Note di Matematica **26,** n. 1, 2006, 131–137.

Composition operators between Fréchet spaces of holomorphic functions

Luciano O. Condori[i]
ICET, Universidade Paulista,
CEP: 06500-000 São Paulo, Brazil
lch@ime.usp.br

M. Lilian Lourenço
Departamento de Matemática, Universidade de São Paulo,
CP 66281 - CEP: 05311-970 São Paulo, Brazil
mllouren@ime.usp.br

Received: 9/8/2004; accepted: 19/5/2005.

Abstract. Let E, F and G be Banach spaces. Let V a balanced open subset of F. The reflexive and Montel composition operator $T_\Phi(f) := f \circ \Phi$ acting between the Fréchet spaces of all G-valued holomorphic functions of bounded type on E is studied in terms of Φ, where Φ is a G-valued holomorphic functions of bounded type on V.

Keywords: holomorphic functions, composition operators

MSC 2000 classification: primary 46G20, 46J05, secondary 47B33

1 Introduction

Let E and G be complex Banach spaces. For an open subset U of E, $H_b(U, G)$ denotes the space of all holomorphic functions from U into G which are bounded on U-bounded subsets of U. It is endowed with the topology τ_b of uniform convergence on U-bounded sets. It is known that $H_b(U, G)$ is a Fréchet space. As usual, we will always omit G in the notation in case $G = \mathbb{C}$. So, for instance, we will write $H_b(U)$ for $H_b(U, \mathbb{C})$.

If F is a complex Banach space and V an open subset of F, given a holomorphic mapping of bounded type $\Phi : V \to E$ with $\Phi(V) \subset U$ we will consider the composition operator $T_\Phi : H_b(U, G) \to H_b(V, G)$ defined by $T_\Phi(f) = f \circ \Phi$. Recently the study of composition operators has deserved some attention. Several results of composition operators between the Fréchet spaces $H_b(U, G)$ of holomorphic functions of bounded type have appeared when $G = \mathbb{C}$ (see for example [2], [4]). For instance, in [3] and [4] M. González and J. Gutiérrez have found results relating the (weak) compactness of holomorphic function

[i]Research supported by a doctoral fellowship from CNPq, Brazil.

of bounded type $\Phi \in H_b(F,E)$ with the (weak) compactness of composition operator T_Φ from $H_b(E)$ into $H_b(V)$.

It seems natural to study Montel and reflexive composition operators between the Fréchet spaces $H_b(U,G)$ of all vector valued holomorphic functions of bounded type.

In this note we study reflexive and Montel composition operators T_Φ between the Fréchet spaces of all G-valued holomorphic functions of bounded type in terms of Φ.

Let us mention that if G is a complex Banach algebra the space $H_b(U,G)$ endowed the τ_b topology of the uniform convergence on U-bounded set is a Fréchet algebra and the composition operator T_Φ is a continuous homomorphism.

Preliminaries. Our notation is standard and we refer to the books of Dineen [1] and Mujica [7] for background information on holomorphic functions on infinite dimensional Banach spaces, to Jarchow [6] and Horvath [5] regarding locally convex spaces theory.

Let E and G be complex Banach spaces. If U is an open subset of E, then a set $A \subset U$ is said to be *U-bounded* if A is bounded and is bounded away from the boundary of U.

If X and Y are complex Hausdorff locally convex spaces and $T : X \to Y$ is a linear map, then $T^* : Y^* \to X^*$ defined by $T^*(y^*) = y^* \circ T$ is a well defined linear map, it is called the algebraic adjoint of T. Under some conditions T^* induces a map $T' : Y' \to X'$ and we call T' *the adjoint or transposed map* of T.

A continuous linear map from X into Y is called *Reflexive* (resp. *Montel*), if it transforms bounded sets into relatively weakly compact (resp. relatively compact) sets.

Let us recall that a continuous linear mappings T from X into Y is called *weakly compact* (resp. *compact*), if it maps some 0-neighborhood into relatively weakly compact (resp. relatively compact) sets. If X and Y are normed space T is weakly compact (resp. compact) if and only if T is Reflexive (resp. Montel).

Let now X be a complex Hausdorff locally convex space and let X' be its topological dual. As usual $\sigma(X,X')$ is the weak topology on X and $\sigma(X',X)$ is the weak-star topology on X'. τ_β denotes the strong topology on X, τ_μ denotes the Mackey topology on X' and τ_c denotes the topology of the uniform convergence on compact subsets of X on X'.

We will denote $\mathcal{B}_X(0)$ a fundamental system of neighborhoods of 0 of the Hausdorff locally convex space X.

Let $\mathcal{A}$ a bounded set in $H_b(U,G)$, we associate the following neighborhood of 0 in $(H_b(U,G)', \tau_\beta)$:

$$U_{0,\mathcal{A},\epsilon} = \{ f' \in H_b(U,G)' / \sup_{f\in\mathcal{A}} |f'(f)| < \epsilon \} \in \mathcal{B}_{(H_b(U,G)',\tau_\beta)}(0).$$

For each finite subset $\{ g_1, \ldots, g_r \}$ of $H_b(U,G)$ and for each $\epsilon > 0$ we consider the following neighborhood of 0

$$V_{0,g_1,\ldots,g_r,\epsilon} = \{ f' \in H_b(U,G)' / |f'(g_1)| < \epsilon, \ldots, |f'(g_r)| < \epsilon \}$$

in $\mathcal{B}_{(H_b(U,G)',\sigma(H_b(U,G))',(H_b(U,G)))}(0)$. To simplify the notation from now we use w^* instead of $\sigma(H_b(U,G)', (H_b(U,G)))$.

2 Composition operators

In this section we study Montel (resp. reflexive) composition operators.

1 Lemma. *Let E and G be Banach spaces. Let $a \in G$ and $a' \in G'$ such that $\|a\| = 1$, $\|a'\| = 1$ and $a'(a) = 1$. Then:*

(i) $J_a : (E', \|\|) \to H_b(E,G)$ given by $J_a(x')(x) = x'(x)a$, for all $x' \in E'$ and for $x \in E$ is a continuous linear mapping. Moreover, the transposed mapping $J_a' : (H_b(E,G)', \tau_\beta) \to (E'', \|\|)$ is continuous.

(ii) Let V be an open set of E and let $\delta_{a'} : V \to (H_b(V,G)', w^*)$ defined by $\delta_{a'}(y)(g) = a'(g(y))$, for all $y \in V$ and for $g \in H_b(V,G)$. Then $\delta_{a'}$ is a continuous mapping and $\delta_{a'}$ maps V-bounded sets into bounded sets for the topology τ_β on $H_b(V,G)'$.

Proof. (i) It is clear that J_a is linear. Now, let B be a bounded set of E and $\epsilon > 0$. So, there is $\lambda_B > 0$ such that $\|x\| \le \lambda_B$ for all $x \in B$. If $x' \in E'$ with $\|x'\| < \frac{\epsilon}{\lambda_B}$ we have $\|J_a(x')\|_B \le \epsilon$ and consequently J_a is continuous at the origin. For the continuity of J_a' we consider $B_\epsilon''(0) \in \mathcal{B}_{E''}(0)$ and the unit ball $B_{E'}$ of E'. Then $J_a(B_{E'})$ is a bounded set in $H_b(E,G)$ and we have $|f'(J_a(x'))| < \epsilon$ for all $x' \in B_{E'}$ and $f' \in \mathcal{W}_{0,J_a(B_{E'}),\epsilon} \in \mathcal{B}_{(H_b(E,G)',\tau_\beta)}(0)$. So $J_a'(f') \in B_\epsilon''(0)$ for all $f' \in \mathcal{W}_{0,J_a(B_{E'}),\epsilon}$.

(ii) Let $y_0 \in V$ and $V_{\delta_{a'}(y_0),g_1,\ldots,g_r,\epsilon} \in \mathcal{B}_{(H_b(V,G)',w^*)}(\delta_{a'}(y_0))$ with $g_i \in H_b(V,G)$. Since each $g_i, 1 \le i \le n$ is continuous in $y_0 \in V$, there exists $B_{\lambda_i}(y_0) \subset V$ such that for each $y \in B_{\lambda_i}(y_0)$ we have $\|g_i(y) - g_i(y_0)\| \le \epsilon$ for each $1 \le i \le n$. Let $\lambda = \min_{1\le i\le r}\{ \lambda_i \}$ and $y \in B_\lambda(y_0)$. So $y \in B_{\lambda_i}(y_0)$ and $|\delta_{a'}(y)(g_i) - \delta_{a'}(y_0)(g_i)| < \epsilon$ for each $i = 1, \ldots, r$. Consequently $\delta_{a'}(y) \in V_{\delta_{a'}(y_0),g_1,\ldots,g_r,\epsilon}$ and $\delta_{a'}$ is continuous.

To complete the proof it suffices to show that $\delta_{a'}$ maps V-bounded sets of V into bounded sets of $(H_b(V,G)', \tau_\beta)$. Let $B \subset V$ be a V-bounded and $V_{0,g_1,\dots,g_r,\epsilon} \in \mathcal{B}_{(H_b(V,G)',w^*)}(0)$ with $g_i \in H_b(V,G)$. So there exist $\lambda_i > 0$ such that $\sup_{y\in B} \|g_i(y)\| < \lambda_i$ for $1 \leq i \leq r$. Let $\lambda = \max_{1\leq i\leq r}\{\lambda_i\}$ and $g' \in \frac{\epsilon}{2\lambda}\delta_{a'}(B)$. Then $|g'(g_i)| < \epsilon, i = 1,\dots,r$ and $g' \in V_{0,g_1,\dots,g_r,\epsilon}$. Consequently $\frac{\epsilon}{2\lambda}\delta_{a'}(B) \subset V_{0,g_1,\dots,g_r,\epsilon}$ and it is $w*$-bounded. Since $H_b(V,G)$ is a barrelled space we have that $\delta_{a'}(B)$ is bounded in the space $(H_b(V,G)', \tau_\beta)$, thus completing the proof.

QED

In the next theorem we study the Montel composition operator.

2 Theorem. *Let E, F and G be Banach spaces. Let $V \subset F$ an open subset, $\Phi \in H_b(V,E)$ and $T_\Phi : H_b(E,G) \to H_b(V,G)$ a composition operator. Consider the following conditions:*

(a) T_Φ is a Montel operator

(b) The adjoint operator $T'_\Phi : (H_b(V,G)', \tau_\beta) \to (H_b(E,G)', \tau_\beta)$ is a Montel operator

(c) Φ maps V-bounded sets into relatively compact sets in E.

Then (a) $\Rightarrow$ (b) $\Rightarrow$ (c).

Proof. $(a) \Rightarrow (b)$ First we show that $T'_\Phi : (H_b(V,G)', \tau_c) \to (H_b(E,G)', \tau_\beta)$ is continuous. Let $\mathcal{X} \subset H_b(E,G)$ be a bounded set and

$$V_{0,\mathcal{X},\epsilon} = \{ f' \in H_b(E,G)' \ / \ \|f'\|_{\mathcal{X}} < \epsilon \} \in \mathcal{B}_{(H_b(E,G)',\tau_\beta)}(0).$$

Since T_Φ is a Montel operator we have that $T_\Phi(\mathcal{X})$ is a relatively compact set of $H_b(V,G)$. Then the closed absorbing convex hull $\Gamma(\overline{T_\Phi(\mathcal{X})})$ is compact, since $H_b(V,G)$ is a Fréchet space.

Now, if we consider

$$W_{0,\Gamma(\overline{T_\Phi(\mathcal{X})}),\epsilon} = \{ h' \in H_b(V,G)' \ / \ \|h'\|_{\Gamma(\overline{T_\Phi(\mathcal{X})})} < \epsilon \} \in \mathcal{B}_{(H_b(V,G)',\tau_c)}(0),$$

we have that $T'_\Phi(W_{0,\Gamma(\overline{T_\Phi(\mathcal{X})}),\epsilon}) \subset V_{0,\mathcal{X},\epsilon}$, since for each $h' \in W_{0,\Gamma(\overline{T_\Phi(\mathcal{X})}),\epsilon}$ we get $|h'(h)| < \epsilon$ for all $h \in \Gamma(\overline{T_\Phi(\mathcal{X})})$. So $|h'(T_\Phi(f))| < \epsilon$ for all $f \in \mathcal{X}$ and $T'_\Phi(h') \in V_{0,\mathcal{X},\epsilon}$. Therefore $T'_\Phi : (H_b(V,G)', \tau_c) \to (H_b(E,G)', \tau_\beta)$ is continuous.

Now, let $\mathcal{A} \subset H_b(V,G)'$ a τ_β-bounded set. Since $H_b(V,G)$ is barrelled we have that $\mathcal{A}$ is an equicontinuous set and consequently $\sigma(H_b(V,G)', H_b(V,G))$-relatively compact. By Banach-Dieudonné theorem we have that $\mathcal{A}$ is τ_c-relatively compact. As T'_Φ is $\tau_c - \tau_\beta$ continuous we have that $T'_\Phi(\mathcal{A}) \subset H_b(E,G)'$ is τ_β-relatively compact on $H_b(E,G)'$.

$(b) \Rightarrow (c)$ Let $a \in G$ with $\|a\| = 1$. By Hahn-Banach theorem there exists $a' \in G'$ with $\|a'\| = 1$ and $a'(a) = 1$. Now, let $\psi : V \to (E'', \|\|)$ defined by $\psi := J'_a \circ T'_\Phi \circ \delta_{a'}$. By Lemma 1 we have that ψ maps V-bounded sets into relatively compact set of E''.

As $\psi(y)(x') = x'(\Phi(y))a'(a) = C(\Phi(y))(x')$, for all $y \in V$, for all $x' \in E'$ where $C : E \to E''$ is the natural inclusion, we have that $\psi(y) = C(\Phi(y))$ for all $y \in V$. So $C \circ \Phi = \psi$ and Φ maps V-bounded sets of V into relatively compact sets in E''. QED

In [3] González-Gutiérrez proved these conditions of Theorem 2 are equivalent when $G = \mathbb{C}$. However, in the general case, the following example shows that the assertions of Theorem 1 are not equivalent.

3 Example. Let Φ be a identity on $\mathbb{C}$, which is a trivially Montel mapping. Give an infinite dimensional Banach space G, we consider the composition operator $T_\Phi : H(\mathbb{C}, G) \longrightarrow H(\mathbb{C}, G)$ given by $T_\Phi(f) = f \circ \Phi = f$ for all $f \in H(\mathbb{C}, G)$. Let (y_n) be a sequence of norm one vectors in G such that $||y_n - y_m|| \geq \delta > 0$ for all $n \neq m$. Define $f_n : \mathbb{C} \longrightarrow G$ by $f_n(\lambda) = \lambda y_n$. Then the sequence (f_n) is bounded in $H(\mathbb{C}, G)$ but is not relatively compact, since $(f_n(1)) = (y_n)$ is not relatively compact.

4 Corollary. *Let E, F and G be Banach spaces. Let $U \subset E$, $V \subset F$ an open set, $\Phi \in H_b(V, E)$ and let $T_\Phi : H_b(U, G) \to H_b(V, G)$ be a composition operator. Then Φ maps V bounded sets into relatively compact sets of U if T_Φ is compact.*

Proof. It suffices to observe that the composition operator $A : H_b(E, G) \to H_b(V, G)$ given by $A(f) = f \circ \Phi$ for all $f \in H_b(E, G)$ is compact if $T_\Phi : H_b(U, G) \to H_b(V, G)$ is compact. QED

The next theorem studies, in the same cases, when the composition operator T_Φ is reflexive.

5 Theorem. *Let E, F and G be Banach spaces. Let V be a balanced open set of F, $\Phi \in H_b(V, E)$ and $T_\Phi : H_b(E, G) \to H_b(V, G)$ be a composition operator. Consider the following conditions:*

(a) T_Φ is a reflexive operator

(b) The adjoint operator $T'_\Phi : H_b(V, G)' \to H_b(E, G)'$ maps τ_β-bounded sets into relatively $\sigma(H_b(E, G)', (H_b(E, G)', \tau_\beta)')$-compact sets

(c) Φ maps V-bounded sets into relatively weakly compact sets in E.

Then $(a) \Rightarrow (b) \Rightarrow (c)$.

Proof. $(a) \Rightarrow (b)$ First we show that $T'_\Phi : (H_b(V,G)', \tau_\mu) \to (H_b(E,G)', \tau_\beta)$ is continuous. Let $\mathcal{X} \subset H_b(E,G)$ be a bounded subset and let

$$V_{0,\mathcal{X},\epsilon} = \{\, f' \in H_b(E,G)' \;/\; \|f'\|_{\mathcal{X}} < \epsilon \,\}$$

be a τ_β-neighborhood of zero in $H_b(E,G)'$. Since T_Φ is a reflexive operator it follows that $T_\Phi(\mathcal{X})$ is a relatively weakly compact set of $H_b(V,G)$. As $H_b(V,G)$ is a Fréchet space we have that the closed convex absolutely hull of $T_\Phi(\mathcal{X})$, $\Gamma(\overline{T_\Phi(\mathcal{X})})$, is a weakly compact set of $H_b(V,G)$.

Now, we consider $W_{0,\Gamma(\overline{T_\Phi(\mathcal{X})}),\epsilon} = \{\, h' \in H_b(V,G)' \;/\; \|h'\|_{\Gamma(\overline{T_\Phi(\mathcal{X})})} < \epsilon \,\} \in \mathcal{B}_{(H_b(V,G)',\tau_\mu)}(0)$. Then it is clear that $T'_\Phi(W_{0,\Gamma(\overline{T_\Phi(\mathcal{X})}),\epsilon}) \subset V_{0,\mathcal{X},\epsilon}$. We claim that

$$T'_\Phi : (H_b(V,G)', w^*) \to (H_b(E,G)', \sigma(H_b(E,G)', (H_b(E,G)', \tau_\beta)'))$$

is a continuous mapping. Indeed let

$$V_{0,f''_1,\dots,f''_k,\epsilon} \in \mathcal{B}_{(H_b(E,G)',\sigma(H_b(E,G)',(H_b(E,G)',\tau_\beta)'))}(0)$$

with $f''_i \in (H_b(E,G)', \tau_\beta)'$ for each $i = 1, 2, \dots, k$. Then we have that $T''_\Phi(f''_i) \in (H_b(V,G)', \tau_\mu)'$. Thus there exist $f_i \in H_b(V,G)$ such that $T''_\Phi(f''_i)(f') = f'(f_i)$ for all $f' \in H_b(V,G)'$ and $i = 1, 2, \dots, k$. If we consider

$$W_{0,f_1,\dots,f_k,\epsilon} \in \mathcal{B}_{(H_b(V,G)',\sigma(H_b(V,G)',H_b(V,G)))}(0),$$

it is easy to see that $T'_\Phi(W_{0,f_1,\dots,f_k,\epsilon}) \subset V_{0,f''_1,\dots,f''_k,\epsilon}$ and consequently T'_Φ is $\sigma(H_b(V,G)', H_b(V,G)) - \sigma(H_b(E,G)', (H_b(E,G)', \tau_\beta)')$ continuous.

Now, let $\mathcal{X} \subset H_b(V,G)'$ be a τ_β-bounded set. As $H_b(V,G)$ is barrelled we have that $\mathcal{X}$ is a w^*-relatively compact.

So $T'_\Phi(\mathcal{X})$ is $\sigma(H_b(E,G)'$, $(H_b(E,G)'$, $\tau_\beta)')$-relatively compact, and the implication $(a) \Rightarrow (b)$ follows.

$(b) \Rightarrow (c)$ Let $a \in G$ with $\|a\| = 1$. Then there is $a' \in G'$ such that $\|a'\| = 1$ and $a'(a) = 1$.

First we show that

$$J'_a : (H_b(E,G)', \sigma(H_b(E,G)', (H_b(E,G)', \tau_\beta)')) \to (E'', \sigma(E'', E'))$$

is a continuous mapping at origin, where J'_a is the adjoint operator of J_a defined in the Lemma 1.

Let $V_{0,x'_1,\dots,x'_r,\epsilon} \in \mathcal{B}_{(E'',\sigma(E'',E'))}(0)$. For each $i = 1, 2, \dots, r$, we consider $x'_i \otimes a \in H_b(E,G)$ defined by $x'_i \otimes a(x) = x'_i(x)a$ for all $x \in E$ and $f''_i : (H_b(E,G)', \tau_\beta) \to \mathbb{C}$ given by $f''_i(f') = f'(x'_i \otimes a)$ for all $f' \in H_b(E,G)'$. Now, it holds that $W_{0,f''_1,\dots,f''_r,\epsilon} \in \mathcal{B}_{(H_b(E,G)',\sigma(H_b(E,G)',(H_b(E,G)',\tau_\beta)'))}(0)$ and we have $J'_a(W_{0,f''_1,\dots,f''_r,\epsilon}) \in V_{0,x'_1,\dots,x'_r,\epsilon}$. So by Lemma 1 (ii) the mapping $\psi : V \to$

$(E'', \sigma(E'', E'))$ given by $\psi = J'_a \circ T'_\Phi \circ \delta_{a'}$ maps V-bounded set into $\sigma(E'', E')$-relatively compact set in E''.

Let $(y_n)_{n\in\mathbb{N}} \subset V$ be a bounded sequence. Since ψ is $\sigma(E'', E')$-relatively compact and $\psi(y)(x') = C(\Phi(y))(x')$ for all $y \in V$ and $x' \in E'$ where C is the natural inclusion from E into E'', there exist a subsequence $(y_{n_k})_k$ of (y_n) and $x'' \in E''$ such that $x'(\Phi(y_{n_k})) \to x''(x')$ for all $x' \in E'$. Consequently, Φ maps bounded sets of V into $\sigma(E, E')$-relatively compact sets in E. QED

6 Remark. Slight modifications of example 1 give an example that shows in general the assertions of Theorem 2 are not equivalent. Indeed, let Φ be a reflexive mapping and let G be a non-reflexive Banach space, and choose a sequence (y_n) of norm one vectors in G without any weakly convergent subsequence. Define (f_n) as the example 1. Then (f_n) is not relatively weakly compact.

In [4] M. González and J. Gutiérrez showed that the conditions of Theorem 2 are equivalent if $G = \mathbb{C}$ and E has Dunford-Pettis property.

References

[1] S. Dineen: Complex Analysis on Infinite Dimensional Spaces, Springer-Verlag, London Berlin Heidelberg 1999.

[2] P. Galindo, M. L. Lourenço, L. A. Moraes: *Compact and weakly compact homomorphisms on Fréchet algebras of holomorphic functions*, Math. Nachr., **236**, (2002), 109–118.

[3] M. González, J. Gutiérrez: *Schauder type theorems for differentiable and holomorphic mappings*, Monatsh. Math., **122**, (1996), 325–343.

[4] M. González, J. Gutiérrez: *Gantmacher type theorems for holomorphic mappings*, Math. Nachr., **186**, (1997), 131–145.

[5] J. Horvath: Topological Vector Spaces and Distributions, Vol. 1. Addison-Wesley, Massachusetts 1966.

[6] H. Jarchow: Locally Convex Spaces, Mathematische Leitfäden, B. G. Teubner, Stuttgart 1981.

[7] J. Mujica: Complex Analysis in Banach Spaces, Math. Stud., **120**, North-Holland 1986.