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Variations on a theme of Cofman

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Abstract. The paper provides a survey on the known results on the collineation groups acting on a line of a projective plane with some transitivity properties. As a new result, it is shown that a group which is faithful and primitive of rank 3 on a line of a projective plane is 1-dimensional semilinear.

Keywords: Projective plane, collineation group, orbit.

MSC 2000 classification: primary 51E15, secondary 20B25.

1 Introduction

A classical subject in finite geometries is the investigation of a finite projective plane Π of order n admitting a collineation group G which acts 2-transitively on a point-subset $\mathcal{O}$ of size v of Π. It dates back to 1967 and it is due to Cofman [6]. It is easily proven that either

(i) the structure of a non trivial 2-$(v, k, 1)$ design is induced on $\mathcal{O}$, or

(ii) $\mathcal{O}$ is an arc, or

(iii) $\mathcal{O}$ is contained in a line.

Although Cofman obtained several interesting results for the three admissible configurations for $\mathcal{O}$ in the paper [6], we will mention only those involving the third one, since this paper focuses entirely on it. Indeed, Cofman proved that if $SL(2, n) \trianglelefteq G$ acts 2-transitively on a line of Π, $n \not\equiv_8 1$ and G contains involutory homologies, then Π is Desarguesian. Starting from Cofman, several papers have been devoted to this subject. Some years later, Schulz [39] and Czerwinski [7] essentially proved that the unique translation planes with a collineation group acting 2-transitively on the line at infinity, are either Desarguesian or Lüneburg

[i]The paper is dedicated to Professor Norman L. Johnson on occasion of his 70^{th} birthday.

planes. Actually, they proved such a characterization under the additional assumption that G does not contain Baer collineations of Π. The assumption was subsequently dropped off by Kallaher [26] in 1987 with the use of the classification of finite 2-transitive groups. In 1981, Korchmáros [29] investigates the general case $v = n + 1$ when $n = 2^r$. Apart from the Desarguesian case, the author proves that either $G \cong Sz(2^s)$ or $G \cong PSU_3(2^s)$. Although in 1985 Korchmáros and Biliotti [2] provided some strong restrictions for the second case, such as the fact that Π cannot be a translation plane, in general, it is still open. Another case which has been widely studied by several authors is that of $\mathcal{O}$ consisting of a line minus a point, i.e. $v = n$. In 1986, Ganley and Jha [22] proved that if $v = n$, Π is a translation plane and l is the line at infinity, then Π is actually a semifield plane. The general case when $v = n$ was investigated by Hiramine [18] in 1993, without any assumption on the structure of Π. Apart from a few numerical values of n, Hiramine shows that the socle of $\bar{G}$, where $\bar{G}$ denotes the group induced by G on $\mathcal{O}$, is an elementary abelian p-group for some prime p, the plane Π has order $n = p^r$ and either $\bar{G}_O \leq \Gamma L(1, p^r)$ or $SL(2, p^r) \leq \bar{G}_O \leq \Gamma L(2, p^r)$, where $\bar{G}_O$ is the the stabilizer in $\bar{G}$ of a point O of the line l. In 1999, Biliotti, Jha and Johnson classified the translation planes Π of order n, $n \neq 2^6$, with a solvable collineation group acting 2-transitively on a point subset of size n of the line at infinity of Π. In 2000, Ganley, Jha and Johnson [23] classified the triple $(\Pi, \mathcal{O}, G)$ for $v = n$, when Π is a translation plane, l is an affine line and G is non solvable. Finally, in 2005, Biliotti and Francot [1] investigated the general case $v \geq n$, and more specifically they completed the case of Cofman without any additional assumption. Actually, in that paper all three possibilities for $\mathcal{O}$ have been investigated, and for each of them the complete structure of G has been determined in detail.

The following theorem represents a synthesis of all the results about the Cofman problem when $\mathcal{O}$ is either a line or a line minus a point.

1 Theorem. *Let Π be a projective plane of order n and let $\mathcal{O}$ be a 2-transitive G-orbit of length v on a line. If $v \geq n$, then one of the following occurs:*

(1) $v = n + 1$, and one of the following occurs:

- *(a) $n = q$, $\Pi \cong PG(2, q)$ and $SL(2, q) \trianglelefteq G$;*
- *(b) $n = q^2$, $q = 2^{2s+1}$, $s \geq 1$, and $Sz(q) \trianglelefteq G$;*
- *(c) $n = q^3$, $q = 2^{2s}$, $s \geq 1$, $PSU(3, q) \trianglelefteq G$ and G fixes a point of Π^l;*
- *(d) $\bar{G}_O \leq \Gamma L(1, v)$;*
- *(e) $v \in \{5^2, 7^2, 11^2, 19^2, 23^2, 29^2, 59^2\}$.*

(2) $v = n$, *and one of the following occurs:*

(a) $\bar{G}_O \leq \Gamma L(1, v)$*;*

(b) $SL(2, q^{d/2}) \trianglelefteq \bar{G}_O$, d *even;*

(c) $v \in \{2^4, 3^2, 3^4, 3^6, 5^2, 7^2, 11^2, 19^2, 23^2, 29^2, 59^2\}$.

For a proof see [1].

Beside the study of the classical cases, another type of investigation was initiated about the triples $(\Pi, \mathcal{O}, G)$ when the length v of $\mathcal{O}$ is smaller than n, but close to n. Indeed, in 2005 Biliotti and Montinaro [3] investigated the case $v = n - 3$. The choice of studying this particular case was justified by the existence of a remarkable example due to Segre and consisting of a complete 6-arc in $PG(2, 9)$ on which $PSL(2, 5)$ acts in its natural 2-transitive permutation representation of degree 6. The paper is mainly based on CFSG, and it is shown that no nontrivial examples occur in the line case (actually, the stated paper provided further interesting examples when $\mathcal{O}$ is an arc). Starting from this paper, Montinaro, in his PhD thesis [34] studied the Cofman problem in all three possible configurations for $\mathcal{O}$ under the assumption $v \geq n/2$. The main tools involved are the Hering-Walker theory on strong irreducibility [15], [16] and [17], the Ho-Goncalves theory on total irregularity [20], and a careful study of the action kernel based on the theory of Schur's multiplier jointly with the Gorestein-Walter [13] theorem. In particular, the following result is obtained.

2 Theorem. *Let* Π *be a projective plane of order* n *and let* $\mathcal{O}$ *be a* 2*-transitive* G*-orbit of length* v *on a line. If* $n > v \geq n/2$*, then one of the following occurs:*

(1) $v = (n+1)/2$, n *odd, and one of the following occurs:*

(a) Π *is the Hall plane of order* 9 *or its dual,* $|\mathcal{O}| = 5$ *and* $SL(2, 5) \trianglelefteq G$*;*

(b) $n = 2q + 1$, $q \equiv_4 3$, $q \neq 7$, $|\mathcal{O}| = q + 1$ *and* $SL(2, q) \trianglelefteq G$.

(2) $v = n/2$, n *even, and one of the following occurs:*

(a) Π *is the Johnson-Walker translation plane of order* 16 *or its dual, and* $PSL(2, 7) \trianglelefteq G$*;*

(b) $n = 2(q + 1)$, $q \equiv_4 3$, $|\mathcal{O}| = q + 1$ *and* $SL(2, q) \trianglelefteq G$.

(3) $n/2 \leq v < n$ *and either*

(a) $\bar{G} \leq A\Gamma L(1, v)$, *or*

(b) $v \in \{2^4, 2^6, 3^2, 3^3, 3^4, 3^6, 5^2, 7^2, 11^2, 19^2, 23^2, 29^2, 59^2\}$

For a proof see [36].

We stress out that, while there are no known examples for the case (3b), examples of type (3a) occur in the Desarguesian plane of order 8, 9, in the Lorimer-Rahilly plane of order 16 and in the Johnson-Walker plane of order 16 and in their duals. A complete description of these examples is given in [15]. The Johnson-Walker plane is particularly rich in examples. Indeed, an affine line contains an orbit of length 8 and one of length 7, both doubly transitive. The second orbit is an example of a 2-transitive orbit of length v in a plane of order n satisfying the relation $v = n/2 - 1$. At this point, it seemed intriguing to tackle the Cofman Problem with the condition $v = n/2 - 1$, obtaining the following theorem.

3 Theorem. *Let* Π *be a projective plane of order* n *and let* $\mathcal{O}$ *be a* 2*-transitive* G*-orbit of length* v*. If* $v = n/2 - 1$*, then either*

(1) $\Pi \cong PG(2, 16)$*,* $\mathcal{O} \cong PG(2, 2)$ *and* $PSL(2, 7) \trianglelefteq G$*, or*

(2) Π *has order* 16*,* $|\mathcal{O}| = 7$*,* $\mathcal{O}$ *is either a* 7*-arc or a subset of a line, and* $G \cong AGL(1, 7)$*, or*

(3) Π *is the Johnson-Walker translation plane of order* 16 *or its dual,* $|\mathcal{O}| = 7$*,* $\mathcal{O}$ *is subset of an affine line* l *and* $PSL(2, 7) \trianglelefteq G$*.*

For a proof see [37]. Note that the cases (1) and (3) really occur, while the case (2) is open.

The first natural generalization of the 2-transitive action on a point subset of a line of a projective plane is represented by the primitive 3-rank action. In this paper we settle down this more general case. In order to do so, we investigate the collineation groups which are transitive on a line of a projective plane, determining the primitive 3-rank actions as a special case. The study is carried out mainly by using the Aschbacher Theorem (e.g. see [28], Main Theorem) when G is classical, and the Main Theorem of [32] when G is exceptional of Lie type. In particular, we obtain the following

4 Theorem. *Let* G *be an almost simple collineation group of a projective plane* Π *of order* n *fixing a line* l*. If the action of* G *on* l *is faithful and transitive, then* G *is* 2*-transitive on* l *and one of the following occurs:*

(1) $n = 2^s$*,* $\Pi \cong PG(2, 2^s)$ *and* $PSL(2, 2^s) \trianglelefteq G$*;*

(2) $n = 2^{2s}$*,* s *odd, and* $Sz(2^s) \trianglelefteq G$*;*

(3) $n = 2^{3s}$, *s even, and* $PSU(3, 2^s) \trianglelefteq G$.

We shall present an outline of the proof, and the reader is referred to [5] for details.
If G contains at least a perspectivity of Π, the assertion easily follows by the Hering [14] trivial normalizer intersection argument. Indeed, despite the fact it was used for 2-transitive actions (e.g. see [29]), it still works with transitive ones. Hence, we may assume that all the involutions in G are Baer collineations of Π. So, let σ be any involution of G. Set $\mathrm{Fix}_l(\sigma) = \mathrm{Fix}(\sigma) \cap l$. It is well known that

$$|\mathrm{Fix}_l(\sigma)| = \frac{|C_G(\sigma)|}{|G_P|} \left|\{\beta \in G_P : \beta^g = \sigma \text{ for some } g \in G\}\right|. \tag{1}$$

(e.g. see [33], relation (9) of page 69). Now, set $\left|\sigma^G\right| = [G : C_G(\sigma)]$ and $K_\sigma = |\{\beta \in G : \beta^g = \sigma \text{ for some } g \in G_P\}|$. Then by (1), we have $|\mathrm{Fix}_l(\sigma)| = \frac{|C_G(\sigma)| K_\sigma}{|G_P|}$. Recall that $|C_G(\sigma)| = \frac{|G|}{|\sigma^G|}$. Hence, $|\mathrm{Fix}_l(\sigma)| = \frac{[G:G_P]K_\sigma}{|\sigma^G|}$. Since σ is a Baer collineation, then $|\mathrm{Fix}_l(\sigma)| = \sqrt{n} + 1$. Furthermore, since G is transitive on l, $[G : G_P] = n + 1$. Then, substituting these values in $|\mathrm{Fix}_l(\sigma)| = \frac{[G:G_P]K_\sigma}{|\sigma^G|}$, we obtain

$$\left|\sigma^G\right| (\sqrt{n} + 1) = K_\sigma(n + 1). \tag{2}$$

Therefore, since $(n + 1, \sqrt{n} + 1) \mid 2$, either n is even and $n + 1 \mid \left|\sigma^G\right|$, or n is odd and $n + 1 \mid 2\left|\sigma^G\right|$. That is $n + 1 \mid \left|\sigma^G\right| (2, n + 1)$.
Let $L = soc(G)$. Since G is faithful on l, then $L \not\leq G_P$. Consequently, $[G/L : G_P L/L] [L : L_P]$, and in particular $[L : L_P]$ divides $\left|\sigma^G\right| (2, n + 1)$. The same holds if we replace L_P with a maximal subgroup M of L containing L_P. That is,

$$[L : M] \mid \left|\sigma^G\right| (2, n + 1).$$

Now, assume that $L = L(q)$, $q = p^h$, is a group of Lie type. When L is classical we may filter the list of maximal subgroups of L provided in [27] and [28], solely by comparing the power of p dividing $[L : M]$ and the one dividing $\left|\sigma^G\right| (2, n+1)$. When G is exceptional of Lie type, the index $[L : M]$ which is evaluated with the information provided in [25] and [32], turns out to be greater than $\left|\sigma^G\right| (2, n+1)$ and is hence ruled out. Therefore, we are able to restrict our study to the case where $\mathrm{Fix}(\sigma)$ is a Baer subplane of Π of order at most q^2, and $C_G(\sigma)$ induces a subgroup isomorphic to $PSL_2(q)$. At this point, the assertion is quite an easy consequence of the following theorem, that we present in its original form because of its indipendent interest (note that, for our purposes we just need to focus on the cases in which $PSL_2(q)$ fixes a line).

5 Theorem. *Let* Π *be a projective plane of order* n *admitting a collineation group* $G \cong PSL(2, q)$, $q > 3$. *If* $n \leq q^2$, *then one of the following occurs:*

(1) $n < q$ *and one of the following occurs:*

a. $\Pi \cong PG(2,4)$ *and* $G \cong PSL(2,5)$*;*
b. $\Pi \cong PG(2,2)$ *or* $PG(2,4)$*, and* $G \cong PSL(2,7)$*;*
c. $\Pi \cong PG(2,4)$ *and* $G \cong PSL(2,9)$*.*

(2) $n = q$, $\Pi \cong PG(2,q)$ *and one of the following occurs:*

a. G *fixes a line or a point and* q *is even;*
b. G *is strongly irreducible and* q *is odd.*

(3) $q < n < q^2$ *and one of the following occurs:*

a. G *fixes a point or a line, and one of the following occurs:*
  i. Π *has order* 16 *and* $G \cong PSL(2,5)$*;*
  ii. Π *is the Lorimer-Rahilly plane of order* 16 *or the Johnson-Walker plane of order* 16*, or their duals, and* $G \cong PSL(2,7)$*;*
b. G *fixes a subplane* Π_0 *of* Π*,* q *is odd and one of the following occurs:*
  i. Π *has order* 16*,* $\Pi_0 \cong PG(2,4)$ *and* $G \cong PSL(2,5)$*;*
  ii. $\Pi_0 \cong PG(2,2)$ *or* $PG(2,4)$*, and* $G \cong PSL(2,7)$*;*
  iii. $\Pi_0 \cong PG(2,4)$ *and* $G \cong PSL(2,9)$*.*
c. G *is strongly irreducible and* q *is odd;*

(4) $n = q^2$ *and one of the following occurs:*

a. G *fixes a point or a line, and one of the following occurs:*
  i. Π *has order* 25 *and* $G \cong PSL(2,5)$*;*
  ii. Π *has order* 81 *and* $G \cong PSL(2,9)$*;*
  iii. Π *has order* q^2 *and* $G \cong PSL(2,q)$ *with* q *even.*
b. G *fixes a subplane* Π_0 *of* Π*,* q *is odd and one of the following occurs:*
  i. $\Pi_0 \cong PG(2,q)$ *and* $G \cong PSL(2,q)$*;*
  ii. $\Pi_0 \cong PG(2,4)$ *and* $G \cong PSL(2,5)$*;*
  iii. $\Pi_0 \cong PG(2,4)$ *and* $G \cong PSL(2,9)$*;*
  iv. Π_0 *is a Hughes plane of order* 9 *and* $G \cong PSL(2,9)$*;*
c. G *is strongly irreducible.*

For further details on this theorem and its consequences the reader is referred to [35] and to [4], respectively.

Finally, from Theorem 4 and by using the O'Nan Scott Theorem (e.g. see [9]), we settle down the case when G is primitive on l.

6 Theorem. *If G is faithful and primitive on l, then the involutions in G are perspectivities of Π and one of the following occurs:*

(1) $soc(G) \cong (Z_p)^d$ and $n+1 = d(G) = p^d$ (note that in this case G might have odd order), or

(2) $K \trianglelefteq G \leq aut(K)$ and one of the following occurs:

a. $n = 2^s$, $\Pi \cong PG(2, 2^s)$ and $K \cong PSL(2, 2^s)$;

b. $n = 2^{2s}$, s odd, and $K \cong Sz(2^s)$;

c. $n = 2^{3s}$, s even, and $K \cong PSU(3, 2^s)$.

For a proof see [5].

Now, we are in a position to cope with the case where G is faithful and primitive of rank 3 on l.
If G is (faithful) primitive of rank 3 on a line l of Π, set Δ_1 and Δ_2 the two nontrivial G-orbits on l. Clearly, we may assume that $|\Delta_1| \leq |\Delta_2|$.

7 Theorem. *If G is faithful and primitive of rank 3 on l, then n is even, $G \leq A\Gamma L(1, n+1)$ and $|\Delta_1| = |\Delta_2| = n/2$.*

Proof. Assume that $p = 2$. Then n is odd and hence at least one among the two orbits Δ_1, Δ_2 has odd length. Since the involutions in G are homologies of axis distinct from l, any Sylow 2-subgroup S of G fixes exactly two points on l, namely O and a point Q lying either in Δ_1 or in Δ_2. Furthermore, S acts semiregularly on the remaining points. If Q lies in Δ_1, then the order of S must divide $|\Delta_1| - 1$ and $|\Delta_2|$. In particular, the maximum power of 2 dividing $|\Delta_1| - 1$ and $|\Delta_2|$ must be the same. Now, if Q lies in Δ_2, by exchanging the roles of Δ_1 and Δ_2 in the previous argument, we obtain that the maximum power of 2 dividing $|\Delta_2| - 1$ and $|\Delta_1|$ must be the same. By filtering the list given in [31], [10] and [11], §5, with respect to one of the previous restrictions, according to whether Q lies in Δ_1 or in Δ_2, respectively, we obtain that either $G_O \leq \Gamma L(1, q^d)$, or $A_5 \trianglelefteq G_O$ and $n+1 = 2^4$ or 2^6. Actually, the numerical cases are ruled out by [19]. Thus, $G_O \leq \Gamma L(1, q^d)$ and $|\Delta_1| = (q^d - 1)/v$ and $|\Delta_2| = (v-1)(q^d-1)/v$, where v is a divisor of $q^d - 1$ (e.g. see [11], §3). As $p = 2$, then v must be odd. Therefore, $|\Delta_1|$ is odd, while $|\Delta_2|$ is even. Note that, $|\Delta_1| - 1 = (q^d - (1+v))/v$, must be divisible by the same power of 2 dividing $|\Delta_2| = (v-1)(q^d-1)/v$, according to the above argument. That is, the power of 2 dividing $q^d - (1+v)$ and $v-1$ must be the same. Nevertheless, this is impossible, since $v-1$ and $v+1$ lie in distinct remainder classes modulo 4. Therefore, the case $p=2$ is completely ruled out.

Assume that p is odd. Let R be a Sylow 2-subgroup of G_O. Since the involutions in R must be elations of Π, then R must be semiregular both on Δ_1 and on Δ_2. Consequently, the maximum power of 2 dividing $|\Delta_1|$ and $|\Delta_2|$ must be the same.

If $q^d \equiv_4 3$, then $n \equiv_4 2$ as $n+1 = q^d$. Then G is solvable by [21], Theorem 13.12. By filtering the list of groups given in [10] with respect to the above restriction, we obtain $G_O \leq \Gamma L(1, q^d)$, where $|\Delta_1|$ and $|\Delta_2|$ are of known length. If $|\Delta_1| < |\Delta_2|$, then $v-1$ is even by [11], Theorem 3.7.(2), which violates the fact that the maximum power of 2 dividing $|\Delta_1|$ and $|\Delta_2|$ must be the same. Hence, $|\Delta_1| = |\Delta_2| = (q^d-1)/2$.

If $p^d \equiv_4 1$, then $n \equiv_4 0$. Now, by filtering Tables 12, 13 and 14 of [31], by filtering [10] and [11], §5, with respect to the above restriction and to $p^d \equiv_4 1$, we see that either $G_O \leq \Gamma L(1, q^d)$ or $n+1 = 31^2$, 47^2, 71^2 or 119^2. Actually, the numerical cases are ruled out by [21], Theorem 3.6. Hence, $G_O \leq \Gamma L(1, q^d)$ in this case as well. By arguing as in the case $q^d \equiv_4 3$, we obtain again $|\Delta_1| = |\Delta_2| = (q^d-1)/2$. This completes the proof. $\boxed{QED}$

Finally, special care is devoted to the case when Π is the projective extension of a translation plane, as shown by the following theorem.

8 Theorem. *Let Π be the projective extension of a translation plane of order $n = u^t$ by the line l. Assume that Π admits a collineation group G fixing the line l and acting faithfully and primitively on it. Then one of the following occurs:*

(1) $n = 2^{2^s}$, *$n+1$ is a Fermat prime and* $Z_{n+1} \trianglelefteq G \leq AGL_1(n+1)$;

(2) $\Pi \cong PG_2(2^s)$ *and* $PSL(2, 2^s) \trianglelefteq G$;

(3) $\Pi \cong L(2^{2s})$ *and* $Sz(2^s) \trianglelefteq G$.

Proof. Assume that $soc(G) \cong (Z_p)^d$. Then $u^t + 1 = p^d$, since $n = u^t$. By [38], either $t=1$, $p=2$ and u is a Mersenne prime, or $d=1$, $u=2$, $t=2^h$, $h \geq 1$ and p is a Fermat prime, or $n = u^t = 8$ a $p^d = 9$.

Assume that $t=1$, $p=2$ and u is a Mersenne prime. Then $\Pi \cong PG_2(u)$ and hence $G \leq PGL_2(u)$. Nevertheless, this case is not admissible since each Sylow 2-subgroup of G has a nontrivial kernel on l consisting of an involutory homology.

Assume that $d=1$, $u=2$, $t=2^s$, $s \geq 1$ and p is a Fermat prime. Hence, $n = 2^{2^h}$ and $Z_p \trianglelefteq G \leq AGL(1,p)$, where $p = 2^{2^h}+1$.

Finally, assume that the latter occurs. Then $\Pi \cong PG_2(8)$ and hence $G \leq P\Gamma L_2(8)$. This is a contradiction, since G contains an elementary abelian subgroup of order 9 acting regularly on l.

Now, assume that G is almost simple. Then the cases (2a)-(2c) of Theorem 4 occur. Actually, the case (2c) is ruled out since Π is the projective extension of a translation plane, whereas the cases (2a)-(2b) really occur (e.g. see [26]). $\boxed{QED}$

We point out that the first case does occur in the Desarguesian planes $PG_2(2^{2^s})$. It should be also remarked that this case is compatible with an open case model of a solvable flag-transitive plane of even order admitting a cyclic group on the line at infinity, due to Suetake [40].

We conclude this paper with the following

9 Problem. Try to solve the analog of the "transitive" Cofman problem when the action of G on the line l of Π is not faithful.

Some suggestions are given below:

(1) The group $SL_2(q)$ inducing $PSL_2(q)$ on a line of a Desarguesian plane of order q is a solution of the problem.

(2) Some arguments developed in [36] may be employed to gain information on the structure of G. Indeed, without loss of generality, we may assume that G is minimal with respect to the property of inducing a primitive group on l. This implies $N = \Phi(G)$, where N is the kernel of the action of G on l and $\Phi(G)$ is the Frattini subgroup of G. So, N is nilpotent. Now, by using the Burnside argument (e.g. see [24]), it is easily seen that either N is central in G, or $G \leq \Gamma L(S/\Phi(S))$ for a Sylow t-subgroup S of N. The latter is ruled out by the semiregularty of N on $\Pi - l$, as N is a group of perspectivities of axis l. Thus, G is a central extension of the group induced by G on l, and the theory of Schur's multipliers can be applied to yield information on the structure of G.

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