)(f)(x) = \int_X f dP_t(x, \cdot) \tag{6}$$

for every $t \geq 0$ and $x \in X$.

Moreover, there exists a right-continuous Markov process

$$(\Omega, \Sigma, (P^x)_{x\in Y_\vartheta}, (Z_t)_{0\leq t\leq +\infty})$$

with state space Y_ϑ and whose paths have left-hand limits almost surely, such that

(i) $P^x_{Z_t}(B) = P_t(x, B)$ for every $t \geq 0$, $x \in X$ and $B \in \mathcal{B}(Y)$,

(ii) w^{-1} as well as each function $f \in C_0^w(X)$ is $P^x_{Z_t}$-integrable,

and

(iii) $T(t)(f)(x) = \int_X f dP^x_{Z_t} = E_x(f^*(Z_t))$

where now f^* denotes the extension of f to Y_ϑ vanishing outside X and $Y_\vartheta = Y \cup \{\vartheta\}$, where ϑ is an isolated point (see [1, Theorem 2.8]; see also [1, Theorem 2.9] and [3, Theorem 2.1]).

On the other hand we point out that there exist positive C_0-semigroups on $C_0^w(X)$ that do not satisfy property (F_3). A simple example is given below.

Example 1. Consider a function $\gamma \in C_b(X)$ such that $\gamma(x) > 0$ for every $x \in X$ and consider the positive bounded linear operator $A : C_0^w(X) \to C_0^w(X)$ defined by

$$Af := \gamma f \qquad (f \in C_0^w(X)). \tag{7}$$

Then A is the generator of the positive C_0-semigroup $(T(t))_{t\geq 0}$ on $C_0^w(X)$ defined by

$$T(t)f := \exp(t\gamma)\, f \qquad (f \in C_0^w(X), t \geq 0), \tag{8}$$

(see, e.g., [16, Chapter II, Section 2.9, p. 65]).
In particular $\|T(t)\| = \|\exp(t\gamma)\|_\infty \leq \exp(\omega t)$ where $\omega := \sup_{x\in X} \gamma(x)$. Clearly $\big(T(t)\,|_{C_0(X)}\big)_{t\geq 0}$ is a positive C_0-semigroup on $C_0(X)$ too but it is not contractive because $\|T(t)\,|_{C_0(X)}\| = \|\exp(t\gamma)\|_\infty > 1$ for every $t > 0$.

The next result provides a necessary and sufficient condition under which a linear operator is the pre-generator (i.e., it is closable and its closure is the generator) of a positive C_0-semigroup satisfying properties (F_1)-(F_3). An important role to this respect is played by the following property that we shall refer to as "generalized positive maximum principle" with respect to a fixed $\omega \in \mathbb{R}$.

Definition 1. We say that a linear operator $A : D(A) \subset C_0^w(X) \to C_0^w(X)$ satisfies the generalized positive maximum principle (briefly, g.p.m. principle) with respect to $\omega \in \mathbb{R}$ if

$$Au(x_0) \leq \omega u(x_0) \tag{9}$$

for every $u \in D(A)$ and $x_0 \in X$ such that $\sup\limits_{x \in X} w(x)u(x) = w(x_0)u(x_0) > 0$.

In such a case, $A - \omega I$ is necessary dissipative (see the proof of Corollary 2.4 of [1]), where I denotes the identity operator on $C_0^w(X)$. Therefore, if in addition $D(A)$ is dense in $C_0^w(X)$, then $A - \omega I$ and, hence A, are closable (see [16, Proposition 3.14, (iv)]).

Theorem 1. *Given a linear operator $A : D(A) \subset C_0^w(X) \longrightarrow C_0^w(X)$ and $\omega \in \mathbb{R}$, the following statements are equivalent:*

(a) the operator $(A, D(A))$ is closable and its closure generates a positive C_0-semigroup $(T(t))_{t\geq 0}$ on $C_0^w(X)$ satisfying the Feller property with respect to E, and

$$\|T(t)\| \leq e^{\omega t} \qquad \textit{for every}\, t \geq 0; \tag{10}$$

(b) (i) $D(A)$ is dense in $C_0^w(X)$ and $(A, D(A))$ satisfies the g.p.m. principle (9) with respect to ω;

(ii) denoted by $(\overline{A}, D(\overline{A}))$ the closure of $(A, D(A))$, there exists a subspace D_0 of $D(\overline{A}) \cap E$ such that $\overline{A}(D_0) \subset E$ and $\left(\overline{A}\mid_{D_0}, D_0\right)$ generates a Feller semigroup on E.

Furthermore, if (a) or (b) holds true, then $\left(T(t)\mid_E\right)_{t\geq 0}$ is the C_0-semigroup generated by $\left(\overline{A}\mid_{D_0}, D_0\right)$.

Proof. $(a) \Rightarrow (b)$. Statement (i) is a consequence of Proposition 2.3 of [1]. As regards statement (ii), consider the generator $(\widetilde{A}, D(\widetilde{A}))$ of the Feller semigroup $\left(T(t)\mid_E\right)_{t\geq 0}$ on E and set $D_0 := D(\widetilde{A}) \subset E$. If $u \in D_0$, then $\lim\limits_{t\to 0^+} \dfrac{T(t)u - u}{t} = \widetilde{A}(u)$ with respect to $\|\cdot\|_\infty$ and hence with respect to $\|\cdot\|_w$, because of (2). Therefore, $u \in D(\overline{A})$ and $\overline{A}u = \widetilde{A}u$. Hence $\overline{A}(D_0) = \widetilde{A}(D_0) \subset E$ and $\left(\overline{A}\mid_{D_0}, D_0\right) = (\widetilde{A}, D(\widetilde{A}))$ so that the result follows.
$(b) \Rightarrow (a)$. First note that there exists $\lambda > 0$ such that

$$E = (\lambda I - \overline{A})(D_0) \subset (\lambda I - \overline{A})(D(\overline{A})),$$

so that $(\lambda I - \overline{A})(D(\overline{A}))$ is dense in $C_0^w(X)$. Therefore, given $f \in C_0^w(X)$ and $\epsilon > 0$, there exists $u \in D(\overline{A})$ such that $\|\lambda u - \overline{A}u - f\|_w \leq \epsilon$ and there exists $v \in D(A)$ such that $\|v - u\|_w \leq \epsilon$ and $\|Av - \overline{A}u\|_w \leq \epsilon$. Then

$$\begin{aligned}\|(\lambda v - Av) - f\|_w &\leq \|(\lambda v - Av) - (\lambda u - \overline{A}u)\|_w + \|\lambda u - \overline{A}u - f\|_w \\ &\leq (\lambda + 2)\epsilon.\end{aligned}$$

Accordingly, $(\lambda I - A)(D(A))$ is dense in $C_0^w(X)$ and hence, by Proposition 2.3 of [1], $(\overline{A}, D(\overline{A}))$ is the generator of a positive C_0-semigroup $(T(t))_{t\geq 0}$ satisfying (10).

In order to show properties (F_1)-(F_3), denote by $(S(t))_{t\geq 0}$ the Feller semigroup on E generated by $\left(\overline{A}\,|_{D_0}, D_0\right)$. Given $u_0 \in D_0$, then the function $u(t) := S(t)u_0$ $(t \geq 0)$ is a solution to the abstract Cauchy problem (in $(E, \|\cdot\|_\infty)$ and hence in $(C_0^w(X), \|\cdot\|_w)$)

$$\begin{cases} \dot{u}(t) &= \overline{A}u(t), \quad t \geq 0, \\ \\ u(0) &= u_0, \end{cases}$$

so that (see, e.g., [16, Section II.6]) $u(t) = T(t)u_0$ for every $t \geq 0$. Thus, for each $t \geq 0$, $T(t) = S(t)$ on the subspace D_0 that is dense in $(E, \|\cdot\|_\infty)$. On the other hand both operators $T(t)$ and $S(t)$ are continuous from $(E, \|\cdot\|_\infty)$ into $(C_0^w(X), \|\cdot\|_w)$ because of (2); hence $T(t) = S(t)$ on E and the result follows. $\boxed{QED}$

A simple situation where condition (ii) of (b) is satisfied is indicated below.

Corollary 1. *Consider a linear operator $A : D(A) \subset C_0^w(X) \longrightarrow C_0^w(X)$ satisfying the g.p.m. principle with respect to some $\omega \in \mathbb{R}$ and assume that there exists a subspace D_0 of $D(A) \cap E$ such that $A(D_0) \subset E$, $(A\,|_{D_0}, D_0)$ is closable in E and its closure generates a Feller semigroup on E.*

Then $(A, D(A))$ is closable in $C_0^w(X)$ and its closure generates a positive C_0-semigroup $(T(t))_{t\geq 0}$ on $C_0^w(X)$ satisfying (10) as well as the Feller property with respect to E.

Moreover, the semigroup $(T(t)\,|_E)_{t\geq 0}$ is generated by the closure of $(A\,|_{D_0}, D_0)$.

Proof. First note that the subspace D_0 is dense in $C_0^w(X)$ and hence $D(A)$ is dense in $C_0^w(X)$ as well. Denote by $(\overline{A}, D(\overline{A}))$ the closure of $(A, D(A))$ in $C_0^w(X)$ and by $(\widetilde{A}, D(\widetilde{A}))$ the closure of $(A\,|_{D_0}, D_0)$ in E. Setting $\widetilde{D_0} := D(\widetilde{A}) \subset E$, we note that, if $u \in \widetilde{D_0}$, there exists $(u_n)_{n\geq 1}$ in $D_0 \subset D(A) \subset D(\overline{A})$ such that $u_n \to u$ and $Au_n \to \widetilde{A}u$ with respect to $\|\cdot\|_\infty$ and, hence, with respect to $\|\cdot\|_w$. Therefore, $u \in D(\overline{A})$ and $\overline{A}u = \widetilde{A}u$. Accordingly, $\widetilde{D_0} \subset D(\overline{A})$ and $\overline{A}\,|_{\widetilde{D_0}} = \widetilde{A}$ and hence the result follows from Theorem 1. $\boxed{QED}$

Another consequence of Theorem 1 is concerned with the extension of positive semigroups from E to $C_0^w(X)$.

Corollary 2. *Let $(S(t))_{t\geq 0}$ be a Feller semigroup on E and denote by $A_0 : D(A_0) \subset E \longrightarrow E$ its generator. Let $A : D(A) \subset C_0^w(X) \longrightarrow C_0^w(X)$ a linear operator satisfying the g.p.m. principle with respect to some $\omega \in \mathbb{R}$ and assume that $D(A_0) \subset D(A)$ and $A\,|_{D(A_0)} = A_0$. Then*

(1) there exists a (unique) positive semigroup $(T(t))_{t\geq 0}$ on $C_0^w(X)$ satisfying (10) such that $T(t)\,|_E = S(t)$ for every $t \geq 0$;

(2) $(A, D(A))$ is closable and its closure is the generator of the semigroup $(T(t))_{t\geq 0}$.

Proof. By the inclusion $D(A_0) \subset D(A)$ and the denseness of $D(A_0)$ in E and hence in $C_0^w(X)$, it follows that $D(A)$ is dense in $C_0^w(X)$. Denoted by $(\overline{A}, D(\overline{A}))$ the closure of $(A, D(A)$ then

$$D(A_0) \subset D(A) \subset D(\overline{A}) \qquad \text{and} \qquad \overline{A}\,|_{D(A_0)} = A\,|_{D(A_0)} = A_0.$$

Therefore, by Theorem 1, $(\overline{A}, D(\overline{A}))$ is the generator of a positive C_0-semigroup $(T(t))_{t\geq 0}$ satisfying (F_1)-(F_3) and (10). Moreover, since $(S(t))_{t\geq 0}$ is generated by $\left(\overline{A}\,|_{D(A_0)}, D(A_0)\right)$, then $S(t) = T(t)\,|_E$ for every $t \geq 0$ and hence part (1) follows as well. Finally, the semigroup in statement (1) is unique because E is dense in $C_0^w(X)$. $\boxed{QED}$

Below we discuss some further applications of Theorem 1 in the particular case where $E = C_0(X)$ or $E = C_*(X)$ (see (3)). In this setting we recall that a linear operator $B : D(B) \subset E \longrightarrow E$ is closable and its closure generates a Feller semigroup on E if and only if

(i) $D(B)$ is dense in E and $(\lambda I - B)(D(B))$ is dense in E for some/every $\lambda > 0$;

(11)

(ii) $(B, D(B))$ verifies the *positive maximum principle*, i.e.,

$$Bu(x_0) \leq 0 \tag{12}$$

for every $u \in D(B)$ and $x_0 \in X$ such that $\sup\limits_{x\in X} u(x) = u(x_0) > 0$ (resp., $x_0 \in X_\infty$ such that $\sup\limits_{x\in X_\infty} u(x) = u(x_0) > 0$, where, if $x_0 = \infty$, then $u(\infty) := \lim\limits_{x\to\infty} u(x)$).

(see [11, Chapter 0, pp. 386-388] or [4, Theorem 2.2]).

Corollary 3. *Let $(A, D(A))$ be the generator of a positive C_0-semigroup on $C_0^w(X)$ satisfying the Feller property with respect to $E = C_0(X)$ or $E = C_*(X)$, provided $w \in C_0(X)$, and (10) with respect to some $\omega_1 \in \mathbb{R}$. Consider a bounded linear operator B on $C_0^w(X)$ such that*

(i) $B(E) \subset E$ and $B\,|_E$ satisfies the positive maximum principle (12).

(ii) B satisfies the g.p.m. principle on $C_0^w(X)$ with respect to some $\omega_2 \in \mathbb{R}$.

Then $(A+B, D(A))$ is the generator of a positive C_0-semigroup on $C_0^w(X)$ satisfying the Feller property with respect to $E = C_0(X)$ or $E = C_*(X)$, resp., and (10) with respect to $\omega := \omega_1+\omega_2$.

Proof. Clearly $A + B$ is closed, it is densely defined and it satisfies the g.p.m. principle with respect to $\omega := \omega_1 + \omega_2$. By Theorem 1 there exists a subspace D_0 of $D(A) \cap E$ such that $A(D_0) \subset E$ and $(A\,|_{D_0}, D_0)$ is the generator of a Feller semigroup on E. Because of assumption (i), it turns out that $((A + B)\,|_{D_0}, D_0)$ is the generator of a Feller semigroup on E and hence the result follows from Theorem 1. $\boxed{QED}$

Remarks 1. 1. An example of bounded linear operator B on $C_0^w(X)$ satisfying condition (i)-(ii) of Corollary 3 is given by $B(f) := \gamma f$ $(f \in C_0^w(X))$ where $\gamma \in C_b(X)$, $\gamma \leq 0$ (resp., if $E = C_*(X)$, $\gamma \in C_*(X)$, $\gamma \leq 0$ and $w \in C_0(X)$). In this case $\omega_2 = \sup\limits_{x\in X} \gamma(x)$.

2. Note that every bounded linear operator B on $C_0^w(X)$ verifies the g.p.m. principle with respect to $\omega = \|B\|$.

Consider, indeed, $u \in C_0^w(X)$ and $x_0 \in X$ such that $\sup\limits_{x\in X} w(x)u(x) = w(x_0)u(x_0) > 0$. Then

$$Bu(x_0) \leq \frac{1}{w(x_0)}\,\|Bu\|_w \leq \frac{\|B\|}{w(x_0)}\,\|u\|_w = \frac{\|B\|}{w(x_0)}\,w(x_0)\,u(x_0) = \|B\|\,u(x_0).$$

Another application is concerned with multiplicative perturbations of generators of positive C_0-semigroups.

Theorem 2. *Consider a linear operator $A : D(A) \subset C_0^w(X) \longrightarrow C_0^w(X)$ satisfying the g.p.m. principle with respect to some $\omega \in \mathbb{R}$ and assume that there exists a subspace D_0 of $D(A) \cap C_0(X)$ such that $A(D_0) \subset C_0(X)$, $(A\,|_{D_0}, D_0)$ is closable and its closure generates a Feller semigroup on $C_0(X)$.*

If $\alpha \in C_b(X)$ and $\alpha(x) > 0$ for every $x \in X$, then $(\alpha A, D(A))$ is closable and its closure generates a positive C_0-semigroup on $C_0^w(X)$ satisfying the Feller property with respect to $C_0(X)$ as well as the inequality (10) with $\omega_1 := \omega^+\,\|\alpha\|_\infty$, where $\omega^+ := \sup\{\omega, 0\}$.

Proof. Clearly $(\alpha A, D(A))$ verifies the g.p.m. principle with respect to $\omega_1 := \omega^+\,\|\alpha\|_\infty$. Denoted by $(B, D(B))$ the closure of $(A\,|_{D_0}, D_0)$ in $C_0(X)$, then both $(B, D(B))$ and $(\alpha B, D(B))$ satisfy the positive maximum principle (12). Therefore, by virtue of Theorem 2 of [18], we infer

that $(\alpha B, D(B))$ is closable and its closure generates a Feller semigroup on $C_0(X)$.
Given $\lambda > 0$, we point out that

$$(\lambda I - \alpha B)(D(B)) \subset \overline{(\lambda I - \alpha A)(D_0)}.$$

Consider indeed $f \in (\lambda I - \alpha B)(D(B))$ and $u \in D(B)$ such that $f = \lambda u - \alpha Bu$. Then there exists a sequence $(u_n)_{n\geq 1}$ in D_0 such that $u_n \to u$ and $Au_n \to Bu$. Since α is bounded, we infer that $\lambda u_n - \alpha A u_n \to \lambda u - \alpha Bu = f$ so that $f \in \overline{(\lambda I - \alpha A)(D_0)}$.
On the other hand, by (11) we know that

$$C_0(X) = \overline{(\lambda I - \alpha B)(D(B))}$$

and hence $\overline{(\lambda I - \alpha A)(D_0)} = C_0(X)$ as well. Since $((\alpha A)\mid_{D_0}, D_0)$ satisfies the positive maximum principle we then conclude that $((\alpha A)\mid_{D_0}, D_0)$ is closable and its closure generates a Feller semigroup on $C_0(X)$ and hence the result follows from Corollary 1. $\boxed{QED}$

As an immediate consequence of Theorem 2 we obtain the following result that should be compared with [5, Theorem 3.1] (see also [18, Théorème 4]).

Consider $X = \mathbb{R}^n$, $n \geq 1$, and $\alpha \in C_b(\mathbb{R}^n)$ such that $\alpha(x) > 0$ for every $x \in \mathbb{R}^n$. Let w be a "smooth" weight on $\mathbb{R}^n$, i.e., $w \in C_b(\mathbb{R}^n) \cap C^2(\mathbb{R}^n)$ and $w(x) > 0$ for every $x \in \mathbb{R}^n$. Further assume that

$$\omega := \sup_{x\in\mathbb{R}^n} \frac{1}{w(x)^2} \sum_{i=1}^{n} \left[2\left(\frac{\partial w}{\partial x_i}(x)\right)^2 - w(x)\frac{\partial^2 w}{\partial x_i^2}(x)\right] < +\infty. \tag{13}$$

We shall consider the Laplace operator

$$\Delta u := \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} \qquad \big(u \in C^2(\mathbb{R}^n)\big) \tag{14}$$

defined on the domain

$$D_w(\Delta) := \{u \in C_0^w(\mathbb{R}^n) \cap C^2(\mathbb{R}^n) : \Delta u \in C_0^w(\mathbb{R}^n)\}. \tag{15}$$

Theorem 3. *The operator $(\alpha\Delta, D_w(\Delta))$ is closable and its closure generates a positive C_0-semigroup on $C_0^w(\mathbb{R}^n)$ satisfying the Feller property with respect to $C_0(\mathbb{R}^n)$ and (10) with respect to $\omega := \omega^+ \|\alpha\|_\infty$ where $\omega^+ = \sup\{\omega, 0\}$ and ω is defined by (13).*

Proof. Let $u \in D_w(\Delta)$ and $x_0 \in \mathbb{R}^n$ such that $\sup_{x\in\mathbb{R}^n} w(x)\,u(x) = w(x_0)\,u(x_0) > 0$. Setting $v := w \cdot u$, then $\Delta v(x_0) \leq 0$ and $\dfrac{\partial v}{\partial x_i}(x_0) = 0$ for every $i = 1, \ldots\ldots, n$. Therefore,

$$\begin{aligned}\Delta u(x_0) &= \frac{1}{w(x_0)^3}\left[w(x_0)^2\,\Delta v(x_0) - w(x_0)\,v(x_0)\,\Delta w(x_0) \ +2v(x_0)\sum_{i=1}^{n}\left(\frac{\partial w(x_0)}{\partial x_i}\right)^2\right] \\ &\leq \frac{v(x_0)}{w(x_0)^3}\left[2\sum_{i=1}^{n}\left(\frac{\partial w(x_0)}{\partial x_i}\right)^2 - w(x_0)\,\Delta w(x_0)\right] \leq \omega\, u(x_0).\end{aligned}$$

Hence $(\Delta, D_w(\Delta))$ verifies the g.p.m. principle with respect to ω.
On the other hand, setting

$$D_0 := \{u \in C_0(\mathbb{R}^n) \cap C^2(\mathbb{R}^n) : \Delta u \in C_0(\mathbb{R}^n)\},$$

then $D_0 \subset D_w(\Delta) \cap C_0(\mathbb{R}^n)$, $\Delta(D_0) \subset C_0(\mathbb{R}^n)$ and $(\Delta\mid_{D_0}, D_0)$ is closable and its closure generates a Feller semigroup on $C_0(\mathbb{R}^n)$ (see, e.g., [12, p.15]). Therefore, the result follows from Theorem 2. $\boxed{QED}$

Remarks 2. (1) Condition (13) is satisfied, e.g., by the weight

$$w(x) := \frac{1}{1+\|x\|^{2m}} \qquad (x \in \mathbb{R}^n,\ m \geq 1).$$

In this case it is not difficult to show that

$$\omega \leq \max\{2(2(m-1)+n)(m-1), 2n\}.$$

(2) A result similar to Theorem 3 holds true by replacing $\mathbb{R}^n$ with an arbitrary bounded open subset of $\mathbb{R}^n$ on which the classical Dirichlet problem is solvable. The relevant proof is the same as the previous one by referring, in this case, to Corollary 2.3 of [12].

2 On a class of degenerate second-order differential operators on $[0,+\infty[$

In this section we shall apply the general results of Section 2 to study a class of degenerate second-order differential operators acting on weighted continuous function spaces defined on the interval $[0,+\infty[$. Our results should also be compared with similar ones obtained in [2], [7], [8], [19], in order to have a quite complete picture of the results that are available in this field.

In particular we point out that the main results of [2] can be obtained more directly by using the methods of Section 2. For the sake of brevity we omit the detail and we proceed to show some new results that differ from those of [2] only when the underlying interval is of the form $[a,+\infty[$ or $]-\infty,a]$.

Without no loss of generality we shall therefore consider the interval $[0,+\infty[$. Consider $w \in C_b([0,+\infty[) \cap C^2(]0,+\infty[)$ such that $w(x) > 0$ for every $x \geq 0$. Consider also $\alpha, \beta \in C(]0,+\infty[)$, $\gamma \in C_b([0,+\infty[)$ and assume that

$$\alpha(x) > 0 \qquad \text{for every } x > 0. \tag{16}$$

For every $u \in C^2(]0,+\infty[)$ set

$$Au(x) := \alpha(x)u''(x) + \beta(x)u'(x) \quad (x > 0) \tag{17}$$

and consider the maximal domain $D_M(A)$ consisting of those functions $u \in C_0^w([0,+\infty[) \cap C^2(]0,+\infty[)$ such that

$$\lim_{x\to 0^+} Au(x) \in \mathbb{R} \quad \text{and} \quad \lim_{x\to+\infty} w(x)Au(x) = 0. \tag{18}$$

If $u \in D_M(A)$, then Au continuously extends to a function in $C_0^w([0,+\infty[)$ that we shall continue to denote by Au.
Finally consider the complete operator $L : D_M(L) \longrightarrow C_0^w([0,+\infty[)$ defined by

$$Lu := Au + \gamma u \tag{19}$$

for every $u \in D_M(L) := D_M(A)$. Thus, if $u \in D_M(L)$, then

$$Lu(x) = \begin{cases} \alpha(x)u''(x) + \beta(x)u'(x) + \gamma(x)u(x) & \text{if } x > 0, \\ \lim\limits_{t\to 0^+} \alpha(t)u''(t) + \beta(t)u'(t) + \gamma(0)u(0) & \text{if } x = 0. \end{cases} \tag{20}$$

In order to provide a generation result for the operator $(L, D_M(L))$ we fix some conditions on α, β, γ and w. First we assume that

$$\lim_{x\to 0^+} \frac{\alpha(x)\left(2w'(x)^2 - w''(x)w(x)\right) - \beta(x)w(x)w'(x)}{w(x)^2} \in \mathbb{R}. \tag{21}$$

Moreover, given $x_0 > 0$ and denoted by

$$W(x) := \exp\left(-\int_{x_0}^{x} \frac{\beta(t)}{\alpha(t)}\, dt\right) \qquad (x > 0) \tag{22}$$

the so-called Wronskian, consider the following properties:

(M_0) $\displaystyle\int_0^{x_0} W(x) \left(\int_x^{x_0} \frac{1}{\alpha(t)\, W(t)}\, dt\right) dx = +\infty,$

(M_0^w) $\displaystyle\int_0^{x_0} w(x)^2\, W(x) \left(\int_x^{x_0} \frac{1}{\alpha(t)\, w(t)^2\, W(t)}\, dt\right) dx = +\infty,$

(V_∞) $\displaystyle\int_{x_0}^{+\infty} W(x)\, dx < +\infty$ or $\displaystyle\int_{x_0}^{+\infty} \frac{1}{\alpha(x)\, W(x)} \left(\int_{x_0}^{x} W(t)\, dt\right) dx = +\infty$

or both,

(V_∞^w) $\displaystyle\int_{x_0}^{+\infty} w(x)^2 W(x) dx < +\infty$ or

$\displaystyle\int_{x_0}^{+\infty} \frac{1}{\alpha(x)w(x)^2 W(x)} \left(\int_{x_0}^{x} w(t)^2 W(t) dt\right) dx = +\infty$ or both.

Finally assume that

$$\overline{\omega} := \sup_{x>0} \frac{\alpha(x)\left(2w'(x)^2 - w''(x)w(x)\right) - \beta(x)w(x)w'(x)}{w(x)^2} < +\infty \tag{23}$$

and consider the following domains

$$D_{MV}(A) := \{u \in C_*([0, +\infty[) \cap C^2(]0, +\infty[) : \lim_{x\to 0^+} Au(x) \in \mathbb{R} \text{ and } \lim_{x\to +\infty} Au(x) = 0\}$$

where $C_*([0, +\infty[) = \{u \in C([0, +\infty[) : \lim_{x\to+\infty} f(x) \in \mathbb{R}\}$, and

$$D^0_{MV}(A) := D_{MV}(A) \cap C_0([0, +\infty[). \tag{24}$$

Proposition 1. *Under assumption (16), if properties (M_0) and (V_∞) hold true, then $(A, D_{MV}(A))$ and $(A, D^0_{MV}(A))$ generate Feller semigroups on $C_*([0, +\infty[)$ and $C_0([0, +\infty[)$, respectively.*

PROOF. By [21, Theorem 4, Lemma 8 and Lemma 9] we know that $(A, D_{MV}(A))$ generates a Feller semigroup on $C_*([0, +\infty[)$ and, hence, it satisfies the positive maximum principle. Since $D^0_{MV}(A) \subset D_{MV}(A)$, $(A, D^0_{MV}(A))$ satisfies the positive maximum principle as well.

Note also that the subalgebra of all functions $u \in C_0([0, +\infty[) \cap C^2(]0, +\infty[)$ that are constant on a neighborhood of 0 and vanish in a neighborhood of $+\infty$ is contained in $D^0_{MV}(A)$ and it is dense in $C_0([0, +\infty[)$ by Stone-Weierstrass theorem. Therefore, $D^0_{MV}(A)$ is dense in $C_0([0, +\infty[)$ as well.

Finally, given $\lambda > 0$ and $f \in C_0([0, +\infty[)$, there exists $u \in D_{MV}(A)$ such that $\lambda u - Au = f$. Since $Au \in C_0([0, +\infty[)$ it turns out that $u \in C_0([0, +\infty[)$ and hence $u \in D^0_{MV}(A)$.

In conclusion, we have proved that $(\lambda I - A)(D^0_{MV}(A)) = C_0([0, +\infty[)$ and hence the result follows by [11, Chapter 0, pp. 386-388]. $\boxed{QED}$

We are now in the position to prove the main result of this section.

Theorem 4. *Assume that properties* (M_0^w) *and* (V_∞^w) *are satisfied. Under assumption (16), (21) and (23), the operator* $(L, D_M(L))$ *is the generator of a positive* C_0*-semigroup on* $C_0^w([0,+\infty[)$ *satisfying (10) with respect to* $\omega = \overline{\omega} + \gamma_\infty$ *where* $\gamma_\infty := \sup_{x\geq 0} \gamma(x)$ *and* $\overline{\omega}$ *is defined by (23).*

Moreover, if $\gamma \leq 0$ *and both properties* (M_0) *and* (V_∞) *are true, then the above semigroup satisfies the Feller property with respect to* $C_0([0,+\infty[)$*, and (10) with respect to* $\omega = \overline{\omega}$.

In such a case, if in addition $\gamma \in C_*([0,+\infty[)$ *and* $w \in C_0([0,+\infty[)$*, then the semigroup verifies the Feller property with respect to* $C_*([0,+\infty[)$ *as well.*

Proof. Consider the lattice isomorphism $\Phi : C_0^w([0,+\infty[) \longrightarrow C_0([0,+\infty[)$ defined by $\Phi(f) := wf$ $(f \in C_0^w([0,+\infty[)$. Then the operator $(A, D_M(A))$ defined by (17) and (18) is transformed by the similarity induced by Φ to the operator $\widetilde{A} : D(\widetilde{A}) \subset C_0([0,+\infty[) \longrightarrow C_0([0,+\infty[)$ defined by

$$\widetilde{A}v(x) := \begin{cases} \widetilde{\alpha}(x)v''(x) + \widetilde{\beta}(x)v'(x) + \widetilde{\gamma}(x)v(x) & \text{if } x > 0, \\ \lim_{t\to 0^+} \widetilde{\alpha}(t)v''(t) + \widetilde{\beta}(t)v'(t) + \widetilde{\gamma}(0)v(0) & \text{if } x = 0, \end{cases}$$

for every $x \geq 0$ and for every $v \in D(\widetilde{A})$, where

$$D(\widetilde{A}) := \{v \in C_0([0,+\infty[) \cap C^2(]0,+\infty[) : \lim_{x\to 0^+} \widetilde{\alpha}(x)v''(x) + \widetilde{\beta}(x)v'(x) \in \mathbb{R} \text{ and } \lim_{x\to+\infty} \widetilde{\alpha}(x)v''(x) + \widetilde{\beta}(x)v'(x) = 0\},$$

$$\widetilde{\alpha}(x) = \alpha(x),\ \widetilde{\beta}(x) = \frac{\beta(x)w(x) - 2\alpha(x)w'(x)}{w(x)}$$

and

$$\widetilde{\gamma}(x) = \frac{\alpha(x)\,[2w'(x)^2 - w''(x)w(x)] - \beta(x)\,w(x)\,w'(x)}{w(x)^2}$$

for every $x > 0$. In other words,

$$D(\widetilde{A}) = \{v \in C_0([0,+\infty[) : \frac{v}{w} \in D_M(A)\}$$

and

$$\widetilde{A} = w\,A\left(\frac{v}{w}\right) \text{ for every } v \in D(\widetilde{A}).$$

Note that the Wronskian $\widetilde{W}$ associated with $\widetilde{A}$ is given by

$$\widetilde{W}(x) = \exp\left(-\int_{x_0}^{x} \frac{\widetilde{\beta}(t)}{\widetilde{\alpha}(t)}\,dt\right) = \frac{W(x)\,w(x)^2}{w(x_0)^2} \qquad (x > 0)$$

where W is defined by (22). Furthermore, by (21) and (23), $\widetilde{\gamma} \in C_b([0,+\infty[)$ and hence the bounded linear operator $\widetilde{B}v := \widetilde{\gamma}v$ $(v \in C_0([0,+\infty[))$ satisfies the g.p.m. principle with respect to $\overline{\omega}$.

Conditions (M_0^w) and (V_∞^w), together with Proposition 1 applied to $(\widetilde{A} - \widetilde{B}, D(\widetilde{A}))$, guarantee that this operator generates a Feller semigroup on $C_0([0,+\infty[)$ and, hence, the operator $\widetilde{A} = (\widetilde{A} - \widetilde{B}) + \widetilde{B}$ defined on $D(\widetilde{A})$ generates a positive C_0-semigroup $(\widetilde{T}(t))_{t\geq 0}$ on $C_0([0,+\infty[)$ such that

$$\|\widetilde{T}(t)\| \leq \exp(\overline{\omega}\,t) \qquad \text{for every } t \geq 0.$$

By similarity (see, e.g., [16, p. 43 and p. 59]) the operator $(A, D_M(A))$ generates a positive C_0-semigroup on $C_0^w([0,+\infty[)$ satisfying the same estimates as above.
Considering now the bounded linear operator $Bu := \gamma\, u$ $(u \in C_0^w([0,+\infty[))$ since $L = A + B$ and B verifies the g.p.m. principle with respect to $\gamma_\infty := \sup_{x>0} \gamma(x)$, then the first part of the theorem follows from Corollary 2.5 of [1].

As regards the second part, under the assumptions (M_0) and (V_∞), by Proposition 1 and Corollary 1, $(A, D(A))$ and B verify all the hypotheses of Corollary 3 (see also Remark 1, 1.) and hence the result follows. $\boxed{QED}$

Remark 1. Note that, if w is decreasing, then (M_0) implies (M_0^w) and the second alternative in (V_∞) implies the corresponding one in (V_∞^w).

We end the paper by showing an example where Theorem 4 can be applied.

Example 2. Consider the differential operator defined as in (20) with

$$\alpha(x) = a\,x^2, \quad \beta(x) = b\,x, \quad w(x) = \frac{1}{1+x^m} \quad (x \geq 0) \tag{25}$$

where $a > 0$, $b \in \mathbb{R}$ and $m \geq 1$, and $\gamma \in C_b([0,+\infty[)$.
Note that the above class of differential operators includes in particular the one related to the Black-Scholes equation (see, e.g., [2, Section 4, (4.3)], [7, Section 3], [8, Section 3.4]).
If α, β and w are given by (25), then, for $x_0 = 1$, the Wronskian turns into

$$W(x) = x^{-\frac{b}{a}} \qquad (x > 0) \tag{26}$$

and (21) and (23) can be easily verified (see also [2, Remark 2.4]). Moreover,

$$\overline{\omega} < am(m-1) + bm. \tag{27}$$

Finally properties (M_0) and (V_∞) can be directly checked without no difficulties and, consequently, the second alternatives of (M_0^w) and (V_∞^w) hold true as well since w is decreasing (see Remark 1). Therefore, Theorem 4 applies to the differential operator

$$Lu(x) = \begin{cases} a\,x^2\,u''(x) + b\,x\,u'(x) + \gamma(x)u(x) & \text{if } x > 0, \\[2ex] \lim\limits_{t \to 0^+} a\,t^2\,u''(t) + b\,t\,u'(t) + \gamma(0)u(0) & \text{if } x = 0, \end{cases} \tag{28}$$

defined on the domain

$$\begin{aligned} D_M(L) = \{u \in C_0^w([0,+\infty[) \cap C^2(]0,+\infty[) : \lim_{x\to 0^+} a\,x^2\,u''(x) + b\,x\,u'(x) \in \mathbb{R} \\ \text{and} \lim_{x \to +\infty} \frac{a\,x^2\,u''(x) + b\,x\,u'(x)}{1+x^m} = 0\}. \end{aligned} \tag{29}$$

References

[1] F. Altomare, A. Attalienti: *Degenerate evolution equations in weighted continuous function spaces, Markov processes and the Black-Scholes equation-Part I,* Result. Math. **42** (2002), 193–211.

[2] F. Altomare, A. Attalienti: *Degenerate evolution equations in weighted continuous function spaces, Markov processes and the Black-Scholes equation-Part II,* Result. Math. **42** (2002), 212–228.

[3] F. Altomare, I. Carbone: *Markov processes and diffusion equations on unbounded intervals*, Taiwanese J. Math. no. 1 **5** (2001), 141–167.

[4] F. Altomare, V. Leonessa, S. Milella: *Cores for second-order differential operators on real intervals,* Commun. Appl. Anal. no. 4 **13** (2009), 477–496.

[5] F. Altomare, S. Milella, G. Musceo: *Multiplicative perturbations of the Laplacian and related approximation problems*, to appear in Journal of Evolution Equations, 2012, DOI: 10.1007/s00028-011-0110-6.

[6] F. Altomare, G. Musceo: *Markov processes and positive semigroups on some classes of weighted continuous function spaces*, Rend. Circ. Mat. Palermo, (2) **57**, (2008), 61–82.

[7] F. Altomare, G. Musceo: *Positive semigroups generated by degenerate second-order differential operators*, Funkcial. Ekvac. **51** (2008), 431–458.

[8] F. Altomare, G. Musceo: *Degenerate second order differential operators with generalized Reflecting Barriers boudary conditions* , Math. Reports, no. 2 **12** (62) (2010), 101–117.

[9] W. Arendt, A. Grabosch, G. Greiner, U. Groh, H. P. Lotz, U. Moustakas, R. Nagel (ed), F. Neubrander, U. Schlotterbeck: One-parameter Semigroups of Positive Operators, Lecture Notes in Math. **1184**, Springer-Verlag, Berlin, 1986.

[10] J. Banasiak, L. Arlotti: Perturbations of Positive Semigroups with Applications, Springer-Verlag, Berlin, 2006.

[11] J. M. Bony, P. Courrège, P. Priouret: *Semi-groupes de Feller sur une variété a borde compacte et problèmes aux limites intégro-différentiels du second ordre donnant lieu au principe du maximum*, Ann. Inst. Fourier, no. 2 **18**, (1968), 369–521.

[12] J. A. van Casteren: Generators of strongly continuous semigroups, Research Notes in Mathematics **115**, Pitman, Boston, 1985.

[13] Ph. Clément, H. J. A. M. Heijmans et al.: One-Parameter Semigroups, North-Holland Publ. Co., Amsterdam-New York, 1987.

[14] E. B. Davies: Linear operators and their spectra, Cambridge Studies in Advanced Mathematics **106**, Cambridge University Press, Cambridge, 2007.

[15] E. B. Dynkin: Markov Processes, Voll. I and II, Springer-Verlag, Berlin, 1965.

[16] K. J. Engel, R. Nagel: One Parameter Semigroups for Linear Evolution Equations, Springer-Verlag, New York, Inc., 2000.

[17] L. Lorenzi, M. Bertoldi: Analytical methods for Markov semigroups, Chapman and Hall/CRC, Boca Raton, 2007.

[18] G. Lumer: *Perturbation de générateurs infinitesimaux du type ≪Changement de temps≫*, Ann. Inst. Fourier (Grenoble), no. 4 **23** (1974), 271–273.

[19] S. Milella, G. Musceo: *On a class of degenerate evolution equations on* $[0, +\infty[$, preprint, 2010.

[20] K. Taira: Diffusion Processes and Partial Differential Equations, Academic Press, San Diego, CA, 1988.

[21] C. A. Timmermans: *On C_0-semigroups in a space of bounded continuous functions in the case of entrance or natural boundary points*, Approximation and Optimization, J. A. Gómez Fernández et al. (Eds), Lecture Notes in Math. **1354**, Springer, Berlin, 1988, 209–216.