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On homotopy Lie algebra structures in the rings of differential operators

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Abstract. We study the Schlessinger–Stasheff's homotopy Lie structures on the associative algebras of differential operators $\mathrm{Diff}_*(\Bbbk^n)$ w.r.t. n independent variables. The Wronskians are proved to provide the relations for the generators of these algebras; two remarkable identities for the Wronskian and the Vandermonde determinants are obtained.

We axiomize the idea of the Hochschild cohomologies and extend the group $\mathbb{Z}_2$ of signs $(-1)^\sigma$ to the circumpherence S^1. Then, the concept of associative homotopy Lie algebras admits nontrivial generalizations.

Keywords: SH algebras, differential operators, Wronskian determinants, CFT.

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Introduction.

In this paper, following [22, 11, 17, 19] we study a special case of N-ary generalizations of the Lie algebras, the *homotopy Lie algebra* structures introduced by Schlessinger and Stasheff [21], within the framework of [22] where the Jacobi identities for an N-linear skew-symmetric bracket are defined by means of the Richardson–Nijenhuis bracket, and using the jet bundles language [2] in order to represent these structures by the Wronskian determinants and the higher order differential operators.

The paper is organized as follows.

In Section 1, we introduce the main algebraic concept of the homotopy N-Lie algebras. We fix notation and define the Richardson–Nijenhuis bracket, the homotopy N-Jacobi identity, and the Hochschild and Koszul cohomologies. Also, we extend the concept of signs from $\pm 1 \in \mathbb{Z}_2$ to $\epsilon(\sigma) \in S^1$ for an associative

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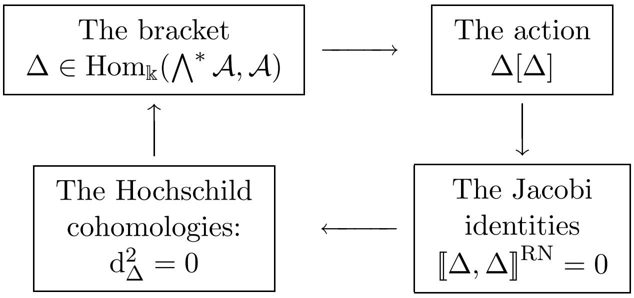
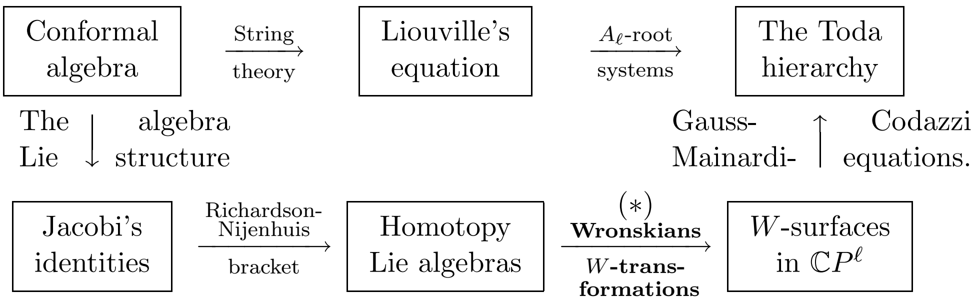
algebra by using the cohomological ideas:



This approach is aimed to contribute the study of related aspects in the cohomological algebra and the field theory.

In Section 2, we fix local coordinates and endow analytic functions $\Bbbk[[x^1, \ldots, x^n]]$ with the homotopy N-Lie algebra structure, such that the Wronskian determinant is the required N-linear skew-symmetric bracket. We discuss 3 basic examples: the finite-dimensional case of polynomials, the case of the Laurent series, and the multidimensional base case which gives us a natural scheme to generalize the notion of the Wronskian. To do that, we recollect basic facts about structures on jet spaces from the geometry of PDE.

In Section 3, we point out areas in mathematical physics where applications of the homotopy Lie structures are expected. First, we reveal the invariant nature of the objects discussed in Section 2: they turn to be the associative algebra of differential operators; also, we justify the concept that extends commutators of vector fields on $\mathbb{C}$ to a homotopy $2N$–Lie algebra of Nth order derivations. Hence, we construct the arrow $(*)$ in the diagram



The diagram brings together preceding researches [16] in geometry of the Liouville and Toda equations with the study [14]–[15] of the conformal Lie algebra generalizations and their representations in differential operators.

Finally, we deal with the free field model from the 2-dimensional CFT, based on the conformal transformations algebra. We make a conjecture that relates the associative algebra of energy–momentum operators, the Ward identities, and the Virasoro algebra, w.r.t. their homotopy Lie properties.

1 Preliminaries: the algebraic concept.

First let us introduce some notation. Let $\mathcal{A}$ be an associative algebra over the field $\Bbbk$ such that $\operatorname{char}\Bbbk = 0$, and let ∂ be a derivation of $\mathcal{A}$. As an illustrative example, one can think $\mathcal{A}$ to be the algebra of smooth functions $f\colon M \to \mathbb{R}$ on a smooth manifold M.

Let $S_m^k \subset S_m$ be the unshuffle permutations such that $\sigma(1) < \sigma(2) < \cdots < \sigma(k)$ and $\sigma(k+1) < \sigma(k+2) < \cdots < \sigma(m)$ for any $\sigma \in S_m^k$. Let $\Delta \in \operatorname{Hom}(\bigwedge^k \mathcal{A}, \mathcal{A})$, $\nabla \in \operatorname{Hom}(\bigwedge^l \mathcal{A}, \mathcal{A})$. By $\Delta[\nabla] \in \operatorname{Hom}(\bigwedge^{k+l-1} \mathcal{A}, \mathcal{A})$ we denote the *action* $\Delta[\cdot]\colon \operatorname{Hom}(\bigwedge^N \mathcal{A}, \mathcal{A}) \to \operatorname{Hom}(\bigwedge^{N+k-1} \mathcal{A}, \mathcal{A})$ of Δ on ∇:

$$\Delta[\nabla](a_1, \ldots, a_{k+l-1}) \stackrel{\text{def}}{=} \sum_{\sigma \in S_{k+l-1}^l} (-1)^\sigma\, \Delta(\nabla(a_{\sigma(1)}, \ldots, a_{\sigma(l)}), a_{\sigma(l+1)}, \ldots, a_{\sigma(k+l-1)}), \quad (1)$$

where $a_j \in \mathcal{A}$.

By $[\![\Delta, \nabla]\!]^{\mathrm{RN}} \in \operatorname{Hom}(\bigwedge^{k+l-1} \mathcal{A}, \mathcal{A})$ we denote the *Richardson–Nijenhuis bracket* of Δ and ∇:

$$[\![\Delta, \nabla]\!]^{\mathrm{RN}} \stackrel{\text{def}}{=} \Delta[\nabla] - (-1)^{(k-1)(l-1)} \nabla[\Delta]. \quad (2)$$

Let $\Delta \in \operatorname{Hom}(\bigwedge^k \mathcal{A}, \mathcal{A})$ and $a_j \in \mathcal{A}$, $1 \le j \le k$; suppose $1 \le l \le k$. The *inner product* $\Delta_{a_1,\ldots,a_l} \in \operatorname{Hom}(\bigwedge^{k-l} \mathcal{A}, \mathcal{A})$ is defined by

$$\Delta_{a_1,\ldots,a_l}(a_{l+1}, \ldots, a_k) \stackrel{\text{def}}{=} \Delta(a_1, \ldots, a_k).$$

1 Definition ([22]). Choose integers N, k, and r such that $0 \le r \le k < N$, and let $a_1, \ldots, a_r, b_1, \ldots, b_k \in \mathcal{A}$. The skew-symmetric map $\Delta \in \operatorname{Hom}(\bigwedge^N \mathcal{A}, \mathcal{A})$ is said to determine the Lie algebra structure of the type (N, k, r) on the $\Bbbk$-vector space $\mathcal{A}$ if the (N, k, r)-Jacobi identity

$$[\![\Delta_{a_1,\ldots,a_r}, \Delta_{b_1,\ldots,b_k}]\!]^{\mathrm{RN}} = 0 \quad (3)$$

holds for any $\vec{a}$ and $\vec{b}$. By $\mathrm{Lie}^{(N,k,r)}(\mathcal{A})$ we denote the set of all type (N, k, r) structures $\Delta \in \operatorname{Hom}(\bigwedge^N \mathcal{A}, \mathcal{A})$ on $\mathcal{A}$.

2 Remark. Note that

$$\mathrm{Lie}^{(N,0,0)}(\mathcal{A}) = \mathrm{Lie}^{(N,1,0)}(\mathcal{A}) \quad (4)$$

for any even N; this is a typical instance of the heredity structures [22]. Really, the following two conditions are equivalent:

$$[\![\Delta, \Delta]\!]^{\mathrm{RN}} = 0 \Leftrightarrow {[\![\Delta, \Delta]\!]^{\mathrm{RN}}}_a = -2\, [\![\Delta, \Delta_a]\!]^{\mathrm{RN}} = 0 \qquad \forall a \in \mathcal{A},$$

owing to Corollary 1.1 in [22]:

$$[\![\Delta,\Delta]\!]^{\mathrm{RN}}{}_a = (-1)^{N-1}[\![\Delta,\Delta_a]\!]^{\mathrm{RN}} + [\![\Delta_a,\Delta]\!]^{\mathrm{RN}}.$$

Finally, $[\![\Delta,\Delta_a]\!]^{\mathrm{RN}} = 0$ for any $a \in \mathcal{A}$.

The preceding papers [14, 15] should be regarded w.r.t. the isomorphism (4).

Let $\Delta \in \mathrm{Hom}(\bigwedge^N \mathcal{A}, \mathcal{A})$ be an N-linear skew-symmetric bracket: $\Delta(a_{\Sigma(1)}, \ldots, a_{\Sigma(N)}) = (-1)^{\Sigma}\Delta(a_1,\ldots,a_N)$ for any rearrangement $\Sigma \in S_N$.

3 Definition. The algebra $\mathcal{A}$ is the *homotopy* N*-Lie* algebra, if the N-Jacobi identity

$$\Delta[\Delta] = 0 \tag{5}$$

holds.

In coordinates, the N-Jacobi identity is

$$\sum_{\sigma\in S^N_{2N-1}} (-1)^{\sigma}\Delta(\Delta(a_{\sigma(1)},\ldots,a_{\sigma(N)}), a_{\sigma(N+1)},\ldots,a_{\sigma(2N-1)}) = 0 \tag{6}$$

for any $a_j \in \mathcal{A}$, $1 \le j \le 2N-1$. Generally, the number of summands in (6) is $\binom{2N-1}{N-1} = \binom{2N-1}{N}$, see [14].

Thus, the Jacobi identity of the type $(N,0,0)$ $[\![\Delta,\Delta]\!]^{\mathrm{RN}} = 2\Delta[\Delta] = 0$ implies Eq. (6) for any even N. If N is odd, then the expression

$$[\![\Delta,\Delta]\!]^{\mathrm{RN}} = 0 \tag{7}$$

is trivial, and we consider Eq. (5) separately from the condition (3).

4 Definition ([11]). Let $\mathcal{A}_* = \mathcal{A}_0 \oplus \mathcal{A}_1$ be a graded vector space. Let $\sigma \in S_N$ be a permutation and let $a_i \in \mathcal{A}_{\varepsilon_i}$ lie in the corresponding components of the algebra. Then the *Koszul sign* $\mathrm{e}(\sigma,\vec{a})$ is

$$\mathrm{e}(\sigma,\vec{a}) \equiv \prod_{\substack{i<j\\ \sigma(i)>\sigma(j)}} (-1)^{\varepsilon_{\sigma(i)}\cdot\varepsilon_{\sigma(j)}}.$$

5 Example. Let $\sigma = 321$, then $\mathrm{e}(\sigma,\vec{a}) = (-1)^{\varepsilon_3\varepsilon_2+\varepsilon_3\varepsilon_1+\varepsilon_2\varepsilon_1}$.

6 Definition ([1]). An SH-*Lie algebra structure* on a graded vector space $\mathcal{A}_*$ is a collection of linear, skew-symmetric maps $\Delta_N\colon \bigwedge^N \mathcal{A}_* \to \mathcal{A}_*$ such that the relations

$$\sum_{i+j=N'+1}\sum_{\sigma\in S^i_{N'}} \mathrm{e}(\sigma)(-1)^{\sigma}(-1)^{i(j-1)}\Delta_j(\Delta_i(a_{\sigma(1)},\ldots,a_{\sigma(i)}),\ldots,a_{\sigma(N')}) = 0$$

hold for any whole $N' \ge 2$ and i, $j \ge 1$.

Definition 6 means that we consider the cardinal set of the homotopy Lie structures at once.

7 Remark. There is a natural independent problem to describe relations between definition of the bracket Δ, definition of the action $\Delta[\Delta]$, and their correlation with the Jacobi identity $[\![\Delta,\Delta]\!]^{\text{RN}}=0$. Really, the signs $(-1)^\sigma$ in any of these constructions can be treated as variables, and having all objects well-defined will provide equations for these coefficients. That is exactly what we do in Eq. (17) and Eq. (18) on page 91.

In the sequel, we study the Jacobi identity (6) of the form (5), where $\Delta\in\operatorname{Hom}(\bigwedge^N\mathcal{A},\,\mathcal{A})$. We conclude that for even Ns the results will be equivalent to the properties of the type $(N,1,0)$ structures, while for odd Ns our results are independent from the scheme proposed in Definition 1.

The Hochschild and the Koszul cohomologies.

The graded Jacobi identity for the Richardson–Nijenhuis bracket provides the Hochschild d_Δ-cohomologies on $\operatorname{Hom}(\bigwedge^*\mathcal{A},\,\mathcal{A})$ for $\Delta\in\operatorname{Hom}(\bigwedge^k\mathcal{A},\,\mathcal{A})$, where k is even:

8 Proposition ([22]). *The Richardson-Nijenhuis bracket satisfies the graded Jacobi identity*

$$[\![\Delta,[\![\nabla,\Box]\!]^{\text{RN}}]\!]^{\text{RN}}=[\![[\![\Delta,\nabla]\!]^{\text{RN}},\Box]\!]^{\text{RN}}+(-1)^{(\Delta-1)(\nabla-1)}[\![\nabla,[\![\Delta,\Box]\!]^{\text{RN}}]\!]^{\text{RN}}.$$

9 Corollary. *Let k be even and an operator $\Delta\in\operatorname{Hom}(\bigwedge^k\mathcal{A},\mathcal{A})$ be such that $[\![\Delta,\Delta]\!]^{\text{RN}}=0$; then the operator $\mathrm{d}_\Delta\equiv[\![\Delta,\cdot]\!]^{\text{RN}}$ is a differential: $\mathrm{d}_\Delta^2=0$.*

Let the bracket $\Delta\in\operatorname{Hom}(\bigwedge^k\mathcal{A},\mathcal{A})$ satisfy the homotopy k-Jacobi identity $\Delta[\Delta]=0$. By ∂_Δ denote the linear map such that $\partial_\Delta\in\operatorname{Hom}(\bigwedge^r\mathcal{A},\bigwedge^{r-k+1}\mathcal{A})$ and

(1) $\partial_\Delta\Big|_{\bigwedge^r\mathcal{A}}=0$ if $r<k$;

(2) $\partial_\Delta(a_1\wedge\ldots\wedge a_r)=\sum\limits_{\sigma\in S_r^k}(-1)^\sigma\Delta[a_{\sigma(1)},\ldots,a_{\sigma(k)}]\wedge a_{\sigma(k+1)}\wedge\ldots\wedge a_{\sigma(r)}$ otherwise.

We obtain the Koszul ∂_Δ-cohomologies for the algebra $\bigwedge^*\mathcal{A}=\bigoplus_{r=2}^\infty\bigwedge^r\mathcal{A}$ over the algebra $\mathcal{A}$ owing to

10 Proposition ([11]). *The operator $\partial_\Delta\colon\bigwedge^*\mathcal{A}\to\bigwedge^*\mathcal{A}$ is a differential: $\partial_\Delta^2=0$.*

By $H^*_\Delta(\mathcal{A})$ we define the Koszul ∂_Δ-cohomologies w.r.t. the differential ∂_Δ. For $N = 2$, the Koszul cohomologies of the Lie algebra of vector fields on the circumpherence S^1 were obtained in [7]. For $N \geq 2$, the Koszul ∂_Δ-cohomologies of free algebras were found in [11].

Examples.

One should notice that algebraic structures (5) have a remarkable geometric motivation to exist. Namely, we have

11 Example ([11]). Let $\mathcal{A} = \Bbbk^{\dim\mathcal{A}}$ and $\Delta\colon \bigwedge^N \mathcal{A} \to \mathcal{A}$ be a skew-symmetric linear mapping of the linear $\Bbbk$-space. If $\dim\mathcal{A} < 2N - 1$, then the identity (6) holds.

In [11], this example is referred as a private communication, and no proof is given there. We offer an easy proof of the example's statement.

Proof. We maximize the number of summands in (6) in order to note its skew-symmetry w.r.t. the transpositions $a_j \mapsto a_{\Sigma(j)}$, $\Sigma \in S_{2N-1}$.

The l.h.s. of Jacobi identity (6) equals

$$\frac{1}{N!(N-1)!}\sum_{\sigma\in S_{2N-1}}(-1)^\sigma \Delta(\Delta(a_{\sigma(1)},\ldots,a_{\sigma(N)}),a_{\sigma(N+1)},\ldots,a_{\sigma(2N-1)}), \quad (8)$$

where *all* elements $\sigma \in S_{2N-1}$ are taken into consideration; see [11, 19]. Expression (8) is skew-symmetric w.r.t. any rearrangement Σ of the elements $a_j \in \vec{a}$:

$$\sum_{\sigma\in S_{2N-1}}(-1)^\sigma \Delta(\Delta(a_{(\sigma\circ\Sigma)(1)},\ldots,a_{(\sigma\circ\Sigma)(N)}),a_{(\sigma\circ\Sigma)(N+1)},\ldots,a_{(\sigma\circ\Sigma)(2N-1)}) =$$
$$= \frac{1}{(-1)^\Sigma}\sum_{\sigma\in S_{2N-1}}(-1)^\sigma \Delta(\Delta(a_{\sigma(1)},\ldots,a_{\sigma(N)}),a_{\sigma(N+1)},\ldots,a_{\sigma(2N-1)}). \quad (9)$$

Consequently, the l.h.s in (8) is skew-symmetric also, and we obtain a $(2N-1)$-linear skew-symmetric operator acting on the vector space of smaller dimension. Thence if $\dim\mathcal{A} < 2N-1 =$ the number of arguments $\sharp\vec{a}$, then (6) holds. QED

• The case $\dim\mathcal{A}_1 = N + 1$, $[a_0, \ldots, \widehat{a_j}, \ldots, a_N] = (-1)^j \cdot a_j$ is well-known to be the cross-product in $\Bbbk^{N+1}$. For $\Bbbk = \mathbb{R}$ and $N = 2$, we have the Lie algebra $\mathfrak{so}(3)$.

• In Theorem 19, we prove the case $\dim\mathcal{A}_2 = N + 1$, $[a_0, \ldots, \widehat{a_j}, \ldots, a_N] = a_{N-j}$, to admit a representation by the Wronskian determinants of scalar fields (smooth functions of one argument). The algebra $\mathcal{A}_2$ will be studied in Section 2.1.

If $\Bbbk = \mathbb{R}$ and $N = 2$, then the algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ are not isomorphic since $\mathcal{A}_1 \simeq \mathfrak{so}(3)$ is simple and $\mathcal{A}_2$ contains two subalgebras: $\operatorname{span}\langle a_0, a_1\rangle$ and $\operatorname{span}\langle a_1, a_2\rangle$. If $N > 2$, then $\mathcal{A}_1$ and $\mathcal{A}_2$ can be deformed one into another; nevertheless, this deformation will not be a homomorphism of the homotopy Lie algebras. Subalgebras $\mathcal{A} \subset \Bbbk[[x]]$ other than $\mathcal{A}_2$ will be discussed in Remark 26.

Associative algebras and the new concept of signs.

Another natural example of the homotopy Lie algebras is given by

12 Proposition ([4, 11]). *Let $\mathcal{A}$ be an associative algebra and let N be even*: $N \equiv 0 \mod 2$; *by definition, put*[1]

$$[a_1, \ldots, a_N] \stackrel{\text{def}}{=} \sum_{\sigma \in S_N} (-1)^\sigma \cdot a_{\sigma(1)} \circ \cdots \circ a_{\sigma(N)}. \tag{10}$$

Then $\mathcal{A}$ is a homotopy Lie algebra w.r.t. this bracket.

Proof ([11]). The crucial idea is using (8) and (9). Let a_1, …, a_{2N-1} lie in $\mathcal{A}$ and $\sigma \in S_{2N-1}$ be a permutation. In order to compute the coefficient of $a_{\sigma(1)} \circ \cdots \circ a_{\sigma(2N-1)}$ in (6) and prove it to be trivial, it is enough to do that for $\alpha = a_1 \circ \cdots \circ a_{2N-1}$ in (6) owing to (9) and (8), successively.

Now we use the assumption $N \equiv 0 \mod 2$. The product α is met N times in (6) in the summands β_j, $1 \leq j \leq N$:

$$\beta_j = (-1)^{N(j-1)} \,[[a_j, \ldots, a_{N+j-1}], a_1, \ldots, a_{j-1}, a_{N+j}, \ldots, a_{2N-1}]. \tag{11}$$

The coefficient of α in β_j equals $(-1)^{j-1}$, and thus the coefficient of α in (6) is

$$\sum_{j=1}^{N} (-1)^{j-1} = 0.$$

The proof is complete. QED

From the proof of Proposition 12 we see that the main obstacle for bracket (10) to provide the homotopy Lie algebra structures for odd Ns are the signs within (10), (1), and in the Richardson–Nijenhuis bracket (2) that defines the Jacobi identity as the cohomological conditions $\mathrm{d}_\Delta^2 = 0$, see Proposition 8. Namely, we have

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[1] Please note that the permutations $\sigma \in S_N$ provide the *direct* left action on $\bigotimes^N \mathcal{A}$, contrary to the inverse action in [13, §II.2.6]. Really, we use σs as shown in Example 5. Thus, by definition, $\sigma(j)$ is the index of the object in an initial ordered set, placed onto jth position after the left action of a permutation σ.

13 Proposition ([4]). *Let the subscript i at the bracket's* (10) *symbol Δ_i denote its number of arguments: $\Delta_i \in \mathrm{Hom}_{\Bbbk}(\bigwedge^i \mathcal{A}, \mathcal{A})$, and let k and ℓ be arbitrary integers. Then the identities*

$$\Delta_{2k}[\Delta_{2\ell}] = 0, \tag{12}$$

$$\Delta_{2k+1}[\Delta_{2\ell}] = \Delta_{2k+2\ell}, \tag{13}$$

$$\Delta_k[\Delta_{2\ell+1}] = k \cdot \Delta_{2\ell+k} \tag{14}$$

hold.

PROOF. The proof of (12) repeats the reasoning in (11) literally. For (13), we note that there is the last summand β_{2k+1} that is not compensated. For (14), the summand $\alpha = a_1 \circ \cdots \circ a_{2\ell+k}$ acquires the coefficient

$$\sum_{j=1}^{k} (-1)^{(2\ell+1)(j-1)} \cdot (-1)^{j-1} = k.$$

This completes the proof. QED

In fact, *a priori* there is no reason for the signs $(-1)^\sigma$ to be preserved, say, for odd Ns, if the condition $\mathrm{d}^2 = 0$ is our target and there is another concept that suits better. For -1 such that $(-1)^2 = 1$, the main technical difficulty is: do we multiply or divide by -1?

In this section, we propose non-standard 'signs' such that all cohomological theory is preserved and the theory for odd Ns is nontrivial. By 'nontrivial' we mean that condition (7) assumed in Proposition 12 holds not because $\Delta[\Delta] - (-1)^{(\text{any odd } -1)^2}\Delta[\Delta] \equiv 0$, but owing to Eq. (5).

First, we fix notation concerning graded algebras; to do that, we follow [3].

Let G be an abelian group. A vector space V is called *G-graded* if $V = \bigoplus_{g\in G} V_g$ for some vector spaces V_g, and for any V_g its *grading* $\mathrm{gr}\, V_g = g$. Vectors $v \in V_g$ are called *homogeneous* of degree g, and then $\mathrm{gr}\, V_g$ is also denoted by $\bar{v}$. Further on, any element $v \in V$ is assumed to be homogeneous.

A $\Bbbk$-algebra $\mathcal{A}_*$ is called *G-graded* if $\mathcal{A}_*$ is a G-graded vector space and $\mathcal{A}_{g_1} \circ \mathcal{A}_{g_2} \subset \mathcal{A}_{g_1+g_2}$ for any g_1, $g_2 \in G$.

A *sign* $\{\cdot,\cdot\}$ is a coupling $G \times G \to \Bbbk \setminus \{0\} \colon (g_1, g_2) \mapsto \{g_1, g_2\}$ such that the conditions

$$\{g_1, g_2\}^{-1} = \{g_2, g_1\}, \qquad \{g_1 + g_2, g_3\} = \{g_1, g_3\} \cdot \{g_2, g_3\} \tag{15}$$

hold for any g_1, g_2, $g_3 \in G$. For an element $g = (g_1, \ldots, g_N) \in G^N = G \oplus \cdots \oplus G$ there is a unique function $\epsilon_g \colon S_N \to \Bbbk \setminus \{0\}$ such that

$$\epsilon(\sigma_{i,i+1}) = \{g_i, g_{i+1}\} \text{ for the transposition } \sigma_{i,i+1} \equiv (i, i+1);$$

$$\epsilon_g(\sigma \circ \Sigma) = \epsilon_{\Sigma(g)}(\sigma) \cdot \epsilon_g(\Sigma),$$

where $\sigma \in S_N$ and $\Sigma(g) = (g_{\Sigma(1)}, \ldots, g_{\Sigma(N)})$. For the super-commutation factor $\{\bar a, \bar b\} = (-1)^{\bar a \bar b}$ the function ϵ_g does not depend on g.

14 Remark. Condition (15) implies an analog of (8) w.r.t. $\epsilon_g(\sigma)$: $\sum_{S^N_{2N-1}} = (N!(N-1)!)^{-1} \sum_{S_{2N-1}}$.

Now we introduce our concept for the complex field $\Bbbk = \mathbb{C}$. Let $p \in \mathbb{N}$ be an arbitrary natural greater than 1 and the grading group be $\mathbb{Z}_{p^{2N}}$. Let $\mathcal{A}_*$ be the free algebra with $2N-1$ generators a_1, …, a_{2N-1} of the degrees $\bar a_j = p^j$, respectively.

By q we denote the primitive Qth root of 1 in $\mathbb{C}$:

$$q = \exp\left(\frac{2\pi i}{Q}\right).$$

Let $a \in \mathcal{A}_*$ be an element such that $\bar a = \bar a^{(2N-1)} \ldots \bar a^{(1)} \bar a^{(0)}$, where $\bar a^{(k)}$ are digits $0 \le \bar a^{(k)} < p$. Then the sign $\{\bar a, \bar a_j\}$ is

$$\{\bar a, \bar a_j\} = q^{\sum_{k<j} \bar a^{(k)} - \sum_{k>j} \bar a^{(k)}}; \tag{16}$$

condition (15) defines the sign $\{\bar a_j, \bar a\} \stackrel{\text{def}}{=} \{\bar a, \bar a_j\}^{-1}$.

We note that the sign defined in (16) is such that $\sigma_{k,\ell} \circ \sigma_{k,\ell}(\bar a) = \bar a$ for any $a \in \mathcal{A}_*$, although $q \cdot q \neq 1$ if $Q > 2$.

Suppose b_1, …, $b_N \in \mathcal{A}_*$ are arbitrary elements; we define the bracket $\Delta \in \mathrm{Hom}_{\Bbbk}(\bigotimes^N \mathcal{A}_*, \mathcal{A}_*)$ by

$$\Delta(b_1, \ldots, b_N) = \sum_{\sigma \in S_N} \epsilon_{\bar b}(\sigma)\, b_{\sigma(1)} \circ \cdots \circ b_{\sigma(N)}. \tag{17}$$

Generally, the degree of the bracket itself $\mathrm{gr}\,\Delta \in \mathbb{Z}_p$ is not zero:

$$\mathrm{gr}\,\Delta(b_1, \ldots, b_N) = \sum_{j=1}^{N} \bar b_j + \mathrm{gr}\,\Delta;$$

from (16), we see that in fact the rightmost digit $\mathrm{gr}\,\Delta$ is always subtracted (see [17]).

Consider the elements a_j, $1 \le j \le 2N-1$. The Jacobi identity is

$$\sum_{\sigma \in S^N_{2N-1}} \epsilon(\sigma)\, \Delta(\Delta(a_{\sigma(1)}, \ldots, a_{\sigma(N)}), a_{\sigma(N+1)}, \ldots, a_{\sigma(2N-1)}) = 0. \tag{18}$$

This identity is the correlation condition on N, p, Q, and $\mathrm{gr}\,\Delta$.

15 Example. The collection

$$N = 2N' \in \mathbb{N}, \quad p = 2, \quad Q = 2, \quad \operatorname{gr}\Delta = 1 \tag{19}$$

is a solution to Eq. (18). The collection $N = 2N' \in \mathbb{N}$, $p = N$, $Q = 2$, and $\operatorname{gr}\Delta = N - 1$ is a solution, too.

Example 15 is an interpretation of Proposition 12. In this case, $\mathcal{A}_*$ is the Grassmann algebra: $a_j \circ a_j = 0$ and $a_k \circ a_\ell + a_\ell \circ a_k = 0$ if $k \neq \ell$.

16 Problem. Are there any other solutions $(N, p, Q, \operatorname{gr}\Delta)$ to Eq. (18)?

The property

$$\sum_{j=0}^{N-1} \exp\left(\frac{2\pi i}{N} j\right) = 0$$

is a motivation for other solutions to exist with $Q > 2$.

17 Remark. Treating the degree $\bar{a}$ of an element $a \in \mathcal{A}_*$ as a number $\bar{a}^{(0)}.\bar{a}^{(1)} \ldots \bar{a}^{(2N-1)}$, we see that for solution (19) the gradings exhaust the segment $[1, 2]$ as $N \to \infty$. If $Q \to \infty$, then the signs $\epsilon(\sigma)$ are dense in the circumpherence S^1.

In the reasonings above, we have preserved the cohomological concept w.r.t. new diffrential d_Δ that involves new signs ϵ.

2 Representations of the homotopy Lie algebras.

2.1 Finite-dimensional case: polynomials.

In this section, we construct finite-dimensional homotopy N-Lie generalizations of the Lie algebra $\mathfrak{sl}_2(\Bbbk)$. Our starting point is the following

18 Example ([14]). The polynomials $\Bbbk_2[x] = \{\alpha x^2 + \beta x + \gamma \mid \alpha, \beta, \gamma \in \Bbbk\}$ of degree 2 with bracket (21) form a Lie algebra isomorphic to $\mathfrak{sl}_2(\Bbbk)$. The commutation relations can be easily checked in the basis $\langle 1, -2x, -x^2 \rangle$:

$$\begin{array}{ccc} [-2x, 1] = 2, & [-2x, -x^2] = 2x^2, & [1, -x^2] = -2x, \\ [h, e] = 2e, & [h, f] = -2f, & [e, f] = h, \end{array}$$

whence the representation $\rho\colon \mathfrak{sl}_2(\Bbbk) \to \Bbbk_2[x]$ is

$$\rho(e) = 1, \qquad \rho(h) = -2x, \qquad \text{and} \quad \rho(f) = -x^2. \tag{20}$$

Consider the space $\Bbbk_N[x] \ni a_j$ of polynomials a_j of degree not greater than N; on this space, there is the N-linear skew-symmetric bracket

$$[a_1, \ldots, a_N] = W(a_1, \ldots, a_N), \tag{21}$$

where W denotes the Wronskian determinant. Since N-ary bracket (21) is N-linear, we consider monomials const $\cdot x^k$ only. We choose $\{a_j^0\} = \{x^k\}$ or $\{a_j^0\} = \{x^k/k!\}$, where $0 \leq k \leq N$ and $1 \leq j \leq 2N-1$, for standard basis in $\Bbbk_N[x]$. Please do not mix the powers $x^0, \ldots, x^N$ and n independent variables $x^1, \ldots, x^n$ introduced in Section 2.3; nevertheless, the notation is absolutely clear from the context. Exact choice of the basis depends on the situation: the monomials x^k are used to demonstrate the presence or absence of certain degrees in N-linear bracket (21), and the monomials $x^k/k!$ are convenient in calculations since they are closed w.r.t. the derivations (and the Wronskian determinants as well).

19 Theorem ([15]). *Let $0 \leq k \leq N$; then the relation*

$$W\left(1, \ldots, \widehat{\frac{x^k}{k!}}, \ldots, \frac{x^N}{N!}\right) = \frac{x^{N-k}}{(N-k)!} \tag{22}$$

holds.

Proof. We have

$$W\left(1, \ldots, \widehat{\frac{x^k}{k!}}, \ldots, \frac{x^N}{N!}\right) = W\left(1, \ldots, \frac{x^{k-1}}{(k-1)!}\right) \cdot W\left(x, \ldots, \frac{x^{N-k}}{(N-k)!}\right), \tag{23}$$

where the first factor in the r.h.s. of (23) equals 1 and has the degree 0. Denote the second factor, the determinant of the $(N-k) \times (N-k)$ matrix, by W_m, $m \equiv N-k$. We claim that W_m is a monomial: $\deg W_m = m$, and prove this fact by induction on $m \equiv N-k$. For $m = 1$, $\deg \det(x) = 1 = m$. Let $m > 1$; the decomposition of W_m w.r.t. the last row gives

$$W_m = W\left(x, \ldots, \frac{x^m}{m!}\right) = x \cdot W\left(x, \ldots, \frac{x^{m-1}}{(m-1)!}\right) - W\left(x, \ldots, \frac{x^{m-2}}{(m-2)!}, \frac{x^m}{m!}\right), \tag{24}$$

where the degree of the first Wronskian in r.h.s. of (24) is $m-1$ by the inductive assumption. Again, decompose the second Wronskian in r.h.s. of (24) w.r.t. the last row and proceed so iteratively using the induction hypothesis. We obtain the recurrence relation

$$W_m = \sum_{l=1}^{m-1} W_{m-l} \cdot (-1)^{l+1} \frac{x^l}{l!} - (-1)^m \frac{x^m}{m!}, \qquad m \geq 1, \tag{25}$$

whence $\deg W_m = m$. We see that the initial Wronskian (23) is a monomial itself of degree $m = N-k$ with yet unknown coefficient.

Now we calculate the coefficient $W_m(x)/x^m \in \Bbbk$ in the Wronskian determinant (23). Consider the generating function

$$f(x) \equiv \sum_{m=1}^{\infty} W_m(x) \tag{26}$$

such that

$$W_m(x) = \frac{x^m}{m!}\,\frac{d^m f}{dx^m}(0), \qquad 1 \le m \in \mathbb{N}.$$

Note that $\exp(x) \equiv \sum_{m=0}^{\infty} x^m/m!$; treating (26) as the formal sum of equations (25), we have

$$f(x) = f(x)\cdot(\exp(-x)-1) - \exp(-x) + 1,$$

whence

$$f(x) = \exp(x) - 1. \tag{27}$$

The proof is complete. QED

20 Definition. The system $(a_1, \ldots, a_{2N-1})$ of monomials is called *the standard system of monomials* $\vec{a}(k_1,k_2)$ if $0 \le \deg a_j \le N$ for all $1 \le j \le 2N-1$ and the set $\{a_j\}$ of elements a_j is

$$\{a_j\} = \{1,\ldots,x^N\} \cup \{1,\ldots,x^N\} \setminus \{x^{k_1}, x^{k_2}, x^{N-k_1}\} \quad \text{where} \quad 0 \le k_1, k_2 \le N. \tag{28}$$

By $J^{N,\infty}_{\vec{a}}(x)$ we denote the l.h.s. in (6) with N-ary bracket (21) acting on analytic functions a_j of $x \in \Bbbk$, and by $J^N_{\vec{a}(k_1,k_2)}(x)$ we denote the same l.h.s. in (6) with N-ary bracket (21) acting on the standard system of monomials $\vec{a}(k_1,k_2)$.

21 Remark ([15]). The number of summands $\sharp J^N_{\vec{a}(k_1,k_2)}(x) = 2^{N-1}$.

Proof. Consider three cases: $k_1 \ne k_2 \ne N-k_1$, $k_2 = k_1$, and $k_2 = N-k_1$.

Case 1. Let $k_1 \ne k_2 \ne N-k_1$; then there are two types of the internal Wronskians in (6). We say that the summands with the internal Wronskians $W\left(\ldots, \widehat{x^{k_1}}, x^{k_2}, x^{N-k_1}, \ldots\right)$ belong to type 1, and the summands containing $W\left(\ldots, x^{k_1}, x^{k_2}, \widehat{x^{N-k_1}}, \ldots\right)$ belong to type 2, where $k_2 \ne N-k_1$ and $k_2 \ne k_1$ by assumption. Other elements x^m are met twice in the set $\{a_j\}$ and their choice for position into the internal (resp., external) Wronskian is arbitrary. So, we have 2 types $\times$ 2^{N-2} arrangements $= 2^{N-1}$ summands.

Case 2. Let $k_1 = k_2 \ne N-k_1$; then there are 2^{N-1} type 1 internal Wronskians $W\Bigl(\ldots, \widehat{x^{k_1}}, x^{N-k_1}, \ldots\Bigr)$, necessarily containing the element x^{N-k_1}; the type 2 summands are not realized for this $\vec{a}(k_1,k_1)$ since these Wronskians are zero identically.

Case 3. This case $k_2 = N-k_1$ repeats literally Case 2 after the transposition type 1 $\leftrightarrow$ type 2.

Thus, in all cases the number of summands $\sharp J^N_{\vec{a}(k_1,k_2)}(x)$ equals 2^{N-1}. QED

22 Corollary. *By the Stirling formula,*

$$\frac{\sharp J_{\vec{a}}^{N,\infty}(x)}{\sharp J_{\vec{a}(k_1,k_2)}^{N}(x)} = \frac{\binom{2N-1}{N-1}}{2^{N-1}} \rightsquigarrow \frac{2^N}{\sqrt{\pi N}} \qquad \textit{as } N \to \infty.$$

We see that the numbers of summands grow exponentially but the case $0 \leq \deg a_j \leq N$ provides exponentially less summands than the general case of analytic functions.

We have shown that the polynomials $\Bbbk_N[x]$ are closed w.r.t. the Wronskian determinant, and we know that any N-linear skew-symmetric bracket Δ on $\Bbbk^{N+1}$ satisfies $\Delta[\Delta] = 0$. Thus, polynomials $\Bbbk_N[x]$ of degree not greater than N form the homotopy N-Lie algebra with N-linear skew-symmetric bracket (21) for any integer $N \geq 2$. In the sequel, we show that the Wronskian $W^{0,1,\ldots,N-1} \in \mathrm{Hom}(\bigwedge^N \Bbbk_N[x], \Bbbk_N[x])$ is the restriction of a *nontrivial* homotopy N-Lie bracket that lies in $\mathrm{Hom}(\bigwedge^N \Bbbk[[x]], \Bbbk[[x]])$.

Thus, we have generalized the Lie algebra $\mathfrak{sl}_2(\Bbbk)$ of quadratic polynomials $\Bbbk_2[x]$ to the homotopy N-Lie algebra of the Nth degree polynomials $\Bbbk_N[x]$ for arbitrary integer $N \geq 2$. Still, the dimension n of the base $\Bbbk \equiv \Bbbk^1 \ni x$ equals 1. In Section 2.3, we generalize the concept to the case $x \in \Bbbk^n$, where integer $n \geq 1$ is arbitrary.

2.1.1 Deformations of the algebra and gauge theory at the limit $N \to \infty$.

Now we discuss the deformation properties of the systems guided by the Jacobi identity (6).

Consider the case given by (21). Let the rescaling be

$$a_j \mapsto \alpha_j a_j, \qquad 0 \leq j \leq N,$$

such that the relations

$$[\alpha_0 a_0, \ldots, \widehat{\alpha_j a_j}, \ldots, \alpha_N a_N] = \alpha_{N-j} a_{N-j}$$

hold for any j. We have

$$\gamma \equiv \prod_{k=0}^{N} \alpha_k = \alpha_j \alpha_{N-j}, \qquad 0 \leq j \leq N.$$

Then $\gamma^2 = \gamma^{N+1}$ and $\gamma^{N-1} = 1$, and for any j we have $\alpha_j \alpha_{N-j} = \gamma$ over the field $\Bbbk$, i.e., a_j and α_{N-j} are arbitrary modulo their product.

We point out the case of even Ns (recall that the Jacobi identity $[\![\Delta,\Delta]\!]^{\mathrm{RN}}$ works then): $N = 2N'$.There exists the special element $a_{N/2}$ such that

$$\alpha_{N'}^2 = \gamma \in \{\ N-1\text{th roots of 1 in } \Bbbk\}.$$

Now is the moment to study the process $N \to \infty$. By $\mathcal{X}$ we denote the formal direct limit $\lim_{N\to\infty} \Bbbk_N[x]$. Of course, the state $a_{N'}$ diverges: $x^{N/2} \not\to x^m$ $\forall m \in \mathbb{N}$. The bracket $[\cdot]$ is the reflections group $\mathbb{Z}_2$ w.r.t. this state, and the factor $\mathcal{X}/\mathbb{Z}_2$ acquires the gauge symmetry group

$$\operatorname{sym} \frac{\mathcal{X}}{\mathbb{Z}_2} \simeq S^1 \simeq SU(1),$$

and thus we obtain some 'gauge' theory.

2.2 Infinite-dimensional case: analytic functions.

23 Remark. We make a conventional remark concerning the underlying field $\Bbbk$. In all the illustrative cases involving the differential calculus we assume $\Bbbk = \mathbb{R}$ thus investigating smooth functions $a_j \in C^\infty(\Bbbk)$. Generally, we deal with analytic functions $a_j \in \Bbbk[[x]]$ treating them as formal series over the field $\Bbbk$ of characteristic 0: $\operatorname{char}\Bbbk = 0$. In the n-dimensional case $x = (x^1, \ldots, x^n) \in \Bbbk^n$, the analytic functions are $a_j \in \Bbbk[[x^1, \ldots, x^n]]$. Still, we preserve the notation '$C^\infty(\Bbbk^n)$' from the case $\Bbbk = \mathbb{R}$ bearing in mind its applications in mathematical physics.

The case when the monomials x^k have an arbitrary degree $k \in \mathbb{N}$ (the Taylor series) or $k \in \mathbb{Z}$ (the Laurent series) is a natural succession of the preceding exposition.

In these cases we generalize the Witt algebra defined by the relations $[a_i, a_j] = (j-i)\, a_{i+j}$: taking into account all our observations on the Wronskians, we start from the representation $a_i = x^{i+1}$, where $x \in \Bbbk$ and $i \in \mathbb{Z}$. For the Wronskian determinant $W^{0,1,\ldots,N-1}$, we consider the relations

$$[a_{i_1}, \ldots, a_{i_N}] = \Omega(i_1, \ldots, i_N)\, a_{i_1+\cdots+i_N}, \tag{29}$$

where the structural constants $\Omega(i_1, \ldots, i_N)$ are skew-symmetric w.r.t. their arguments, and use the representation $a_i = x^{i+N/2}$. We claim that the function Ω is the Vandermonde determinant.

24 Theorem. *Let ν_1, …, $\nu_N \in \Bbbk$ be constants and denote $\nu = \sum_{i=1}^N \nu_i$; then the equality*

$$W^{0,1,\ldots,N-1}(x^{\nu_1}, \ldots, x^{\nu_N}) = \prod_{1\le i<j\le N} (\nu_j - \nu_i) \cdot x^{\nu - N(N-1)/2} \tag{30}$$

holds, i.e., the Wronskian determinant of monomials is a monomial itself, and the coefficient is the Vandermonde determinant.

Proof. Consider the determinant (30): $A = \det \|a_{ij}\, x^{\nu_j - i+1}\|$. From jth column take the monomial $x^{\nu_j - N+1}$ outside the determinant:

$$A = x^{\nu - N(N-1)} \cdot \det \|a_{ij}\, x^{N-i}\|;$$

all rows acquire common degrees in x: deg(any element in ith row) $= N - i$. From ith row take this common factor x^{N-i} outside the determinant:

$$A = x^{\nu - N(N-1)/2} \cdot \det \|a_{ij}\|,$$

where the coefficients a_{ij} originate from the initial derivations in a very special way: for any i such that $2 \leq i \leq N$, we have

$$a_{1,j} = 1 \qquad \text{and} \qquad a_{ij} = (\nu_j \underline{- i + 2}) \cdot a_{i-1,j} \quad \text{for } 1 < i \leq N.$$

The underlined summand does not depend on j, and thus for any $k = N, \ldots, 2$ the determinant $\det \|a_{ij}\|$ can be splitted into the sum:

$$\det \|a_{ij}\| = \det \|a'_{k,j} = \nu_j \cdot a_{k-1,j}; \quad a'_{ij} = a_{ij} \text{ if } i \neq k\| + \\ + \det \|a''_{k,j} = (2 - i) \cdot a_{k-1,j}; \quad a''_{ij} = a_{ij} \text{ if } i \neq k\|,$$

where the last determinant is trivial.

Solving the recurrence relation, we obtain

$$\det \|a_{ij}\| = \det \|\nu_j^{i-1}\| = \prod_{1 \leq k < l \leq N} (\nu_l - \nu_k).$$

This completes the proof. QED

25 Remark. We have computed the structural constants in (29) using another basis $a'_i = x^i$ such that the resulting degree is not $\sum_{k=1}^{N} \deg a'_k$. Nevertheless, the result is correct since we use the translation invariance of the Vandermonde determinant:

$$\Omega(i_1, \ldots, i_N) = \Omega(i_1 + \frac{N}{2}, \ldots, i_N + \frac{N}{2}),$$

and all reasonings hold.

26 Remark (Subalgebras of the generalized Witt algebra). Of course, the homotopy N-Lie algebra of polynomials (22) is not a unique subalgebra in the generalized Witt algebra (29)–(30) of analytic functions. Now we point out subalgebras (both finite- and infinite-dimensional) of this algebra.

For $N = 2$, we have the special case of two-dimensional subalgebras $\mathrm{span}_{\Bbbk}\langle a_0, a_k \mid k = \mathrm{const} \in \mathbb{Z}\rangle$.

Let $N \geq 2$ be fixed and consider the cardinal basis a_i, $i \in \mathbb{Z}$; note that Theorem 24 is valid for $i \in \Bbbk$, but now we analyze the discrete situation. In what follows, we point out the generators such that the subalgebras are their linear span over $\Bbbk$.

If $N = 2N'$ is even, then

$$\mathrm{span}_{\Bbbk}\langle a_{-qN'}, a_{(-q+1)N'}, \ldots, a_0, \ldots, a_{qN'} \mid q = \mathrm{const} \in \mathbb{N}\rangle$$

is an $(N+1)$-dimensional homotopy Lie subalgebra for any constant q. If $N = 2N' - 1$ is odd, then

$$\mathrm{span}_{\Bbbk}\langle a_{(-2N'+1)q}, a_{(-2N'+3)q}, \ldots, a_{-q}, a_q \ldots, a_{(2N'-1)q} \mid q = \mathrm{const} \in \mathbb{N}\rangle$$

are the subalgebras.

As for the infinite-dimensional subalgebras, they are $\mathrm{span}_{\Bbbk}\langle a_{qj} \mid q = \mathrm{const} \in \mathbb{N}, j \in \mathbb{Z}\rangle$ for any N and $\mathrm{span}_{\Bbbk}\langle a_{(2j+1)q} \mid q = \mathrm{const} \in \mathbb{N}, j \in \mathbb{Z}, \quad N \equiv 1 \mod 2\rangle$ for odd Ns. Besides, there are algebras with indices of the same sign. Without loss of generality, assume $0 < j_0 \in \mathbb{Z}$. If $k \in \mathbb{N}$ is a divisor of j_0 or $k = 1$, then $\mathrm{span}_{\Bbbk}\langle a_{j_0+jk} \mid j \in \mathbb{N}\rangle$ is a subalgebra. If N is odd, then $\mathrm{span}_{\Bbbk}\langle a_{2j+1} \mid j \in \mathbb{Z}, 2j+1 \geq j_0\rangle$ is a subalgebra too.

Now we recall the behaviour of bracket (21) w.r.t. a change of coordinates $y = y(x)$.

27 Theorem. *The relation*

$$\det \left\| \frac{d^j \phi^i}{dx^j} \right\|_{\substack{i = 1,\ldots,N \\ j = 0,\ldots,N-1 \\ \phi^i = \phi^i(y(x))}} = \left(\frac{dy}{dx}\right)^{\frac{N(N-1)}{2}} \det \left\| \frac{d^j \phi^i}{dy^j} \right\|_{\substack{\phi^i = \phi^i(y) \\ y = y(x)}}$$

holds and thus the conformal weight $\Delta(N)$ *for the Wronskian determinant of* N *scalar fields* ϕ^i *of weight* 0 *is* $\Delta(N) = N(N-1)/2$.

Proof. Consider a function $\phi^i(y(x))$ and apply the total derivative D_x^j using the chain rule. The result is

$$\frac{d^j \phi^i}{dy^j} \cdot \left(\frac{dy}{dx}\right)^j + \text{terms of lower order derivatives } \frac{d^{j'}}{dy^{j'}}, \quad j' < j.$$

These lower order terms differ from the leading terms in $D_x^{j'}\,\phi^i(y(x))$, $0 \leq j' < j$, by the factors common for all i and thus they produce no effect since a determinant with coinciding (or proportional) lines equals zero. From ith row of the Wronskian we extract $(i-1)$th power of dy/dx, their total number being $N(N-1)/2$. Refer Section 3.2 for definition of the conformal weight. QED

28 Remark. We see that the Wronskian determinant of N functions is *not* a function itself. Thus, the objects we deal with are of some different nature, and the functions are their coefficients w.r.t. some basis. In Section 3.1, we show that these objects are in fact higher order differential operators.

Theorem 27 can be generalized to the case $n \geq 1$. We also see that the statement is generally not true if the generalized Wronskian is $\partial^{\sigma_1} \wedge \ldots \wedge \partial^{\sigma_N} \neq \mathrm{const} \cdot \mathbf{1} \wedge \partial \wedge \ldots \wedge \partial^{N-1}$.

In the study of homotopy Lie structures, our starting point is the following experimental fact.

29 Remark ([14]). Let $a_1, \ldots, a_5$ be arbitrary analytic functions; then the homotopy 3-Jacobi identity holds for the Wronskian determinants: $J^{3,\infty}_{\vec{a}}(x) \equiv 0$. The proof is by direct calculation of 20 Wronskians.

We claim that analytic functions admit the homotopy N-Lie algebra structure.

30 Claim. *Choose an integer $N \geq 2$ and let $a_1, \ldots, a_{2N-1}$ be arbitrary analytic functions; then the N-Jacobi identity*

$$W^{0,1,\ldots,N-1}[W^{0,1,\ldots,N-1}](a_1, \ldots, a_{2N-1}) = 0$$

holds.

Recently, Claim 30 was proved in [5]; we sketch the proof for consistency and then, in Section 2.3, we generalize the concept of the Wronskian determinant to the case of arbitrary number $n \in \mathbb{N}$ of independent variables $x^1, \ldots, x^n$.

Consider the generalized Wronskians $W^{\vec{\imath}} = \partial^{i_1} \wedge \ldots \wedge \partial^{i_N} \in \mathrm{Hom}(\bigwedge^N \mathcal{A}, \mathcal{A})$. Let a multiindex $\vec{\imath} \in \mathbb{Z}^N_+$ be such that $0 \leq i_1 < \ldots < i_N$. By $\mathrm{Hom}_t(\bigwedge^N \mathcal{A}, \mathcal{A})$ we denote the linear span of the generalized Wronskians $W^{\vec{\imath}}$ such that $|\vec{\imath}| = t$. By definition, put $|W^{\vec{\imath}}| = |\vec{\imath}|$.

31 Lemma ([5]). *Let $t < N(N-1)/2$, then* $\mathrm{Hom}_t(\bigwedge^N \mathcal{A}, \mathcal{A}) = 0$.

Proof [5]. If $0 \neq \partial^{i_1} \wedge \ldots \wedge \partial^{i_N} \in \mathrm{Hom}_t(\bigwedge^N \mathcal{A}, \mathcal{A})$, then $t = \sum_{j=1}^N i_N \geq 0 + 1 + 2 + \cdots + (N-1) = N(N-1)/2$. QED

32 Lemma ([4]). *Let $\vec{\imath}$ and $\vec{\jmath}$ be multiindexes: $\vec{\imath} \in \mathbb{Z}^k_+$ and $\vec{\jmath} \in \mathbb{Z}^l_+$; then the relation*

$$W^{\vec{\imath}}[W^{\vec{\jmath}}] \in \mathrm{Hom}_{|\vec{\imath}|+|\vec{\jmath}|}\Bigl(\bigwedge^{k+l-1} \mathcal{A}, \mathcal{A}\Bigr)$$

holds.

33 Corollary ([5]). *Let k and l be positive integers, then the identity*

$$W^{0,1,\ldots,k}[W^{0,1,\ldots,l}] = 0$$

holds.

PROOF [5]. First note that $|W^{0,1,\ldots,k}| = k(k+1)/2$ for any k. Thence,

$$W^{0,1,\ldots,k}[W^{0,1,\ldots,l}] \in \mathrm{Hom}_{\frac{k(k+1)+l(l+1)}{2}}(\bigwedge^{k+l+1}\mathcal{A},\mathcal{A}).$$

Nevertheless, $k^2+k+l^2+l < (k+l+1)(k+l)$ for any k and l. Consequently, by Lemma 31,

$$\mathrm{Hom}_{\frac{k^2+k+l^2+l}{2}}(\bigwedge^{k+l+1}\mathcal{A},\mathcal{A}) = 0 \quad \text{and} \quad W^{0,1,\ldots,k}[W^{0,1,\ldots,l}] = 0.$$

This completes the proof of Claim 30. *QED*

Now is the moment to observe that the Wronskians $\partial^{\vec{\sigma}}$ provide the structure essentially more restrictive than the homotopy N-Lie algebra, namely, the SH (strongly homotopy) Lie structure (also named the L_∞ algebra structure). We see that for a fixed system of coordinates Corollary 33 states exactly the following

34 Theorem. *The Wronskian determinants $W^{0,\ldots,k}$, $k \in \mathbb{N}$, provide the SH-Lie algebra structure on smooth functions $C^\infty(\Bbbk)$.*

From Corollary 33 we also obtain

35 Theorem. *Let k and l be positive integers, then the relation*

$$[\![W^{0,1,\ldots,k}, W^{0,1,\ldots,l}]\!]^{\mathrm{RN}} = 0$$

holds.

36 Corollary. *The Hochschild d_W-cohomologies are $H^*_{\mathrm{d}_W} = \mathrm{span}_\Bbbk\langle W^{0,\ldots,l}$, $l \geq 1\rangle$, since the differential $\mathrm{d}_{W^{0,\ldots,k}} = [\![W^{0,\ldots,k}, \cdot]\!]^{\mathrm{RN}}$ is trivial.*

2.2.1 On exterior multiplication.

In this section, we discuss the scheme that generates the Wronskian

$$W^{0,1,\ldots,N+1}$$

starting from the Wronskians $W^{0,1,\ldots,N-1}$ and $W^{N,N+1}$.

In [7, 22], the notion of the exterior multiplication $\wedge$ in $\mathrm{Hom}(\bigwedge^*\mathcal{A},\mathcal{A})$ is introduced.

37 Definition. Let $\Delta \in \mathrm{Hom}(\bigwedge^k\mathcal{A},\mathcal{A})$ and $\nabla \in \mathrm{Hom}(\bigwedge^l\mathcal{A},\mathcal{A})$ be two operators; by definition, put

$$(\Delta\wedge\nabla)(a_1,\ldots,a_{k+l}) = \sum_{\sigma\in S^k_{k+l}} (-1)^\sigma\,\Delta(a_{\sigma(1)},\ldots,a_{\sigma(k)})\cdot\nabla(a_{\sigma(k+1)},\ldots,a_{\sigma(k+l)})$$

for any $a_1,\ldots,a_{k+l} \in \mathcal{A}$.

We see that the Wronskians $W^{0,1,\ldots,N+1} = \partial^0 \wedge \ldots \wedge \partial^{N+1}$ are obtained in the fashion of Definition 37, e.g., $W^{0,1,\ldots,N+1} = W^{0,1,\ldots,N-1} \wedge W^{N,N+1}$. In Section 2.3, we construct generalizations of the Wronskian determinants for analytic functions $\Bbbk[[x^1,\ldots,x^n]]$ using Definition 37 essentially.

38 Definition. The structure $\Delta \in \mathrm{Hom}(\bigwedge^N \mathcal{A},\mathcal{A})$ is called a *multi-derivation* if the Leibnitz rule

$$\Delta(ab,a_2,\ldots,a_N) = a\,\Delta(b,a_2,\ldots,a_N) + \Delta(a,a_2,\ldots,a_N)\,b$$

is valid for any a, b, $a_j \in \mathcal{A}$.

Taking into account Remark 2, we restrict ourselves to the case of even $N = 2N'$.

39 Proposition ([22]). *Let* $\Delta \in \mathrm{Lie}^{(2k,0,0)}(\mathcal{A})$ *and* $\nabla \in \mathrm{Lie}^{(2l,0,0)}(\mathcal{A})$ *be multi-derivations. If* $[\![\Delta,\nabla]\!]^{\mathrm{RN}} = 0$*, then* $\Delta \wedge \nabla \in \mathrm{Lie}^{(2k+2l,0,0)}(\mathcal{A})$*.*

From Corollary 33 we know that the Wronskians $W^{0,1,\ldots,N+1}$ do satisfy the identity (5) for any even N, in particular. Thus, the conclusion of Proposition 39 is valid; still, we stress that the Wronskian determinants are *not* multi-derivations; besides, by a simple argument we observe the discouraging fact that, generally, $[\![W^{0,1,\ldots,N-1}, W^{N,N+1}]\!]^{\mathrm{RN}} \neq 0$. So, the example of the Wronskians demonstrates that there is another mechanism that generates higher order structures by using the exteriour multiplication rather than the reasoning in Proposition 39.

The task to describe all pairs of the generalized Wronskians $W^{\vec{\imath}}$, $W^{\vec{\jmath}} \in \mathrm{Hom}(\bigwedge^* \mathcal{A},\mathcal{A})$, such that the identity $[\![W^{\vec{\imath}}, W^{\vec{\jmath}}]\!]^{\mathrm{RN}} = 0$ holds, i.e., computation of the Hochschild $\mathrm{d}_{W^{\vec{\imath}}}$-cohomologies w.r.t. the differential $\mathrm{d}_{W^{\vec{\imath}}} = [\![W^{\vec{\imath}}, \cdot]\!]^{\mathrm{RN}}$ defined on the whole $\mathrm{Hom}(\bigwedge^* \mathcal{A},\mathcal{A})$, remains an independent problem.

2.3 Multidimensional base $n \geq 1$: generalizations of the Wronskians.

In this section, we generalize the concept of the Wronskian determinants to the multidimensional case of the base $\Bbbk^n$: further on, we consider the kth order jets $J^k(n,1)$ over the bundle $\pi\colon \Bbbk^n \times \Bbbk \to \Bbbk^n$, where the base dimension is $n \geq 1$ and the algebra $\mathcal{A}$ is the associative commutative algebra $C^\infty(\Bbbk^n)$ of smooth functions.

In order to construct a natural n-dimensional base generalization of the Wronskians, we pass to the geometrical standpoint and make an experimental observation first.

Let $\varkappa(\pi)$ be the $\mathcal{F}(\pi)$-module of evolutionary derivations $\partial_a = \sum_{j,\sigma} D_\sigma(a^j) \cdot \cdot \partial/\partial u^j_\sigma$, $a^j \in \mathcal{F}(\pi)$, where $\mathcal{F}(\pi)$ is the algebra of smooth functions $C^\infty(J^\infty(\pi))$.

To each Cartan N-form $\omega \in \mathcal{C}^N\Lambda(\pi)$ we assign the operator $\nabla_\omega \in \mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(N)}(\varkappa(\pi), \mathcal{F}(\pi))$ such that

$$\nabla_\omega(a_1, \ldots, a_N) = \ni_{a_N} \rfloor (\ldots(\ni_{a_1} \rfloor \omega)\ldots), \tag{31}$$

where $a_i \in \varkappa(\pi)$.

40 Proposition ([2, Chapter 5]). *Correspondence* (31) *is the isomorphism of the $\mathcal{F}(\pi)$-modules*:

$$\mathcal{C}^N\Lambda(\pi) \simeq \mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(N)}(\varkappa(\pi), \mathcal{F}(\pi)).$$

Further on, we use the notation $\omega(\ni_{a_1}, \ldots, \ni_{a_N}) \stackrel{\mathrm{def}}{=} \ni_{a_N} \rfloor (\ldots(\ni_{a_1} \rfloor \omega)\ldots)$, where $\omega \in \mathcal{C}^N\Lambda(\pi)$ and $a_i \in \varkappa(\pi)$.

41 Remark ([14]). Consider the infinite jets $J^\infty(\pi)$ over the bundle $\pi\colon \Bbbk \times \Bbbk \to \Bbbk$. Let $x \in \Bbbk$ be the independent base variable, u be the dependent fiber variable, D_x be the total derivative w.r.t. x, and $u^{(k)} \equiv D_x^k u$ be the coordinates in $J^\infty(\pi)$ for any $k \geq 0$. By d_C we denote the Cartan differential, $d_C\colon C^\infty(J^\infty(\pi)) \to \mathcal{C}\Lambda(J^\infty(\pi))$, that maps $u^{(k)} \mapsto du^{(k)} - D_x u^{(k)}\,dx$. The Wronskian determinants (21) can be interpreted as action of the N-forms $d_C u \wedge \ldots d_C u^{(N-1)} \in \mathcal{C}\Lambda^*(J^{N-1}(\pi)) \subset \mathcal{C}\Lambda^*(J^\infty(\pi))$ upon the evolutionary vector fields $\ni_{a_j} \equiv \sum_{k=0}^\infty D_x^k(a_j)\,\partial/\partial u^{(k)}$:

$$[a_1, a_2] = d_C u \wedge d_C(u')\,(\ni_{a_1}, \ni_{a_2}),$$
$$[a_1, a_2, a_3] = d_C u \wedge d_C(u') \wedge d_C(u'')\,(\ni_{a_1}, \ni_{a_2}, \ni_{a_3}), \qquad \text{etc.}$$

for any $a_j \in C^\infty(\Bbbk)$. We emphasize that $a_j \in C^\infty(\Bbbk) \subset \varkappa(\pi)$, i.e., we restrict ourselves to the functions on the base M.

42 Remark. Consider the ternary bracket $\square_3 \in \mathrm{Hom}(\bigwedge^3 C^\infty(\mathbb{R}^2), C^\infty(\mathbb{R}^2))$:

$$\square_3(a_1 \wedge a_2 \wedge a_3) = d_C u \wedge d_C u_x \wedge d_C u_y(\ni_{a_1}, \ni_{a_2}, \ni_{a_3}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ D_x(a_1) & D_x(a_2) & D_x(a_3) \\ D_y(a_1) & D_y(a_2) & D_y(a_3) \end{vmatrix}.$$

For the bracket $\square_3$, the homotopy ternary Jacobi identity $\square_3[\square_3] = 0$ of the form (5) holds. We prove this fact by direct calculations using the Jet software [18].

43 Proposition ([2]). *The dimension of the jets space $J^k(n, 1)$ vertical part $J^k(n, 1)/\Bbbk^n$ is*

$$\dim \frac{J^k(n,1)}{\Bbbk^n} = \dim J^k(n,1) - n = \sum_{i=0}^{k} \binom{n+i-1}{n-1} = \binom{n+k}{n}.$$

We also note that the dimension $N \equiv \binom{n+k}{n}$ is such that the inequality

$$\dim J^{2k}(n,1) - n - 1 > 2(\dim J^k(n,1) - n - 1)$$

is valid; in what follows, we need to subtract the dimension $\dim J^0(n,1) = n+1$ in order to deal with non-trivial multiindexes $\sigma \neq \varnothing$ such that $|\sigma| > 0$.

Choose arbitrary positive integers n and k; then $N \equiv \binom{n+k}{n}$ is the dimension $\dim(J^k(n,\,1)/\Bbbk^n)$. Let $\mathcal{A} = C^\infty(\Bbbk^n)$ be the algebra of smooth functions $a_j \in \mathcal{A}$, $1 \leq j \leq N$. Now we define the N-linear skew-symmetric bracket $\Box \in \operatorname{Hom}(\bigwedge^N \mathcal{A}, \mathcal{A})$: by definition, put

$$\Box(a_1, \ldots, a_N) = \bigwedge_{l=0}^{k} \Big(\bigwedge_{|\sigma|=l} d_C \cdot D_\sigma u \Big) (\ni_{a_1}, \ldots, \ni_{a_N}). \tag{32}$$

44 Theorem. *The N-linear skew-symmetric bracket $\Box \in \operatorname{Hom}(\bigwedge^N \mathcal{A}, \mathcal{A})$ defined in* (32) *satisfies the homotopy N-Lie Jacobi identity*

$$\Box[\Box] = 0. \tag{33}$$

Proof. In contrast with the reasoning in Section 2.2, we deal with $D_{\vec{\sigma}} = D_{\sigma_1} \wedge \ldots \wedge D_{\sigma_N}$, where σ_j is a multiindex $(\sharp x^1, \ldots, \sharp x^n) \in \mathbb{Z}_+^n$ for any j, $1 \leq j \leq N = \binom{n+k}{n}$. By definition, put $|D_{\vec{\sigma}}| = |\vec{\sigma}| = \sum_{j=1}^N |\sigma_j|$. We see that $|\Box[\Box]| = 2|\Box|$, c.f. Lemma 32.

Now we note that the non-trivial skew-symmetric $(2N-1)$-linear bracket $\Box_{\min} \in \operatorname{Hom}(\bigwedge^{2N-1} \mathcal{A}, \mathcal{A})$ with the minimal norm is

$$\Box_{\min} = \Box \wedge \Big(\sum_{\bar{\jmath} \in \Lambda^{N-1}\left(J^{2k}(n,1)/J^k(n,1)\right)} \operatorname{const}(\bar{\jmath}) \cdot D_{\sigma_{\bar{\jmath}}} \Big), \tag{34}$$

where $\operatorname{const}(\bar{\jmath}) \in \Bbbk$ are some constant coefficients.

We claim that $|\Box_{\min}| > 2|\Box|$, and thence $\Box[\Box] = 0$. Really, consider the r.h.s. in (34) and note that $|\Delta \wedge \nabla| = |\Delta| + |\nabla|$, see page 100 for definition of the wedge product $\wedge$ in this case. The set of N different derivatives in the first wedge factor $\Box$ admits the canonical splitting:

$$\Box = D_{\vec{\tau}} = \mathbf{1} \wedge \underbrace{D_{\tau_2} \wedge \ldots \wedge D_{\tau_N}}_{N-1 \text{ factors}},$$

where $\vec{\tau}$ contains all multiindexes in $J^k(n,1)$, and those under-braced derivatives are in bijective correspondence with $N-1$ different derivatives in any summand in the second wedge factor (there is the correspondence owing to the equal numbers of elements). Still,

$$1 \le |D_{\tau_i}| = |\tau_i| \le k < k+1 \le |\sigma_j| = |D_{\sigma_j}| \le 2k \qquad \forall i \ne 1, \quad \forall j.$$

Indeed if a multiindex σ_j is such that u_{σ_j} is a coordinate in $J^{2k}(n,1)/J^k(n,1)$, then σ_j is *longer* than any multiindex τ_i such that u_{τ_i} is a coordinate in $J^k(n,1)$. Consequently, the norm $|\cdot|$ of the second wedge factor in the r.h.s. of (34) is strictly greater than $|\square|$, and thus $\square[\square]$ is trivial. This completes the proof. *QED*

From the proof, we also conclude that the inequality $|\square_{\min}| > 2|\square| = |\square[\square]|$ strengthens as $k \to \infty$ and $N = \binom{n+k}{n}$.

We give an example of the binary homotopy 3-Lie algebra of polynomials:

45 Example. The polynomials $\operatorname{span}_{\Bbbk}\langle 1, x, y, xy\rangle \subset \Bbbk_2[x,y]$ endowed with the ternary bracket $\mathbf{1} \wedge D_x \wedge D_y$ form a homotopy 3-Lie algebra. The commutation relations of this algebra are

$$[1,x,y] = 1, \qquad [1,x,xy] = x, \qquad [1,y,xy] = -y, \quad \text{and} \quad [x,y,xy] = -xy,$$

and we see that the structural constants are such that the generators x and y are mixed.

In this section, we have realized the continualization scheme: the N-ary bracket $W^{0,1,\ldots,N-1}$ is defined on the sequence of $\Bbbk$-algebras

$$\Bbbk_N[x] \hookrightarrow \Bbbk[[x]] \hookrightarrow \left[\begin{array}{c} \Bbbk[[x^1,\ldots,x^n]] \\ \{\sum_\alpha c_\alpha \cdot x^\alpha \mid \alpha \in \Bbbk,\ c_\alpha \in \Bbbk\}. \end{array}\right.$$

The sets of indexes are finite, cardinal, cardinal w.r.t. any of n generators, and continuous, respectively.

We also note that the definition of the Koszul ∂_Δ-cohomologies is invariant w.r.t. the number of derivations $\partial_i\colon \mathcal{A} \to \mathcal{A}$, $i = 1,\ldots,n$, so that the cohomological constructions are preserved for the Wronskian determinant in (32).

3 Applications and examples.

3.1 Differential operators and the W–geometry.

In this section, the field $\Bbbk$ is the complex field $\mathbb{C}$: $\Bbbk = \mathbb{C}$, and z is the holomorphic coordinate in $\mathbb{C}$. Consider the holomorphic differential operators $\mathrm{Diff}_*(\mathbb{C})$ w.r.t. this coordinate:

$$\nabla_{\vec{w}} = \sum_{j=0}^{p} w_j(z) \cdot \partial^j. \tag{35}$$

We claim that these operators admit the homotopy Lie structure for arbitrary order p, and for any p this algebra contains a homotopy $2p$-Lie subalgebra.

Let $a_j \in \mathrm{Diff}_*(\mathbb{C})$ be $a_j = w_j(z)\,\partial^{k_j}$ for $1 \leq j \leq N$. By (10), put

$$[w_1 \cdot \partial^{k_1}, \ldots, w_N \cdot \partial^{k_N}] \stackrel{\text{def}}{=} \sum_{\sigma \in S_N} (-1)^\sigma\, w_{\sigma(1)}\, \partial^{k_{\sigma(1)}} \circ \cdots \circ w_{\sigma(N)} \cdot \partial^{k_{\sigma(N)}}. \quad (36)$$

This bracket is N-linear over $\mathbb{C}$ and skew-symmetric w.r.t. permutations of its arguments.

First, we count derivatives: Consider the special case $k_j \equiv p = \text{const}\ \forall j$ and solve the equation

$$Np = \frac{N(N-1)}{2} + p \quad \text{for } p\text{:} \quad p = \frac{N}{2}; \quad (37)$$

note that $N(N-1)/2 = |W^{0,1,\ldots,N-1}|$.

Further on, we restrict ourselves to the case $N \equiv 0 \mod 2$. It turns out that for odd Ns we need to consider half-integer powers of the derivation ∂: ∂^0, $\partial^{1/2}$, ∂, $\partial^{3/2}$, …. So, in case N is odd, one can choose between using a theory involving $\sqrt{\partial}$ or recalling (17)-(18) in order to preserve the cohomological concept by using an advanced notion of the sign.

46 Theorem. *Let N be even and $w_j \in \mathbb{C}[[z]]$ for $0 \leq j \leq N$; put $p = N/2$. Then we have*

$$[w_1\,\partial^p, \ldots, w_N\,\partial^p] = W^{0,1,\ldots,N-1}(w_1, \ldots, w_N) \cdot \partial^p.$$

Proof. Permutations of arguments in the r.h.s. of (36) are reduced to permutations of f_js since $k_j \equiv p$. Let $\sigma \in S_N$ be a permutation and $\vec{\jmath} \in \mathbb{Z}^N \cap [0, Np]^N$ be a vector in the integral lattice. Suppose that the r.h.s. in (36) is expanded from left to right and all possible derivation combinations

$$S_{\sigma,\vec{\jmath}} \stackrel{\text{def}}{=} (-1)^\sigma\, \partial^{j_1}(w_{\sigma(1)}) \cdot \cdots \cdot \partial^{j_N}(w_{\sigma(N)})$$

are obtained; we stress that *not* all vectors $\vec{\jmath} \in \mathbb{Z}^N \cap [0, Np]^N$ can be realized: at least, $|\vec{\jmath}| \leq Np$. Still, the set $J = \{\vec{\jmath}\} \subset \mathbb{Z}^N \cap [0, Np]^N$ does not depend on σ. Assume there is the summand such that two functions w_a and w_b acquire equal numbers of derivations for some combination $\vec{\jmath} \in J$. Then, for the same combination $\vec{\jmath}$ and the transposition τ_{ab} there is the permutation $\tau_{ab} \circ \sigma$ such that the order of w_a and w_b is reversed and $S_{\sigma,\vec{\jmath}} + S_{\tau_{ab}\circ\sigma,\vec{\jmath}} = 0$. Thus, only the Wronskian remains at ∂^p owing to Eq. (37). QED

Theorem 46 is a generalization of a perfectly familiar fact: the commutator of two vector fields is a vector field. We stress that Theorem 46 forbids the naive approach that combines N vector fields (e.g., symmetries of a PDE) in an attempt to obtain some vector field again.

47 Remark. Unfortunately, for arbitrary operators (35) of order $p = N/2$ this mechanism of compensations does not work. Really, suppose that the powers $k_j \le p$ are arbitrary; then the sets $J \subset \mathbb{Z}^N \cap [0, Np]^N$ do *depend on* σ, and generally $J(\sigma) \neq J(\tau_{ab} \circ \sigma)$, if two functions w_a and w_b are differentiated w.r.t. z equal number of times in a summand $S_{\sigma,\vec{\jmath}(\sigma)}$. Of course, we can obtain the Wronskian determinant at some suitable power of ∂, but there can be much more summands, even at ∂^ℓ for $\ell \ge p$.

The same difficulty occurs for $k_j \ge p$, when we consider formal differential operators $\nabla = \sum_{j=p}^{\infty} w_j(z) \cdot \partial^j$.

Nevertheless, for arbitrary whole $p' \ge (N-1)/2$ we have

$$[w_1\,\partial^{p'}, \ldots, w_N\,\partial^{p'}] = W^{0,1,\ldots,N-1}(w_1, \ldots, w_N) \cdot \partial^{Np' - N(N-1)/2}.$$

For various pairs $(N, p) \in \mathbb{N} \times \mathbb{N}$, one can deduce many extravagant phenomena. In [6], the following proposition is proved: for $p = 1$, vector fields $\mathrm{D}(M^n)$ on a smooth n-dimensional manifold M^n are closed w.r.t N-ary bracket (36) and form the homotopy N-Lie algebra if $N = n^2 + 2n - 2$.

We pose a combinatorial problem to obtain the number of summands $\sharp J(\sigma)$ for the set of powers $\{k_1, \ldots, k_N\}$. We need further study of the 'compensations mechanism' in order to compute the image of (36) for arbitrary $\vec{k}$.

As a corollary to Theorem 35, we have

48 Theorem. *Let N be even; consider the $\mathbb{C}[[z]]$-module $W_{N/2} \stackrel{\text{def}}{=} \mathrm{span}_{\mathbb{C}}\langle w(z)\,\partial^{N/2}\rangle$ of the order $N/2$ holomorphic operators. Then $W_{N/2}$ is endowed with the homotopy N-Lie algebra structure w.r.t. bracket* (36)*.*

We note that the SH-Lie algebra structure on the differential operators $W_{N/2}$ is missing if we assume that the brackets $\Delta_{2N'+1}$ with odd numbers of arguments are nontrivial, and restrict ourselves to odd formalism.

Nevertheless, the difficulties in complete description of the r.h.s. in (36) do not influence upon our ability to observe the homotopy N-Lie structure on the associative algebra of operators (35).

Really, as a corollary to Proposition 12 on page 89 we obtain

49 Theorem. *Let N be even, then differential operators* (35) *of arbitrary orders compose the homotopy N-Lie algebra w.r.t. bracket* (36)*.*

Now we note that the concept of the homotopy Lie structures for differential operators (35) has a nice application in the W-geometry.

The A_ℓ-W-geometry [8]–[10] is the geometry of one-dimensional complex curves Σ: $\dim_{\mathbb{C}}(\Sigma) = 1$, $\dim_{\overline{\mathbb{C}}}(\Sigma) = 1$, chirally embedded into the Kähler manifold $\mathbb{C}P^\ell$: $f^A(z)$ and $\bar{f}^{\bar{A}}(\bar{z})$ are the embedding functions, $0 \le A \le \ell$ and $0 \le \bar{A} \le \ell$.

50 Definition ([9, §3.2]). A general infinitesimal *W–transformation* δ_W is a change of the embedding functions f^A, $\bar{f}^{\bar{A}}$ of the form

$$\delta_W\, f^A(z) = \sum_{j=0}^{k} w_j(z)\, \partial^j f^A(z), \qquad \delta_W\, \bar{f}^{\bar{A}}(\bar{z}) = \sum_{j=0}^{k} \bar{w}_j(\bar{z})\, \bar{\partial}^j \bar{f}^{\bar{A}}(\bar{z}), \tag{38}$$

where $w_j \in \mathbb{C}[[z]]$ and $\bar{w}_j \in \bar{\mathbb{C}}[[\bar{z}]]$.

We see that a W-transformation is uniquely determined by the higher order differential operator $\sum_j w_j(z) \cdot \partial^j \in \mathrm{Diff}_*(\mathbb{C})$.

From Theorems 49 follows

51 Corollary. *The higher-order W-transformations compose the homotopy N-Lie algebras for even natural Ns.*

Consider the differential operators generated by the derivations $\partial_j \colon \mathbb{C}[[z^1, \ldots, z^n]] \to \mathbb{C}[[z^1, \ldots, z^n]]$ w.r.t. n independent variables z^1, $\ldots$, z^n and construct N-ary bracket Δ (10) such that n-dimensional homotopy Jacobi identities $[\![\Delta, \Delta]\!]^{\mathrm{RN}} = 0$ hold. We note that $2N$-ary bracket (10) provides structures much more general than ones given by $\binom{n+k}{n}$-ary bracket (32) defined on their coefficients $\mathbb{C}[[z^1, \ldots, z^n]] \subset \varkappa = \mathcal{F}(\pi)$. Really, there is no natural associative algebra structure on the module $\varkappa$ of evolutionary derivations, such that this structure agrees with bracket (32) (of course, we can multiply $\varphi_1, \varphi_2 \in \varkappa$ componentwise).

The notion of the homotopy Lie algebras hints to seek for the n-dimensional base generalizations of the Toda equations in a special way, starting from chiral embeddings of n-dimensional complex manifolds into some Kähler manifold:

52 Problem ($\dim_{\mathbb{C}} \Sigma = n$, $\dim_{\overline{\mathbb{C}}} \Sigma = n$). Will the operators $\sum_{|\vec{\tau}|\geq 0} w_{\vec{\tau}}(z^1, \ldots, z^n) \cdot D_{\vec{\tau}}$ define any transformations of chiral embeddings of n-dimensional complex manifolds Σ into $\mathbb{C}P^\ell$? If the answer is positive, what are the compatibility equations for these embeddings (for $n = 1$, they are the A_ℓ-Toda hierarchy)?

3.2 The homotopy Lie structures and the Ward identities.

We recall that the starting point, namely, $n = 1$, $N = 2$, and $\Bbbk = \mathbb{C}$, is the conformal transformations algebra. Consider the free field $\mathbf{x}(z \equiv x^1)$ and its traceless energy-momentum tensor $T(z) = (\partial_z \mathbf{x})^2$. By $\phi_{\Delta_j}(z_j)$ we denote primary fields of conformal weights Δ_j, respectively, where $z_j \in \mathbb{C}$.

Consider a transformation $\varepsilon(z)\, \partial/\partial z + \cdots$ of the model; the transformation laws are

$$z \mapsto z + \varepsilon(z) + o(|z|) \qquad \delta_\varepsilon \phi_\Delta(z) = \varepsilon(z) \partial_z \phi_\Delta(z) + \Delta \cdot (\partial_z \varepsilon(z))\, \phi_\Delta(z)$$

$$\delta_\varepsilon S = \int d^2 z\, T(z) \partial_{\bar{z}} \varepsilon(z), \qquad \delta_\varepsilon T(z) = \varepsilon(z) \partial_z T + 2(\partial_z \varepsilon(z))\, T(z) + \frac{c}{12}\, \partial_z^3 \varepsilon(z)$$

in the infinitesimal form; in the finite form, we have

$$z \mapsto f(z), \qquad \phi_\Delta(z) \mapsto \left(\frac{df}{dz}\right)^\Delta \cdot \phi_\Delta(f(z)),$$
$$T(z) \mapsto \left(\frac{df}{dz}\right)^2 T(f(z)) + \frac{c}{12}\,\{f,z\},$$

where c is the central charge and $\{f,z\} = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2$ is the Schwarzian derivative, $f' \equiv df/dz \equiv \partial_z f(z)$.

53 Proposition ([20, §9.2]). *Let $n = 1$: $x^1 \equiv z$, and $N = 2$; then the following two facts are equivalent*:

- *there is the differential operator $\mathcal{L}_{(3)}(\xi_i; z_j)$ such that the Ward identity*
$$\langle T(\xi_1)T(\xi_2)T_(\xi_3)\,\phi_{\Delta_1}(z_1)\ldots\phi_{\Delta_k}(z_k)\rangle = \mathcal{L}_{(3)}(\xi_i; z_j) \circ \langle\phi_{\Delta_1}(z_1)\ldots\phi_{\Delta_k}(z_k)\rangle \tag{39}$$
holds and the operator $\mathcal{L}$ does not depend on the sequence ξ_{i_1}, ξ_{i_2}, ξ_{i_3} used to expand the l.h.s. in Eq. (39);

- *the Jacobi identity holds for the Virasoro algebra of the conformal transformation generators*
$$T_{\varepsilon(z)} \equiv \oint_C \frac{dz}{2\pi i}\,\varepsilon(z)T(z).$$

54 Conjecture. *Let $n = 1$ and choose an arbitrary whole $N \geq 2$; then these two statements are equivalent*:

- *there is the differential operator $\mathcal{L}_{(2N-1)}(\xi_1, \ldots, \xi_{2N-1}; z_j)$ such that the Ward identities*
$$\langle T(\xi_1)\ldots T(\xi_{2N-1})\,\phi_{\Delta_1}(z_1)\ldots\phi_{\Delta_k}(z_k)\rangle = \\ = \mathcal{L}_{(2N-1)}(\xi_i; z_j) \circ \langle\phi_{\Delta_1}(z_1)\ldots\phi_{\Delta_k}(z_k)\rangle \tag{40}$$
hold and the operator $\mathcal{L}$ does not depend upon the ordered sequence ξ_{i_1}, …, $\xi_{i_{2N-1}}$ used to expand the l.h.s. in Eq. (40) *w.r.t. the associative algebra elements $T(\xi_1)$, …, $T(\xi_{2N-1})$*;

- *the Virasoro algebra of the conformal transformation generators admits homotopy N-Lie generalizations, i.e., there exist one-dimensional central extensions of the homotopy N-Lie algebras of smooth functions, and the homotopy N-Jacobi identities are valid for them.*

Conclusion.

In this paper, the following results are obtained:

1. The concept of homotopy N-Lie algebras w.r.t. the signs $\epsilon(\sigma) \in S^1$ is introduced in order to construct a nontrivial generalization of the homotopy Lie structures on associative algebras for odd Ns, preserving the idea of the Hochschild cohomologies.

2. The structural constants, expressed in the Vandermonde determinants, and the conformal weight of the bracket are calculated for the homotopy Lie algebras of analytic functions such that the required brackets are the Wronskian determinants. Thus, a wide class of N-ary generalizations (both discrete and continuous) is described for the Witt algebra.

3. Natural extension of the Wronskian determinants for the case of n independent variables is proposed such that the homotopy Jacobi identities hold for the algebra of analytic functions in n variables.

4. The concept of the vector fields Lie algebra $\mathrm{Vect}(\mathbb{C})$ is enlarged to the case of the homotopy $2N$-Lie algebras of order N differential monomials.

5. The higher order W-transformations are proved to compose the homotopy N-Lie algebras for for even Ns.

The concepts of the Hochschild and the Koszul cohomologies allow further, purely algebraic studies of these matters. The language of jet bundles and the initial standpoint both hint us to seek the homotopy Lie structures in the geometry of PDE and in the CFT models. Finally, we note that there are two famous mechanisms that provide the associative algebra structures, namely, the Yang-Baxter equation and the WDVV equation.

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