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Generalized j-planes

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Abstract. We construct and study a class of translation planes with kernel $K \cong GF(q)$, order q^n, and $n > 2$. These planes generalize the j-planes discovered by Johnson, Pomareda and Wilke in [14]. We show these planes are actually $jj \cdots j$-planes. Hence, most of the results obtained in this article are on $jj \cdots j$-planes. In fact, our study shows that these planes are either nearfield or new.

An infinite class of nearfield $jj \cdots j$-planes is shown to exist, and a finite set of sporadic non-André $jj \cdots j$-planes is presented.

Keywords: Translation planes, homology groups, nets, j-planes

MSC 2000 classification: Primary 51, 05; Secondary 20

1 Introduction

One of the most important objects in Finite Geometries are translation planes, they have been extensively studied given their connection to almost every known finite geometric structure. However, this study has been notoriously restricted to translation planes of dimension two over their kernel, that is, with spreads in $PG(3, q)$. One of the few classes of large dimensional translation planes that is well-known is the class of André planes.

A collineation of an affine plane Π is an injective map that preserves the incidence relation. The group of all collineations of a given translation plane Π is called 'the full collineation group of Π' and will be denoted as $Aut(\Pi)$. Any subgroup of $Aut(\Pi)$ will be called a 'collineation group'. The group of collineations fixing the zero vector is called 'the translation complement'. The group of all linear transformations in the linear complement is called 'the linear translation complement'. It is known that the collineation group of Π is the semidirect product of the translation group and the translation complement. Moreover, the translation complement is a subgroup of $\Gamma L(V)$- the group of bijective semilinear transformations of V onto itself.

[i] To N.L. Johnson on his 70^{th} birthday

If a collineation Ψ of Π fixes a line ℓ pointwise and all the lines through a point P setwise, then Ψ is called a perspectivity of Π, or (P, l)-perspectivity. Moreover, if $P \in \ell$, then Ψ is called and elation. On the other hand, if $P \notin \ell$, then Ψ is called a homology. In either case, P is called the center of Ψ and ℓ is called the axis of Ψ.

It follows that if Ψ is a perspectivity of Π with axis l so that $0 \in l$ (Ψ is called an affine homology), then there exists a component C of S such that $l = C$. Also, if the center P of the affine homology Ψ is in ℓ_∞ (in the projective extension of Π), then the component C_P that contains P is called the co-axis of Ψ.

André planes of order q^n admit two affine homology groups of order $(q^n - 1)/(q-1)$ that fix a pair of components L and M such that one group has axis L and coaxis M and the remaining group has axis M and coaxis L (they are symmetric homology groups). If the groups are cyclic, then the planes may be characterized. For instance, when $n = 2$, Johnson proved:

1 Theorem. *[Johnson [9]] Let π be a translation plane with spread in $PG(3, q)$ that admits at least two homology groups of order $q + 1$. Then one of the following occurs:*

(1) $q \in \{5, 7, 11, 19, 23\}$ (the irregular nearfield planes and the exceptional Lüneburg planes are examples),

(2) π is André,

(3) q is odd and π is constructed from a Desarguesian spread by $(q+1)$-nest replacement (actually $q = 5$ or 7 for the irregular nearfield planes also occur here),

(4) q is odd and π is constructed from a Desarguesian spread by a combination of $(q + 1)$-nest and André net-replacement,

(5) $q \equiv -1 \bmod 4$ and the axis/coaxis pair is invariant under the full collineation group (in this case there is a non-cyclic homology group of order $q+1$),

(6) $q = 7$ and the plane is the Heimbeck plane of type III with 10 homology axes of quaternion groups of order 8.

However, when $n > 3$ the situation is different.

2 Theorem. *[Johnson and Pomareda [13]] Let π be a translation plane of order q^n that admits symmetric cyclic affine homology groups of orders $(q^n - 1)/(q - 1)$, $n > 2$. Then, the plane π is André.*

Moreover, we know that

3 Theorem. *[Johnson–Pomareda [12]] Let π be a translation plane of order q^n and kernel containing $GF(q)$. Assume that the plane admits a cyclic homology group of order $(q^n-1)/(q-1)$. Then one of the following must occur:*

(i) The axis and the coaxis of the group are both invariant or

(ii) The plane is André.

Thus, a natural question arises: is every plane admitting just one affine homology group of order $(q^n-1)/(q-1)$ an André plane? An example of a known class of non-André planes of order q^2 satisfying that property is the j-planes of Johnson, Pomareda and Wilke (see [14]) , these planes are translation planes that, following André's theory of spreads (see [1]), are represented by the spreadset:

$$S=\left\{\begin{bmatrix}1 & 0\\ 0 & \partial^j\end{bmatrix}\begin{bmatrix}t & u\\ bu & t+au\end{bmatrix};\ t,u\in GF(q)\ \textit{and}\ \partial=\det\begin{bmatrix}t & u\\ bu & t+au\end{bmatrix}\right\}$$

for some irreducible polynomial $p(x)=x^2-ax-b\in GF(q)[x]$.

An easy way to check whether or not this set actually defines a spread is using the following well-known result.

4 Theorem. *A set of q^n matrices $S=\{0\}\cup\{M_1,\dots M_{q^n}\}$, together with $x=0$ defines a spread of V_{2n} if and only if each M_i and every non-zero difference M_i-M_j is non-singular.*

Most of what is known about j-planes is on either [14], [6], [11] or in the handbook of finite translation planes [3]. The most important results on j-planes are:

(1) The regular nearfield planes are $(q-1)/2$-planes,

(2) Any translation plane or order q^2, q odd, admitting a cyclic collineation group (in the translation complement) that intersects trivially the kernel homologies must be a j-plane.

(3) There are many infinite classes of j-planes of order p^{2r}, where p is an odd prime, and many sporadic examples of j-planes of small order. The latter were found using a computer.

(4) The set of j-planes with spread in $PG(3,q)$ is equivalent to the set of monomial conical flocks in $PG(3,q)$.

A first generalization of j-planes can be found in [16], where $jj\cdots j$-planes of order 4^3 are constructed and studied. These planes are shown to be either

André or new. Also, André net replacements and derivation are used to obtain even more new planes. Finally, $jj\cdots j$-planes of order 4^3 are associated to flat flocks, which are partitions of the Segre variety $\mathcal{S}_{n,n}$ by Veroneseans $\mathcal{V}_n$.

This article extends the result in [16] to $jj\cdots j$-planes of order q^n. Moreover, an apparently larger family of translation planes, called generalized j-planes, is proved to be equal to the family of $jj\cdots j$-planes.

We close this section by mentioning an important fact that will be used later in the definition of $jj\cdots j$-planes and in other parts of this article.

Let F be a field of matrices of order q^n considered as an extension of $GF(q)$, then the determinant of a matrix $M \in F$ can be thought of as the norm (from $GF(q^n)$ onto $GF(q)$) of M (see, for example, [4], p. 585). This shows that the determinant $det : F^* \to GF(q)^*$ is a surjective homomorphism.

2 Definition and basic properties

Let $F \subset M_n(q)$ be a field of matrices isomorphic to $GF(q^n)$ that contains $GF(q)$ as a set of scalar matrices. Consider a function

$$f : M_n(q) \to M_n(q)$$

such that

(1) $f(MN) = f(M)f(N)$, for all $M, N \in F^*$,

(2) $f(M)^{q-1} = Id$ for all $M \in F^*$

Note that $f(M)^{-1} = f(M)^{q-2}$, for all $M \in F^*$.

For any function f as above we define:

$$G = \left\{ \begin{bmatrix} f(M)^{-1} & 0 \\ 0 & M \end{bmatrix};\ M \in F^* \right\} \subset GL(2n, q)$$

Clearly, $G \cong \mathbb{Z}_{q^n-1}$.

5 Definition. Let $S = \{x = 0, y = 0\} \cup \mathcal{O}_G(y = x)$, where $\mathcal{O}_G(y = x)$ is the orbit of the line $y = x$ under the group G. Alternatively,

$$S = \{y = xf(M)M;\ M \in F\} \cup \{x = 0\}$$

In case S is a spread, the associated translation plane Π, which would have order q^n, will be called a generalized j-plane.

6 Remark. Let j be a fixed integer and $n = 2$. Using the previous definition, with a suitable field F, we can construct a j-plane by considering the function

$$f_j : M_2(q) \to M_2(q)$$

defined by

$$f_j(M) = \begin{bmatrix} 1 & 0 \\ 0 & \det(M)^j \end{bmatrix}$$

7 Definition. Let S be a spread of the vector space V and let Π be its associated translation plane. The kernel of Π is the division ring K formed by all endomorphisms of V that stabilize each of the components of S. It can be shown that V is a vector space over K and that $GF(q)$ can be considered as a subring of K.

8 Remark. Since generalized j-planes have spreads in $GL(n, q)$ their kernels always contain $GF(q)$.

Using theorem 4 and the fact that G is cyclic, it is easy to show that the set S in definition 5 is a spread if and only if

$$\det(f(M)M - Id) \neq 0 \quad \text{for every} \quad M \neq Id \text{ in } F^*$$

Moreover, as

$$\det(f(M)M - Id) \neq 0 \iff \det(f(M^{-1})M^{-1} - Id) \neq 0$$

then, for S to be a spread one just needs to check $\det(f(M)M - Id) \neq 0$ for only, roughly, half of the elements in F.

The previous condition, although simple and not very useful theoretically, can be easily implemented in a computer. By doing this, using Maple ©, we have found several examples of the planes discussed in this article, these examples will be surveyed in section 5.

9 Definition. Let $F \subset M_n(q)$ be a field of order q^n, and let $j_2, j_3, \ldots, j_n$ be elements of $\{0, 1, 2, \cdots, q-2\}$, we consider

$$f(M) = \operatorname{diag}(1, \partial^{j_2}, \cdots, \partial^{j_n})$$

for all $M \in M_n(q)$, where $\partial = \det(M)$.

In case this function yields a generalized j-plane, then the plane is called a $j_2 j_3 \cdots j_n$-plane, or simply a $jj \cdots j$-plane.

10 Theorem. *Every generalized j-plane is isomorphic to some $jj \cdots j$-plane.*

Proof. Let Π be a generalized j-plane associated to a function f and a field F. Let M be a generator of F^*. Note that, because of the way f is defined, the minimal polynomial of $f(M)$ divides $x^{q-1}-1$. It follows that all eigenvalues of $f(M)$ are distinct, and thus $f(M)$ is diagonalizable.

Let A be an invertible matrix such that $Af(M)A^{-1}$ is a diagonal matrix. It is easy to see that the map ϕ given by $(x, y) \mapsto (Ax, Ay)$ maps the spread S of Π to

$$\phi(S) = \{y = x[Af(M)A^{-1}][AMA^{-1}];\ M \in F\}$$

Note that $\tilde{F} = AFA^{-1}$ is a field isomorphic to F that contains $GF(q)$ as a set of scalar matrices, and that $Af(M)A^{-1}$ is diagonal. Thus Π is isomorphic to a generalized j-plane with the 'same' field and function that is almost the one used for $jj\cdots j$-planes.

Let $\partial = \det(M)$. Since $< M >= F^*$, then $< \partial >= GF(q)^*$. It follows that

$$Af(M)A^{-1} = \mathrm{diag}(\partial^{j_1}, \partial^{j_2}, \partial^{j_3}, \cdots, \partial^{j_n})$$

for some $j_1, j_2, j_3, \cdots, j_n \in \{0, 1, 2, \cdots, q-2\}$.

So, the only thing left to show is that it is enough to consider j_1 equal to zero. We will do this in two cases.

Case 1) If $(nj_1+1, q-1) = 1$, then there is a number t such that $(\partial^{nj_1+1})^t = \partial$, for all $\partial \in GF(q)^*$.

For every matrix $M \in \tilde{F}^*$, consider $\tilde{\Delta} = \mathrm{diag}(1, \partial^{k_2}, \ldots, \partial^{k_n})$, where ∂ is the determinant of M and $k_i = t(j_i - j_1)$.

Using that $GF(q)$ injects in $\tilde{F}$ as a set of scalar matrices we obtain that

$$\tilde{S} = \{\tilde{\Delta}M;\ M \in F^*\}$$

is the spread of a $jj\cdots j$-plane.

Case 2) If $(nj_1 + 1, q-1) = d \neq 1$, let $\Gamma_\alpha = \{M \in F;\ \det(M) = \alpha\}$ and let α be of order $(q-1)/d$.

Note that $\alpha^{j_1}\Gamma_\alpha = \Gamma_1$ and that $(1)^{j_1}\Gamma_1 = \Gamma_1$, then $|\cup \alpha^{j_1}\Gamma_\alpha| < q^n - 1$. Hence, there is an element $N \in \Gamma_\alpha$ so that $(\tilde{\partial}^{j_1}N) = Id$, where $\tilde{\partial} = \det(N)$. It follows that $\det(\tilde{\Delta}N - Id) = \det(\Delta - Id) = 0$ for $\tilde{\Delta} = \mathrm{diag}(\tilde{\partial}^{j_1}, \tilde{\partial}^{j_2}, \ldots, \tilde{\partial}^{j_n})$, and $\Delta = \mathrm{diag}(1, \tilde{\partial}^{j_2-j_1}, \ldots, \tilde{\partial}^{j_n-j_1})$. Thus, $S = \{\Delta M;\ M \in F^*\}$ would not be a spread, a contradiction. $\boxed{QED}$

As it is implied by the previous theorem, we will now work only with $jj\cdots j$-planes, thus we will need to change the notation a little. Instead of using a function f to construct $jj\cdots j$-planes, we will use Δ_M to denote

$$\Delta_M = \mathrm{diag}(1, \partial_M^{j_2}, \partial_M^{j_3}, \cdots, \partial_M^{j_n})$$

where $\partial_M = \det(M)$, for all $M \in M_n(q)$. We will not use subindices when it is clear what matrix M we are talking about.

With this notation we have that a set $S = \{x = 0\} \cup \{y = x\Delta M;\ M \in F\}$ is a spread if and only if $\det(\Delta_M M - Id) \neq 0$ for all $M \in F$, but that it is enough to check this for 'half' of the elements in F. Using this, we are able to show

11 Corollary. *Let* Π *be a* $j_2 j_3 \cdots j_n -$ *plane of order* q^n*, then* $\gcd(nj_i + 1, q - 1) = 1$ *for every* $i > 1$*.*

Proof. Let $S = \{\Delta M;\ M \in F^*\}$ be the spread of Π.

Without loss of generality, assume $\gcd(nj_2 + 1, q - 1) = d \neq 1$.

Let $M = \alpha Id$, where the order of α is $\frac{q-1}{d}$. Then

$$\Delta M = \mathrm{diag}(\alpha, \alpha^{nj_2+1}, \ldots, \alpha^{nj_n+1}) = \mathrm{diag}(\alpha, 1, \ldots, \alpha^{nj_n+1}).$$

Thus, $\det(\Delta M - Id) = 0$, which is a contradiction. $\boxed{QED}$

12 Corollary. *Let* Π *be a* $j_2 j_3 \cdots j_n -$ *plane of order* q^n*. Then, if both* q *and* n *are odd, then all the* j_i*'s are even.*

13 Remark. A $00 \cdots 0$-plane is Desarguesian because the spreadset has the structure of a field.

14 Corollary. *There are no* $jj \cdots j$*-planes of order* 3^n *with* n *odd but the Desarguesian.*

Proof. Since n and q are odd, then the previous corollary forces each of the j_i's to be even. Now we use that $q = 3$ and thus the j_i's can only take the values 0 and 1. It follows that all the j_i's are zero and the plane is Desarguesian. $\boxed{QED}$

3 The full collineation group of a $jj \cdots j$-plane

15 Lemma. *Let* σ *be an automorphism of* $GF(q)$*, then for every tuple* $(j_2, j_3, \ldots, j_n)$*, the collineation* Ψ*, defined by*

$$(x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n) \mapsto (x_1^\sigma, x_2^\sigma, \cdots, x_n^\sigma, y_1^\sigma, y_2^\sigma, \cdots, y_n^\sigma),$$

induces an isomorphism between two (not necessarily distinct) $j_2 j_3 \cdots j_n$*-planes.*

Proof. It is clear that the image under Ψ of a $jj \cdots j$-plane is also a translation plane.

Now let Π be a $j_2 j_3 \cdots j_n$-plane with spread

$$S = \{y = x\Delta_M M;\ M \in F\} \cup \{x = 0\}$$

for some field F.

We first note that Ψ maps the subspaces $y = xM$, for $M \in F$, to subspaces of the form $y = xN$, where N lives in a field F'. So, we can think Ψ mapping a field F to a field F', which contains $GF(q)$ as a set of scalar matrices.

Now, using that $\sigma \in Aut(GF(q))$ we get that $\det(\Psi(M)) = \Psi(\det M)$, for all $M \in F$. Finally, by just performing a few simple computations, we get that,

$$\Psi(y = x\Delta_M M) = (y = x\Delta_{\Psi(M)}\Psi(M)).$$

for all $M \in F$.

As the Δ's in the previous formula are all associated to the same tuple $(j_2, j_3, \ldots, j_n)$, then we obtain an isomorphism of $j_2 j_3 \cdots j_n$-planes. $\boxed{QED}$

The previous lemma shows what automorphisms of $GF(q)$ do to $jj\cdots j$-planes, thus our study of collineations of $jj\cdots j$-planes will now focus on linear collineations.

All $jj\cdots j$-planes admit at least two homology groups. These groups will play an important role in the following sections. The most important of them is the following subgroup of G, see p6:

$$H_y = \left\{ \begin{bmatrix} \Delta^{-1} & 0 \\ 0 & M \end{bmatrix};\ \det(M) = 1 \right\}$$

This group is a cyclic $((\infty), y = 0)$-homology group of order $(q^n - 1)/(q - 1)$ that intersects trivially the kernel homologies.

The second group is H_x, a $((0), x = 0)$-homology group of order $q-1$ induced by

$$H_0 = \left\{ N \in G;\ N = \begin{bmatrix} \Delta^{-1} & 0 \\ 0 & rId \end{bmatrix},\ r \in GF(q)^* \right\}$$

Specifically,

$$H_x = \left\{ \begin{bmatrix} r\Delta^{-1} & 0 \\ 0 & Id \end{bmatrix};\ r \in GF(q)^* \text{ and } \Delta = \Delta_{rId} \right\}$$

It is easy to see that H_x and H_y are symmetric homology groups. It is known this implies the groups commute with each other, and thus, $H_x H_y = H_x \times H_y$.

For a fixed $jj\cdots j$-plane Π, consider the homology group H_y. Since its order is $(q^n - 1)/(q - 1)$, then its line orbits define $q - 1$ André nets in the plane. Together with the lines $y = 0$ and $x = 0$, these nets partition the spread. All such orbits look like:

$$N_v = \{y = x(\Delta_L LM);\ M \in F^* \text{ and } \det(M) = 1\}$$

where L is fixed with $\det(L) = v \in GF(q)^*$.

In a future article we will consider replacements of these nets and the planes they induce.

16 Definition. Let Π be a projective plane and let P be a point and l a line of Π. The plane Π is called (P,l)-transitive, if for any two points X and Y with $P \neq X, Y$ and X, Y not in l and $PX = PY$ there exists a homology Φ with center P and axis l such that $\Phi(X) = Y$.

17 Theorem. *[Gingerich, see [19], p.14] Let Π be a translation plane. If P and Q are distinct points on ℓ_∞ and if O is an affine point of Π, then Π is (P, OQ)-transitive, if and only if Π is (Q, OP)-transitive.*

Recall that $jj\cdots j$-planes admit two symmetric homology groups (H_y and H_x) of orders $(q^n-1)/(q-1)$ and $q-1$ respectively with axis/coaxis (0) and (∞), and note that if the plane is also André then there is a second homology group H of order $(q^n-1)/(q-1)$, which is symmetric to H_y and has the same orbits in ℓ_∞ that H_y has. It follows that the group H_xH acts transitively on $\ell_\infty \backslash \{(0),(\infty)\}$, and that (using Gingerich's theorem) there is a symmetric homology group acting transitively on $\ell_\infty \backslash \{(0),(\infty)\}$. We have proved.

18 Theorem. *Every André $jj\cdots j$-plane is a nearfield plane.*

19 Corollary. *The translation complement of a non-André $jj\cdots j$-plane Π fixes the lines $x = 0$ and $y = 0$.*

Proof. We notice that $x = 0$ and $y = 0$ are the axis and coaxis of the cyclic homology group H_y. Since the matrices of the spreadset of Π may be written using entries in $GF(q)$, then the kernel of the plane contains $GF(q)$.

The result follows from theorem 3. $\boxed{QED}$

20 Remark. Since the group G has an orbit of size q^n-1 in ℓ_∞ and $y = x$ is a component of every $jj\cdots j$-plane, then any collineation Ψ in the linear translation complement will be in a coset of G having a representative looking like:

$$\Psi = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}$$

for some $A \in GL(n,q)$.

21 Lemma. *Let Ψ be a collineation of a $jj\cdots j$-plane Π that is represented as in the previous remark. Then, A is an element of F.*

Proof. Let N_1 be the net of Π formed by all the matrices in F with determinant one, and let M be a generator of N_1 as a subgroup of F^*.

Since Ψ maps $y = xM$ to an element in the spread of Π, then

$$A^{-1}MA = \Delta_P P$$

for some $P \in F$.

Now recall that the map $(x,y) \mapsto (x,yM)$ is a homology of Π, then the map

$$\Phi : (x,y) \mapsto (x, yA^{-1}MA) \qquad \text{or} \qquad \Phi : (x,y) \mapsto (x, y\Delta_P P)$$

is also a homology of Π.

As this putative new homology has the same order of M, then it follows that Π admits two homology groups of order $(q^n-1)/(q-1)$, namely H_y and the group generated by Φ. Since the order of the product of these groups is

$$|H_y < \Phi > | = \frac{|H_y|| < \Phi > |}{|H_y \cap < \Phi > |} = \frac{\left(\frac{q^n-1}{q-1}\right)^2}{|H_y \cap < \Phi > |}$$

then $(q^n-1)/(q-1)^2$ divides $|H_y \cap < \Phi > |$. This means that the fields F and $A^{-1}FA$ intersect in at least $(q^n-1)/(q-1)^2$ elements, thus these fields must be the same (as $n > 2$).

Hence, A normalizes F, and thus Ψ may be interpreted as kernel homology of the Desarguesian plane determined by F, then A must be an element of F. $\boxed{QED}$

22 Lemma. *If a collineation Ψ of a $jj\cdots j$-plane Π is represented as in the previous lemma. Then, the group generated by Ψ contains an element that acts on the nets of Π, and that fixes at least two nets.*

Proof. Let $\Psi = \mathrm{diag}(A, A)$ with $A \in F$. The action of Ψ on the lines of the spread is given by

$$\begin{aligned}\Psi(y = x\Delta M) &= (y = xA^{-1}\Delta MA)\\ &= (y = xA^{-1}\Delta AM)\end{aligned}$$

Thus, we can say $\Psi(\Delta) = A^{-1}\Delta A$. Moreover, since Ψ is a collineation, then $A^{-1}\Delta A = \Delta\Delta_N N$, for some $N \in F$ (note that $\det(\Delta_N N) = 1$). It follows that Ψ induces an action on the nets of the plane Π.

It is easy to see that the net N_1 is fixed by Ψ, then the group generated by Ψ acts on a set with $q-2$ elements. We just need to show that Ψ fixes another net.

Assume that $gcd(1 + j_2 + \cdots + j_n, q-1) = 1$. Then $\det(\Delta_N N) = 1$ implies $\det(N) = 1$, as $\det(\Delta_N N) = \det(N)^{1+j_2+\cdots+j_n}$. In this case, the action of Ψ is

$$\Psi(\Delta) = A^{-1}\Delta A = \Delta$$

It follows that Ψ fixes all nets.

Now assume that $gcd(1 + j_2 + \cdots + j_n, q-1) = d \neq 1$. In this case, we still have the group generated by Ψ acting on a set with $q-2$ elements. Let's say that $|A| = |\Psi| = a$. Moreover, there are $(q-1)/d$ classes of nets containing exactly d nets each, these classes are formed by matrices having the same determinant.

We note that Ψ fixes each of these classes. In case that $gcd(a, d) = 1$, then Ψ would fix an element of each class having d elements. Similarly, since the net

N_1 is fixed, then if $gcd(a, d-1) = 1$ implies that Ψ would fix an element of the class that contains the net N_1. Hence, the only way to not fix a second net is to have $gcd(a, d) \neq 1$ and $gcd(a, d-1) \neq 1$.

Finally, notice that if there is an orbit of nets under the group generated by Ψ that is larger than others, then we can find an element in $< \Psi >$, that fixes a net. Thus, no second net fixed means that all orbits have the same size, and this implies that a divides both d and $d-1$. Thus, $a = 1$. $\boxed{QED}$

23 Lemma. *Let Ψ be a collineation of a $jj \cdots j$-plane Π as in the previous two lemmas. Then, A is a scalar matrix. That is, Ψ is a kernel homology.*

Proof. Let $\Psi = \mathrm{diag}(A, A)$ with $A \in F$. If Ψ is not a kernel homology, then we can consider a non-trivial element $\Phi \in < \Psi >$ such that the group $< \Phi >$ does not contain kernel homologies.

Moreover, using the previous lemma and that all groups considered are cyclic, we can assume Φ, given by $\Phi = \mathrm{diag}(B, B)$, fixes at least two nets and has prime order b (which must divide $q^n - 1$).

Using that Φ fixes a non-trivial net we get that $B^{-1}\Delta B = \Delta N$, with $N \in F$. Note that $B^{-i}\Delta B^i = \Delta N^i$ for all i. This implies that the order of N divides the order of B, so $N \in < B >$ (using that $GF(q)^*$ is cyclic), thus $N = B^i$ for some i.

We know that $B^{-1}\Delta B = \Delta B^i$, which implies $\Delta^{-1}B^{-1}\Delta = B^{i-1}$. It follows that $\Delta^{-j}B^{-1}\Delta^j = B^{-1^{j-1}(i-1)^j}$. We now consider $j = q-1$ and we get $B = B^{-1^{q-1}(i-1)^{q-1}}$. But b (the order of B) is prime, so there is no way to find j such that $-1^{q-1}(i-1)^{q-1} \equiv 0 \bmod b$. So, $i = 1$.

Finally we obtain $B^{-1}\Delta B = \Delta B$, which forces $B = Id$, a contradiction. $\boxed{QED}$

24 Theorem. *The linear part of the translation complement of a non-André $jj \cdots j$-plane Π is isomorphic to the direct product of G and the kernel homologies.*

Proof. We are looking for collineations not in G. So, we use the previous lemma and remark to restrict ourselves to find a linear element Ψ in the translation complement of the $jj \cdots j$-plane Π that fixes $y = x$, $x = 0$ and $y = 0$. That is, $\Psi = \mathrm{diag}(A, A)$.

Note that the net N_1 goes under Ψ to $A^{-1}N_1A$, which is a subset of size $(q^n - 1)/(q - 1)$ of a field. As, N_1 is the only subset of the spreadset of Π with these conditions, we get $\Psi(N_1) = N_1$. Moreover, since $A^{-1}FA$ is a field that intersects F in at least $(q^n - 1)/(q - 1)$ elements (determinant one matrices) and $n > 2$, then $A^{-1}FA = F$, which means that A normalizes F. It follows that Ψ is a collineation of the Desarguesian plane with spreadset F, which forces A to be in F (Ψ is linear).

The previous lemma and the fact that the kernel homologies commute with G finish the proof. $\boxed{QED}$

25 Corollary. *The linear part of the translation complement of a non-André $jj \cdots j$-plane Π contains no p-elements.*

26 Corollary. *A $j_2 j_3 \cdots j_n$-plane of order q^n with $(j_1, j_2, \ldots, j_n) \neq (0, 0, \ldots, 0)$ is not Desarguesian.*

Proof. Let Π be a Desarguesian $j_2 j_3 \cdots j_n$-plane of order q^n. Consider Φ, a collineation of Π interchanging $x = 0$ and $y = 0$, then we use G to assume that Φ fixes $y = x$ as well. Thus Φ looks like $\Phi = \begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix}$ for some $A \in GL(n, q)$.

It follows that $\Phi(y = x\Delta M) = (y = xB^{-1}(\Delta M)^{-1}B)$. Now we proceed as we did in the proof of theorem 24 to see that B normalizes the field F and thus Φ must be a kernel homology. Hence, Π is not Desarguesian. $\boxed{QED}$

27 Remark. We had already learned that the kernel of a $jj \cdots j$-plane contains $GF(q)$. Now that we know the translation complement of any non-André f-plane it is clear that non-Desarguesian $jj \cdots j$-planes have kernel isomorphic to $GF(q)$.

Now we will rule out a few classes of planes as candidates to be non-André $jj \cdots j$-planes. The rest of the classes of known translation planes can be also excluded either trivially or following the same procedures used in [16]. Hence, the following results will show that the class of non-André $jj \cdots j$-planes is new. However, we must not forget that there are also planes that are obtained by transposition, net replacement and derivation on known planes. Some of these are studied later in this paper, the others will be investigated in a future paper.

28 Remark. Since $jj \cdots j$-planes are of dimension larger than two over their kernels, then several known planes such as conical flock planes are not even mentioned in the list, as these are two dimensional. The list of known planes is fairly short when planes of dimension two are excluded, this happens because high-dimensional planes have not yet been studied as thoroughly as the ones with spread in $PG(3, q)$.

Recall that in corollary 26 we showed that $jj \cdots j$-planes cannot be Desarguesian (unless all the j's are 0). So, most of the results that follow assume the plane to consider is non-André.

29 Remark. Since the collineation group of any given non-André $jj \cdots j$-plane Π fixes the lines $x = 0$ and $y = 0$ (Lemma 19), the plane Π is not flag-transitive. Also, as the collineation group of the Culbert - Ebert planes of order q^3 (see [5]) admit a group of order 3 induced by the automorphism of the underlying field of matrices of order q^3 and non-André $jj \cdots j$-planes do not, then a Culbert – Ebert plane cannot be a non-André $jj \cdots j$-plane.

30 Remark. A $jj\cdots j$-plane Π does not admit collineations fixing the lines in ℓ_∞ of a Baer subplane. Thus, Π cannot be a generalized Hall plane.

Also, $jj\cdots j$-planes cannot be Johnson non-André net (hyper regulus) replacement planes (see [7] and [8]), as these planes have a very small collineation group, "induced" by the plane they were constructed from.

Finally, a derived lifted plane (see [10]) admits p-elements as collineations, which may be elations or Baer elements. If we assume that a non-André $jj\cdots j$-planes is a derived lifted plane then we obtain contradiction with corollary 25.

In general, transposing the spread of a translation plane, described below, may yield a different translation plane. However, we shall see that $jj\cdots j$-planes of order q^n do not induce new planes under transposition.

31 Definition. Let $S = \{y = xM\} \cup \{(x = 0)\}$ be a spread. Then the spread given by $S^t = \{y = xM^t\} \cup \{x = 0\}$ is called the transposed spread of S. It is known that, after a change of basis, the collineations of S^t look like

$$(x, y) \to (x, y)\begin{bmatrix} D^t & B^t \\ C^t & A^t \end{bmatrix},$$

where

$$(x, y) \to (x, y)\begin{bmatrix} A & C \\ B & D \end{bmatrix}$$

is a collineation of S.

32 Lemma. *If Π be a $j_2 j_3 \cdots j_n$-plane, then its transposed plane is also a $j_2 j_3 \cdots j_n$-plane.*

Proof. Let $S = \{y = x\Delta M;\ M \in F^*\} \cup \{x = 0\}$ be the spreadset for Π. Then S^t looks like

$$S^t = \{y = xM^t\Delta^t;\ M \in F^*\} \cup \{x = 0\}.$$

After the change of basis $(x, y) \mapsto (y, x)$,

$$S^t = \{y = x\Delta^{-t}M^{-t};\ M \in F^*\} \cup \{x = 0\}.$$

Since $\Delta^t = \Delta$ and $\det(M^{-1}) = \det(M)^{-1}$ for $M \in F^*$, the transposed spread is

$$S^t = \{y = x\Delta M^t;\ M \in F^*\} \cup \{x = 0\}.$$

Recall that F is a field obtained by extending $GF(q)$ by using θ (see section 2). Note that by extending K by θ^t we obtain $F^t = \{M^t;\ M \in F\}$, since θ^t satisfies $p(x)$, then $F^t \cong F$. $\boxed{QED}$

33 Remark. With the previous result we know that transposition of planes is not going to yield a $jj\cdots j$-plane unless one starts with a $jj\cdots j$-plane. Moreover, we know that the transposed plane cannot be André as its collineation group is conjugated to the collineation group of the original plane, and this plane was not André.

As odd as it may seem, there are cases when a $jj\cdots j$-plane is isomorphic to its transposed plane, this will be proved in lemma 42.

We now want to show that $jj\cdots j$-planes are not symplectic, in order to do this we need to learn a little more about this class of planes.

34 Definition. A translation plane Π is said to be symplectic (see [17], [2], [20] and [21]) if it admits a spreadset of symmetric matrices, which we will call a symplectic spread.

Recently Kantor [18] showed that every symplectic spread is also symplectic over its kernel. Using this result, we can assume that a $jj\cdots j$-plane or a replaced $jj\cdots j$-plane can be represented in some basis by a set of $n\times n$ matrices that are symmetric.

35 Theorem. *[Johnson and Vega [15]] Let π be a symplectic translation plane of order q^n and kernel isomorphic to $GF(q)$. Then any affine homology of π must have order dividing $q-1$.*

36 Corollary. *A non-Desarguesian symplectic plane of order q^n and $n>1$ cannot be a $jj\cdots j$-plane.*

Proof. We now have an affine homology group of order $(q^n-1)/(q-1)$ which must divide $(q-1)$, impossible unless $n=1$. $\boxed{QED}$

4 Isomorphisms of $jj\cdots j$-planes

37 Lemma. *Let Π_1 and Π_2 be two non-André $jj\cdots j$-planes. Assume that Ψ is an isomorphism between Π_1 and Π_2. Let H_i be the homology group of order $(q^n-1)/(q-1)$ of the plane Π_i for $i=1,2$. Then H_2 and the homology group induced by Ψ and H_1 in Π_2 have the same axis and coaxis.*

Proof. Firstly, because of lemma 15 we consider Ψ to be represented by an element of $GL(2n,q)$.

Recall that the homology groups H_1 and H_2 are cyclic and note that there is another cyclic homology group of order $(q^n-1)/(q-1)$ acting on Π_2: the group induced by Ψ and H_1. Let us call this group $\Psi(H_1)$. It is clear that we can assume that the groups H_1 and H_2 have same axis $y=0$ and coaxis $x=0$.

Let l and m be the axis and coaxis of $\Psi(H_1)$ respectively. If the set of lines $\{x=0,\ y=0,\ l,\ m\}$ has size at least 3, then there is going to be a collineation of Π that fixes neither $x = 0$ nor $y = 0$. This contradicts lemma 19.

It follows that the axes are either interchanged or fixed. The first case contradicts lemma 19; thus the latter case holds. $\boxed{QED}$

Let $\{N_{i,v};\ v \in \{1, \alpha, \ldots, \alpha^{n-1}\}\}$ be the set of nets of the $jj\cdots j$-plane Π_i that are induced by the homology group H_i, for $i = 1, 2$.

38 Corollary. *With the same hypothesis as in the previous lemma, Ψ can be chosen so that $\Psi(N_{1,v}) = N_{2,v}$ for all $v \in \{1, \alpha, \ldots, \alpha^{n-1}\}$.*

Proof. Since Ψ fixes $x = 0$ and $y = 0$, the matrix for Ψ looks like $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$. However, since Π_2 has a single orbit in $\ell_\infty \setminus \{x = 0, y = 0\}$, we can assume that the line $y = x$ is fixed. Thus, Ψ is given by $\begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}$.

It follows that Ψ sends the line $y = x\Delta_M M$ to the line $y = xA^{-1}\Delta_M MA$.

If the orbits induced by the mapping of H_1 by Ψ are distinct to the orbits of H_2, then as in the proof of lemma 21 we will get that Ψ sends $N_{1,1}$ onto $N_{1,2}$ (and thus Ψ is an isomorphism of fields). Finally, since

$$A^{-1}\Delta_M MA = A^{-1}\Delta_M AA^{-1}MA$$

and $det(M) = det(A^{-1}MA)$, then we conclude that $\Psi(N_{1,v}) = N_{2,v}$ for all v. $\boxed{QED}$

Now, in order to obtain more isomorphisms between $jj\cdots j$-planes, we consider a particular family of fields of matrices constructed as follows.

Let $K = \{\alpha Id \ / \ \alpha \in GF(q)\} \subset M_n(q)$ be a field of matrices representing $GF(q)$, and let $p(x) = x^n - a_{n-1}x^{n-1} - \cdots - a_1 x - a_0$ be a monic polynomial that is irreducible over $GF(q)[x]$. We consider the matrix

$$\theta = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & 1 \\ a_0 & a_1 & \cdots & a_{n-2} & a_{n-1} \end{bmatrix}$$

and note that θ is the companion matrix of $p(x)$, and thus $F = K(\theta) \cong GF(q^n)$.

If K is any field of order q^n containing $GF(q)$ as diagonal matrices, then we can consider K to be a simple extension of $GF(q)$. Let us say $K = GF(q)(\mu)$, where μ satisfies an irreducible polynomial $p(x)$ of degree n over $GF(q)$. Then, if $\theta = C_{p(x)}$, $K(\mu)$ and $K(\theta)$ are conjugate to each other.

With the previous idea in mind we obtain an easy corollary of lemma 15.

39 Corollary. *With the notation used above. Let σ be an automorphism of K and $\theta' = C_{p(x)^\sigma}$, where*

$$p(x)^\sigma = x^n - a_{n-1}^\sigma x^{n-1} - \cdots - a_1^\sigma x - a_0^\sigma$$

If F' is a field of order q^n conjugated to $K(\theta')$ and F is a field conjugated to $K(\theta)$ then, for every tuple $(j_2, j_3, \cdots, j_n)$, the collineation Ψ, defined by

$$(x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n) \mapsto (x_1^\sigma, x_2^\sigma, \cdots, x_n^\sigma, y_1^\sigma, y_2^\sigma, \cdots, y_n^\sigma),$$

induces an isomorphism between two $j_2 j_3 \cdots j_n$-planes with fields F and F'.

40 Lemma. *With the same notation as above. Let $A = \mathrm{diag}(\gamma^{1-n}, \gamma^{2-n}, \ldots, \ldots, \gamma^{-1}, 1) \in M_n(q)$, where $\gamma \in GF(q)^*$.*

Let $q(x) = x^n - \gamma a_{n-1} x^{n-1} - \cdots - \gamma^{n-1} a_1 x - \gamma^n a_0$.

Then, whenever $q(x)$ is irreducible and for every $(j_2, j_3, \ldots, j_n)$, the block matrix:

$$\Psi = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}$$

defines an isomorphism between two $j_2 j_3 \cdots j_n$-planes with fields that are conjugated to fields with associated polynomials $p(x)$ and $q(x)$ respectively.

Proof. The proof is similar to the proof of the lemma 15.

Consider $\theta = C_{p(x)}$.

Then, $A^{-1}\theta A = \gamma^{-1} C_{q(x)}$.

Then, if $q(x)$ is irreducible, then Ψ maps the field $F = K(\theta)$ to the field $E = K(\theta'')$ where $\theta'' = C_{q(x)}$.

Since A is a diagonal matrix, it commutes with any other diagonal matrix. In particular, it commutes with Δ. Using this, it is easy to show that Ψ induces the desired isomorphism. $\boxed{QED}$

Finally, a couple of results that use the particular presentation for fields we have used in the previous two results. The first one is a corollary of corollary 12.

41 Corollary. *There are no $j_2 j_3 -$planes of order 5^3 but the Desarguesian.*

Proof. Corollary 12 restricts us to study the cases $(j_2, j_3) = (0, 0), (0, 2), (2, 0)$ and $(2, 2)$.

For the latter three cases, we wrote a program in Maple © to compute all the determinants $\det(\Delta M - Id)$ for every $M \neq Id$ in F and for every field F of the form used in the previous results. These computations demonstrated that there was always a matrix $M \neq Id$ in F so that $\det(\Delta M - Id) = 0$. Hence, the only possible case is $(j_2, j_3) = (0, 0)$, which yields a Desarguesian plane. $\boxed{QED}$

And now the continuation of lemma 32

42 Lemma. *Let Π be a $jj\cdots j$-plane, Then Π is isomorphic to its transposed plane.*

Proof. Let $S=\{y=x\Delta M;\ M\in F^*\}\cup\{x=0\}$ be the spreadset for Π. Then S^t looks like

$$S^t=\{y=xM^t\Delta^t;\ M\in F^*\}\cup\{x=0\}.$$

After the change of basis $(x,y)\mapsto(y,x)$,

$$S^t=\{y=x\Delta^{-t}M^{-t};\ M\in F^*\}\cup\{x=0\}.$$

Since $\Delta^t=\Delta$ and $\det(M^{-1})=\det(M)^{-1}$ for $M\in F^*$, the transposed spread is

$$S^t=\{y=x\Delta M^t;\ M\in F^*\}\cup\{x=0\}.$$

Denote by F^t the set $F^t=\{M^t;\ M\in F\}$.

Let F be obtained by extending $GF(q)$ by using θ, a root of $p(x)$ as described above. Note that by extending K by θ^t we obtain F^t, since θ^t satisfies $p(x)$, then $F^t\cong F$, and thus $S\cong S^t$. $\boxed{QED}$

5 Existence, particular cases and examples

Let $q-1=nk$ and $gcd(n,k)=1$. Consider $c\in GF(q)^*$ having order $q-1$.

Let $p(x)=x^n-c$. If we assume that there is an element in $GF(q)$ such that $p(x_0)=0$, then $x_0^n=c$, which implies that $1=(x_0)^{q-1}=(x_0^n)^k=c^k=1$. That is a contradiction with $q-1$ being the order of c (that or $k=1$).

Since there are exactly n elements in $GF(q)$ so that $x^n=1$, then $K=GF(q)(\theta)$ contains all the roots of $p(x)$, where $\theta^n=c$. Thus, K is a Galois extension of $GF(q)$ that admits n automorphisms that fix $GF(q)$, namely $\sigma_i: F\longrightarrow F$ defined by $\sigma_i(\theta)=\theta\omega^i$, where ω is a primitive n^{th} root of 1.

Since $|Aut(F_{/GF(q)}|\geq n$, then $[F:GF(q)]=n$ implying that $p(x)$ is irreducible. We have shown:

43 Lemma. *Let $q-1=nk$ and $gcd(n,k)=1$. Consider $c\in GF(q)^*$ having order $q-1$. Then, $p(x)=x^n-c$ is irreducible over $GF(q)$.*

44 Remark. Under the hypothesis of the previous lemma. By using the irreducible polynomial $p(x)=x^n-c$ we can construct a field of matrices of order q^n obtained by extending $GF(q)$ using $\theta=C_{p(x)}$.

The set $\mathbb{B}=\{1,\theta,\theta^2,\dots,\theta^{n-1}\}$ is a basis of F over $GF(q)$. It is not hard to see, using $q-1=nk$, that the matrix for the function $x\mapsto x^q$ with respect to $\mathbb{B}$ is $M_q=\operatorname{diag}(1,c^k,c^{2k},\dots,c^{(n-1)k})$.

Consider Π to be a $j_2 j_3 \cdots j_n$-plane, with $j_i = (i-1)k$. We know the lines of Π look like $(y = x\,\mathrm{diag}(1, \partial^k, \partial^{2k}, \ldots, \partial^{(n-1)k})M)$, where $\det(M) = \partial$.

Since the order of c is $q-1$, then there is an i_∂ such that $c^{i_\partial} = \partial$, thus $M_q^{i_\partial} = \mathrm{diag}(1, \partial^k, \partial^{2k}, \ldots, \partial^{(n-1)k})$.

It follows that the André plane with nets $\{(y = xM_q^{i_\partial}M); \det(M) = \partial\}$ and Π are the same plane.

Note that this infinite family of $jj\cdots j$-planes is a subclass of the known family of André planes. For an example of a family of new $jj\cdots j$-planes the reader is referred to the end of this section.

We will now look at a new condition on the j_i's for a $j_2 j_3 \cdots j_n$-planes to exist. We will work with $jj\cdots j$-planes having a very particular associated field of matrices. The field will be an extension of $GF(q)$ by the companion matrix θ of a polynomial $p(x) = x^n - \alpha$, irreducible in $GF(q)[x]$.

Consider $\Delta = \Delta_\theta$. It is easy to check (by induction) that $\partial = \det(\theta) = (-1)^{n+1}\alpha$ and that $\det(\theta - \Delta)$ equals

$$(-1)\det\begin{bmatrix} -\partial^{-j_2} & 1 & \ddots & \vdots \\ & \ddots & \ddots & 0 \\ \cdots & 0 & -\partial^{-j_{n-1}} & 1 \\ 0 & \cdots & 0 & -\partial^{-j_n} \end{bmatrix} + (-1)^{n+1}\alpha \begin{bmatrix} 1 & 0 & \cdots & 0 \\ -\partial^{-j_2} & 1 & \ddots & \vdots \\ & \ddots & \ddots & 0 \\ \cdots & 0 & -\partial^{-j_{n-1}} & 1 \end{bmatrix}$$

which is

$$\begin{aligned} \det(\theta - \Delta) &= (-1)^n \partial^{-(j_2 + j_3 + \cdots + j_{n-1} + j_n)} + (-1)^{n+1}\alpha \\ &= (-1)^n \partial^{-(j_2 + j_3 + \cdots + j_{n-1} + j_n)} + \partial \\ &= (-1)^n \partial \left[\partial^{-(j_2 + j_3 + \cdots + j_{n-1} + j_n + 1)} + (-1)^n\right] \end{aligned}$$

So, in order to have a plane we need $\partial^{-(j_2+j_3+\cdots+j_{n-1}+j_n+1)} \neq (-1)^{n+1}$. It follows that if n is odd we want $j_2 + j_3 + \cdots + j_{n-1} + j_n + 1$ not to be a multiple of $q-1$, and when n is even $j_2 + j_3 + \cdots + j_{n-1} + j_n + 1$ should not be a multiple of $(q-1)/2$.

Now we will look at specific examples of $jj\cdots j$-planes, all these examples will be of small order. In particular, we will find non-André $jj\cdots j$-planes.

In order to obtain these examples, we wrote a small program in Maple © to help us compute the determinants in the condition that is after remark 8. Also, we have used the presentation for fields discussed before corollary 39 for the search of planes to be more expedite. Of course, our program considers the restrictions on the j_i's obtained in corollary 11 and in section 5.

Note that more planes might be obtained by allowing fields associated to $jj\cdots j$-planes to be not of the 'standard' form described above.

5.1 jj-planes of order 4^3

Planes of order 4^3 were studied in detail in a previous article by the author, Johnson and Wilke. Some results are mentioned here, but the reader is referred to [16] for more details.

The computer search yielded 16 non-Desarguesian $j_2 j_3$-planes of order 4^3. However, by studying the isomorphism classes we proved that there are exactly three isomorphism classes of non-Desarguesian $j_2 j_3$-planes of order 4^3. The planes are given by:

(a, b, c)	(j_2, j_3)
$(0, 0, \alpha)$	$(0, 1)$
$(0, 0, \alpha)$	$(1, 2)$
$(0, 0, \alpha)$	$(2, 2)$

where the triple (a, b, c) represents the field of matrices (via the coefficients of $p(x)$) over which the j_2, j_3-plane is constructed.

We proved that the plane $(0, 0, \alpha)-(1, 2)$ is André, and thus a nearfield plane. However, the other two planes are not. Also, we showed that all j, k-planes of order 4^3 are isomorphic to their transposed plane.

5.2 jj-planes of order 7^3

Our little program found all possible $j_2 j_3$-planes of order 7^3, then we used lemma 40 with $(-1)^3 5 = 2$, and $(-1)^3 4 = 3$ in $\mathbb{F}_7$ to restrict all planes of the form $(0, 0, 2) - (j_2, j_3)$ to planes $(0, 0, 5) - (j_2, j_3)$, and all planes of the form $(0, 0, 3) - (j_2, j_3)$ to planes $(0, 0, 4) - (j_2, j_3)$.

We get more isomorphisms by using a generic Ψ defined by $\Psi(x, y) = (xA, yA)$. The following are the distinct matrices A together with the isomorphism they generate.

For $A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 0 & 5 \\ 0 & 1 & 0 \end{bmatrix}$, Ψ is an isomorphism between $(0, 0, 5) - (0, 4)$ and $(0, 0, 5) - (4, 0)$ and from $(0, 0, 5) - (2, 4)$ and $(0, 0, 5) - (4, 2)$.

For $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & -1 & 0 \end{bmatrix}$, Ψ is an isomorphism between $(0, 0, 5) - (0, 4)$ and $(0, 0, 4) - (4, 0)$ and $(0, 0, 5) - (2, 4)$ and $(0, 0, 4) - (4, 2)$.

For $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 5 \\ 0 & 1 & 0 \end{bmatrix}$, Ψ is an isomorphism between $(0, 0, 4) - (0, 4)$ and $(0, 0, 5) - (4, 0)$ and $(0, 0, 4) - (2, 4)$ and $(0, 0, 5) - (4, 2)$.

Thus, so far all j_2j_3-planes of order 7^3 are:

(a,b,c)	(j_2,j_3)
(0,0,5)	(0,4)
(0,0,5)	(2,4)

where (a,b,c) determines the field used to construct the j_2j_3-plane via the coefficients of the irreducible polynomial used.

We use that an isomorphism of these two planes would imply the existence of a matrix A that normalizes the field, this yields a contradiction with the spectrum of the Δ's associated to the matrices of the field. Thus, these planes are not isomorphic to each other. Moreover, following the same techniques used before, we can show that $(0,0,5)-(2,4)$ is André, and therefore nearfield. This plane is an example of the planes discussed in remark 44.

5.3 jjj-planes of order 3^4

The computer search only yields two j_2,j_3,j_4-planes of order 3^4, they are:

(a,b,c,d)	(j_2,j_3,j_4)
(0,2,0,1)	(1,0,1)
(0,1,0,1)	(1,0,1)

These planes are extremely interesting (besides being new), as they admit two different types of derivable nets and also admit a homology group half the maximum possible size for a plane of that order. We intend to study these planes and all their properties in a future article.

5.4 jjj-planes of order 4^4

As we have done before, we find all the j_2,j_3,j_4-planes of order 4^4 and then we use lemma 15 to determine that $(1,1,0,\alpha)-(2,0,1) \cong (1,1,0,\alpha+1)-(2,0,1)$ and $(\alpha,\alpha+1,0,\alpha+1)-(2,0,1) \cong (\alpha+1,\alpha,0,\alpha)-(2,0,1)$. This leaves us with:

(a,b,c,d)	(j_2,j_3,j_4)
$(1,1,0,\alpha)$	(2,0,1)
$(\alpha+1,\alpha,0,\alpha)$	(2,0,1)

It is not hard to see that neither of these planes is André.

5.5 jjj-planes of order 5^4

With the help of a computer we found all $j_2j_3j_4$-planes of order 5^4, then we use lemma 40 and $\gamma = 2$ to get $(0,4,0,3)-(2,0,2) \cong (0,1,0,3)-(2,0,2)$ and $(0,3,0,2)-(2,0,2) \cong (0,1,0,3)-(2,0,2)$. Thus, we have left:

We get more isomorphisms by using a generic Ψ defined by $\Psi(x,y) = (xA, yA)$. The following are the distinct matrices A together with the isomorphism they generate.

For $A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$, Ψ is an isomorphism between $(0,0,0,3) - (2,0,2)$ and $(0,0,0,2) - (2,0,2)$.

For $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 3 & 0 \\ 0 & 4 & 0 & 0 \end{bmatrix}$, Ψ is an isomorphism between $(0,0,0,3) - (1,2,3)$ and $(0,0,0,2) - (3,2,1)$ and from $(0,0,0,3) - (3,2,1)$ and $(0,0,0,2) - (1,2,3)$.

For $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$, Ψ is an isomorphism between $(0,0,0,3) - (1,2,3)$ and $(0,0,0,3) - (3,2,1)$.

Thus, we have left:

(a,b,c,d)	(j_2,j_3,j_4)
(0,0,0,3)	(2,0,2)
(0,0,0,3)	(1,2,3)
(0,2,0,2)	(2,0,2)
(0,2,0,2)	(1,0,1)
(0,2,0,2)	(3,0,3)
(0,1,0,3)	(2,0,2)
(0,1,0,3)	(1,0,1)
(0,1,0,3)	(3,0,3)

As we did in the case of $j \cdots j$-planes of order 4^3, we can see that the plane $(0,0,0,3) - (1,2,3)$ is André, and therefore nearfield. This plane is another example of the planes discussed in remark 44. Also, it is easy to check that the other planes are non-André.

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