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Invariants on primary abelian groups and a problem of Nunke

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Abstract. If G is an arbitrary abelian p-group, an invariant K_G is defined which measures how closely G resembles a direct sum of cyclic groups. This invariant consists of a class of finite sets of regular cardinals, and is inductively constructed using filtrations of various subgroups of G; K_G can also be considered to be a measure of the presence of non-zero elements of infinite height in G. This construction is particularly useful when the group has final rank less than the smallest weakly Mahlo cardinal; and in this case, G is a direct sum of cyclics iff K_G is empty. These deliberations are then used to place several of the most significant results relating to direct sums of cyclics into a significantly broader context. For example, G is shown to be almost a direct sum of cyclics iff every set in K_G has at least two elements. Finally, K_G is used to give a more complete and concrete answer to a classical problem of Nunke, which asks when the torsion product of two abelian p-groups is a direct sum of cyclics.

Keywords: primary abelian groups, direct sums of cyclics, invariants, filtrations, torsion product, almost direct sums of cyclics

MSC 2000 classification: 20K10, 20K25

1 Introduction

One of the most frequently encountered questions in the study of abelian p-groups is to determine when a group G is isomorphic to a direct sum of cyclics; such groups we will refer to as Σ-*cyclic*. This leads to the following interesting question: Is there a way to *describe*, or even *measure*, just how far an *arbitrary* abelian p-group G is from being Σ-cyclic? In this paper, one strikingly compact and concrete answer is given to this question.

To begin, by the term "group" we will mean an abelian p-group, where p is a prime fixed for the duration. Our terminology and notation will generally follow

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[5], and we will on occasion refer the reader to [4] for set-theoretic material. Note that if a group G is Σ-cyclic, then it can have no non-zero elements of infinite height, and in fact, if G is countable, then this condition is not only necessary, but also sufficient. The main purpose of this paper is to inductively generalize this observation to groups of larger cardinalities. We begin with some terminology:

Suppose $\mathcal{Q}$ is a class of ordinals (which may or may not be a set). Usually $\mathcal{Q}$ will be the class of uncountable regular cardinals, which we denote by $\mathcal{R}$, but $\mathcal{Q}$ might also be a particular ordinal α, which we identify with the collection of all smaller ordinals. We now let $\mathcal{Q}_f$ denote the class of finite subsets of $\mathcal{Q}$. By a *$\mathcal{Q}_f$-antichain* we mean a *set* M of finite subsets of $\mathcal{Q}$ (i.e., a subset of $\mathcal{Q}_f$) such that whenever $S, T \in M$ and $S \subseteq T$, then $S = T$. Given a $\mathcal{Q}_f$-antichain M, let K be the class of all $T \in \mathcal{Q}_f$ such that $S \subseteq T$ for some $S \in M$. We call such a class a *$\mathcal{Q}_K$-invariant* and we say M *generates* K. Note that if K is a $\mathcal{Q}_K$-invariant, then the set of minimal subsets of K under the inclusion ordering is precisely M, and if $S \in K$ and $S \subseteq T \in \mathcal{Q}_f$, then $T \in K$. Conversely, if K is a subclass of $\mathcal{Q}_f$ such that $S \in K$ and $S \subseteq T \in \mathcal{Q}_f$ implies $T \in K$, then the collection of minimal sets for K is a set exactly when there is an α such that $T \in K$ iff $T \cap \alpha \in K$; and if this happens, then K is the $\mathcal{Q}_K$-invariant generated by these minimal sets. We will write M_K for the $\mathcal{Q}_f$-antichain corresponding to $K \in \mathcal{Q}_K$ and K_M for the $\mathcal{Q}_K$-invariant generated by M.

We now point out two special, and extreme, cases of the above notions: If $M = \emptyset$, then $K_M = \emptyset$, which we denote by $0_{\mathcal{Q}}$, and if $M = \{\emptyset\}$, then $K_M = \mathcal{Q}_f$, which we denote by $1_{\mathcal{Q}}$. Note that the class of all $\mathcal{Q}_K$-invariants, which we denote by $\mathcal{Q}_K$, is partially ordered by inclusion and $0_{\mathcal{Q}}$ is its least element and $1_{\mathcal{Q}}$ is its greatest element.

Putting this terminology to work, for any group G, we will inductively define an $\mathcal{R}_K$-invariant, which we denote by K_G, whose corresponding $\mathcal{R}_f$-antichain of minimal sets we will denote by M_G. The elements of K_G or M_G can be viewed as “obstructions” to G breaking apart into a direct sum of cyclics.

The first section of the paper is devoted to setting up this definition and exploring its basic properties; e.g., G will be separable iff $K_G \neq 1_{\mathcal{R}}$ (Lemma 1(a)), and these $\mathcal{R}_K$-invariants behave well with respect to subgroups (Lemma 1(c)) and direct sums (Theorem 3(a)). If G is Σ-cyclic, then it easily follows that $K_G = 0_G$ (Theorem 3(b)). On the other hand, if $K_G = 0_G$ and the final rank of G is strictly less than the first *weakly Mahlo cardinal*, then G is Σ-cyclic (Theorem 6 - an uncountable regular cardinal κ is weakly Mahlo if the collection of regular cardinals $\tau < \kappa$ is stationary in κ; in particular, a weakly Mahlo cardinal is *weakly inaccessible*, i.e., a regular limit cardinal). We let δ_m denote the least weakly Mahlo cardinal, if that exists, and otherwise, we let $\delta_m = \infty$. We say

G is a δ_m*-group* if its final rank is strictly less than δ_m. Assuming the axiom of constructibility (V=L), we prove that if M is an $\mathcal{R}_f$-antichain such that no element of $\cup M$ is (weakly) Mahlo, then there is a group G such that $M_G = M$ (Theorem 10).

In the second section we relate these invariants to some other well known aspects of the theory of Σ-cyclic groups. In particular, we present generalizations of the following results:

(Hill - see [8]) *If G is the ascending union of a (countable) sequence of pure subgroups G_n which are all Σ-cyclic, then the same holds for G.* We generalize this by showing that if κ is a regular cardinal (i.e., $\kappa \in \{\aleph_0\} \cup \mathcal{R}$) and G is the smoothly ascending union of pure subgroups G_i for $i < \kappa$, then $\{\, T \in K_G \mid \kappa \notin T\,\}$ is the union of the classes $\{\, T \in K_{G_i} \mid \kappa \notin T\,\}$ (Theorem 14). Note that if $\kappa = \aleph_0$, this just says $K_G = \cup K_{G_i}$; and so if G is a δ_m-group, then Hill's result follows as a special case (where each $K_{G_i} = 0_{\mathcal{R}}$).

(Danchev and Keef - see [1]) *If A and G are separable groups, and $g : A \to G$ is an ω_1-bijection (i.e., the kernel and cokernel of g are countable), then A is Σ-cyclic iff G is Σ-cyclic.* We generalize this to show that if κ is a cardinal and g is a κ*-bijection* (i.e., the kernel and cokernel of g have cardinality less than κ), then K_A and K_G essentially only differ by elements of κ_f (Theorem 16). In particular, if $\kappa = \aleph_1$, then this implies that $K_A = K_G$; and again, if in addition A and G are δ_m-groups, then the above result of [1] follows.

(Dieudonne - see [2]) *If A is a subgroup of G and $C = G/A$ is Σ-cyclic, then G is Σ-cyclic iff A is contained in a pure subgroup B of G which is Σ-cyclic.* We generalize this to show that if $K \in \mathcal{R}_K$, and $K_C \subseteq K$, then $K_G \subseteq K$ iff A is contained in a pure subgroup B of G such that $K_B \subseteq K$ (Theorem 20). Again, if G is a δ_m-group and $K = 0_{\mathcal{R}}$, then the above result of Dieudonne follows.

It should perhaps be emphasized that our approach allows all three results pertaining to Σ-cyclic groups to be generalized in such a way that they apply to the entire class of groups, though admittedly, these work best in the case of δ_m-groups.

We next relate our invariants to an interesting class of groups defined by Hill in [9]: The separable group G is said to be *almost* Σ*-cyclic* if it has a collection of closed (in the p-adic topology) subgroups which is closed under ascending unions and such that for any countable $X \subseteq G$, there is a countable member of the collection C such that $X \subseteq C$ (Hill used the equivalent terminology "almost a coproduct of cyclics"). We characterize this class using $\mathcal{R}_K$-invariants by showing that a group G is almost Σ-cyclic iff every element of K_G has at least two elements (Theorem 26). In particular, this characterization allows us to answer the following question of [9]: Is a summand of an almost Σ-cyclic group also a member of this class? In fact, we show that the class of almost

Σ-cyclic groups is actually closed under *arbitrary* subgroups (Corollary 27). We also use this characterization to verify that the all the above results for Σ-cyclic groups can be generalized to results pertaining to almost Σ-cyclic groups (Corollaries 28, 29 and 30).

Since M_G is an invariant of G, it follows that so is $\cap M_G$, and in Theorem 33 we describe the elements of this set. We use this to obtain a new characterization of Σ-cyclic δ_m-groups using ascending chain of subgroups, at least in the context of Gödel axiom of constructibility (Corollary 34).

This paper grew out of and generalizes [15], where the following fundamental problem of R. Nunke on the torsion product was considered: Under what circumstances is the torsion product of two groups Σ-cyclic (see also [16], [17], [18], [7], [14], [11], [12] and [13])? The approach to this problem given in this paper has at least three important advantages, however. First, it is considerably more concrete, and correspondingly less abstract. In addition, several of the key results in [15] were limited to groups whose final ranks did not exceed the first weakly inaccessible cardinal, whereas in this paper, we are able to handle all groups whose final ranks do not exceed the first weakly Mahlo cardinal, which is considerably larger. And finally, the techniques of this paper are much easier to apply to questions not involving the torsion p! roduct, such as the study of almost Σ-cyclic groups. These two approaches are specifically related in Theorem 12.

In the torsion-free case, Nunke's problem is somewhat reminiscent of the famous *Whitehead problem*, which asks if $\mathrm{Ext}(G,\ \mathbf{Z}) = \{0\}$ implies that G is free; both questions ask when certain homologically defined groups are Σ-cyclic. In his seminal work on the undecidability of the Whitehead problem, Shelah utilized a construction referred to as a λ-*system* (see section VII.3 of [4]). Consequently, it is perhaps unsurprising that there is a more than passing similarity between the techniques used to construct K_G and the construction of a λ-system of a torsion-free abelian group.

We will denote the torsion product of the groups G and H by the (admittedly non-standard) notation $G \bigtriangledown H$. One of the main reasons for this convention (originally suggested by Claudia Metelli) is that it emphasizes the multiplicative nature of the functor. For example, from the very beginning of the study of Tor, the consideration of iterated products, $G_1 \bigtriangledown \cdots \bigtriangledown G_n$, has played an important role (see [16], [18]). In particular, if we define $G^n = G \bigtriangledown \cdots \bigtriangledown G$, it is natural to ask the nilpotent version of Nunke's problem: For what groups G does there exist an integer n such that G^n is Σ-cyclic? Such a group we call *K-nilpotent* and the smallest such n (if it exists) we refer to as the *K-index* of G.

We define a natural product structure on $\mathcal{R}_K$ and show that for all groups G and H, $K_{G \bigtriangledown H} = K_G K_H$ (Theorem 4). It follows that when G and H are

δ_m-groups, that $G \bigtriangledown H$ is Σ-cyclic iff K_G and K_H are never disjoint (i.e., for all $S \in K_G$ and $T \in K_H$, $S \cap T \neq \emptyset$ - Corollary 8).

In the third section, we use $\mathcal{R}_K$-invariants to give a more detailed analysis of Nunke's problem. If $K \in \mathcal{Q}_K$, we are led to define its *K-complement,* denoted $K^\perp \in \mathcal{Q}_K$, as the set of all $S \in \mathcal{Q}_f$ such that $S \cap T \neq \emptyset$ for all $T \in K$. The assignment $K \to K^{\perp\perp}$ determines a closure operation on $\mathcal{Q}_K$ (Lemma 36) and the topological properties of $\mathcal{Q}_K$ are closely related to its multiplicative properties. For example, we show (Theorem 50) that K has *infinite K-index* (i.e., $K^n \neq 0_\mathcal{Q}$ for every $n < \omega$) iff K is *dense* (i.e., $\overline{K} = K^{\perp\perp} = 1_\mathcal{Q}$).

Nunke's problem naturally leads to the consideration of $\mathcal{R}_K$-invariants of the form $K = J^\perp$ for some $J \in \mathcal{R}_K$, which are precisely those that are closed in the above topology. If $K \in \mathcal{Q}_K$, then K is closed whenever M_K is finite (Corollary 43). This is not the case when M_K is infinite but we are able to characterize precisely when a given $\mathcal{Q}_K$-invariant is closed using subsets of M_K which are Δ-*systems* (i.e., families of sets, any distinct pair of which intersect in a fixed set, called the *root* of the system – Theorem 42). Perhaps surprisingly, this characterization implies that the property of being closed only depends upon a countable subset of $\mathcal{Q}$ (Corollary 44). In addition, we are able to prove that closure preserves products (Theorem 47).

We give a particularly satisfying answer to the nilpotent version of Nunke's problem (Theorem 55), again, at least for δ_m-groups. The general and nilpotent versions of Nunke's problem are closely related due to the following fact: A δ_m-group G is a "zero divisor" (i.e., there is a group H which is not Σ-cyclic such that $G \bigtriangledown H$ is Σ-cyclic) iff it is K-nilpotent. In other words, the groups that arise in answering Nunke's problem are precisely the K-nilpotent groups.

2 $\mathcal{R}_f$-antichains and $\mathcal{R}_K$-invariants

Given a group G, we inductively define $K_G \subseteq \mathcal{R}_f$ as follows: If $p^\omega G \neq \{0\}$, then $T \in K_G$ for all $T \in \mathcal{R}_f$; and if $p^\omega G = \{0\}$, then $T \in K_G$ iff

(†) There is a $\kappa \in T$ such that if $T' = \kappa \cap T = \{\, \beta \in T \mid \beta < \kappa \,\}$, then G has a subgroup A of cardinality κ, with a filtration $\mathcal{A} = \{\, A_i \mid 0 < i < \kappa \,\}$ such that

$$\Gamma_{T'}(A) = \{\, 0 < i < \kappa \mid T' \in K_{A/A_i} \,\}$$

is stationary in κ, i.e., for any CUB (closed and unbounded) subset $C \subseteq \kappa$, $C \cap \Gamma_{T'}(A)$ is non-empty. Observe that we do not assume the filtration starts at $A_0 = \{0\}$, or is even defined for $i = 0$, though there will be occasions when we demand these conditions be satisfied. Of course, we let M_G be the minimal elements of K_G under set inclusion.

Note that we will only be concerned with whether $\Gamma_{T'}(A)$ is stationary in κ. If $\mathcal{A}'$ is another filtration of A, it follows that $\mathcal{A}$ and $\mathcal{A}'$ will agree on a CUB subset of κ, so that the property that $\Gamma_{T'}(A)$ is stationary does not depend upon which filtration is chosen. As a result, we will often, without comment, replace one filtration by another, e.g., one composed of pure subgroups.

1 Lemma. *Suppose G is a group. The following hold:*

(a) $K_G = 1_{\mathcal{R}}$ *iff* $p^\omega G \neq \{0\}$ *(i.e., G is not separable);*

(b) $K_G \in \mathcal{R}_f$*;*

(c) If H is a subgroup of G, then $K_H \subseteq K_G$;

(d) If $T \in M_G$ is non-empty and κ is the largest element of T, then there is a subgroup A of G of cardinality κ such that $T \in M_A$;

Proof. (a): If G is not separable, then it immediately follows from the definition that $K_G = 1_{\mathcal{R}}$. On the other and, if G is separable, then (†) implies that every $T \in K_G$ is non-empty, so that $K_G \neq 1_G$.

(b): We use induction, so suppose $T \in K_G$, $T \subseteq S \in \mathcal{R}_f$ with $n = |S|$. If G is not separable, then $K_G = 1_{\mathcal{R}}$, and $S \in K_G$. On the other hand, if G is separable, then select $\kappa \in T$ and $A \subseteq G$ as in (†). By induction on n, for all $i \in \Gamma_{T'}(A)$, $T' \subseteq S' = S \cap \kappa$ implies $S' \in K_{A/A_i}$, so that $S \in K_G$, as required. In addition, if γ is any cardinal greater than $|G|$, then it is easy to check that $T \in K_G$ iff $T \cap \gamma \in K_G$, so that M_G is a set, as required.

(c): If G is not separable, then $K_G = 1_{\mathcal{R}}$, and the result follows. If G is separable, then so is H. In this case, if $T \in K_H$ together with $A \subseteq H$ and $\kappa \in T$ satisfies (†) for H, then it also satisfies (†) for G, so that $T \in K_G$, as required

(d): Note $\emptyset \notin K_G$, so G is separable. Choose $\kappa' \in K_G$ and $A \subseteq G$ satisfying (†). It follows that $T'' = \{\, \tau \in T \mid \tau \leq \kappa' \,\}$ together with A also satisfies (†), so that $T'' \in K_A \subseteq K_G$. The minimality of T, however, implies that $T = T'' \in M_A$, and that $\kappa = \kappa' = |A|$, as required. $\boxed{QED}$

Mimicking (†), if $T \in \mathcal{R}_f$, then a group H of regular cardinality κ will be said to be *T-stationary* if for some filtration $\{H_i\}_{i<\kappa}$ of H,

$$\Gamma_T(H) = \{\, i < \kappa \mid T \in K_{H/H_i} \,\}$$

is stationary in κ.

2 Lemma. *Suppose $T \in \mathcal{R}_f$ and H is a group of cardinality $\kappa \in \mathcal{R}$.*

(a) If H is T-stationary and $\kappa \notin T$, then $T \cup \{\kappa\} \in K_H \in \mathcal{R}_K$;

(b) If $T \in M_H$, $\kappa \in T$ and $T' = T - \{\kappa\}$ then H is T'-stationary.

PROOF. Regarding (a), if H is not separable, then this follows from $K_H = 1_{\mathcal{R}}$. Otherwise, $T \cup \{k\}$ and $A = H$ clearly satisfies (†). As to (b), choose $\kappa_0 \in T$ and A as in (†). Note that if $\kappa_0 < \kappa$, then $T_0 = \{\, t \in T \mid t \leq \kappa_0 \,\}$ will be in K_H contradicting the minimality of T. We may therefore assume that $\kappa_0 = \kappa$. If $\mathcal{H} = \{H_i\}_{i<\kappa}$ is a filtration of H, then $\mathcal{A} = \{H_i \cap A\}_{i<\kappa}$ is a filtration of A. Since $A/(H_i \cap A)$ embeds in H/H_i, it follows that $\Gamma_{T'}(A) \subseteq \Gamma_{T'}(H)$, so that H is T'-stationary. $\boxed{QED}$

3 Theorem. *Suppose G is a group.*

(a) If $G = \oplus_{i\in I} G_i$, then $K_G = \cup_{i\in I} K_{G_i}$;

(b) If G is Σ-cyclic, then $K_G = 0_{\mathcal{R}}$.

PROOF. Regarding (a), the containment $\supseteq$ follows from Lemma 1(c). Once again we prove the converse by inducting on $n = |T|$, so suppose $T \in M_G$; we will be done if we can show that T is in K_{G_i} for some $i \in I$. Note G is not separable iff some G_i is not separable, so the result follows in this case; so we may assume all these groups are separable and $T \neq \emptyset$. If κ is the largest element of T, choose A as in Lemma 1(d). In fact, after possibly expanding A without altering its cardinality, we may assume $A = \oplus_{j\in J} A_j$, where A_j is a subgroup of G_j of cardinality at most κ and $J \subseteq I$ also has cardinality at most κ. In fact, we will assume that $J = \lambda \leq \kappa$. We claim that there is a $j < \lambda$ such that $T \in K_{A_j}$, which will imply that $T \in K_{G_j}$, proving the result. So assume that this does not hold. Note that we certainly cannot have $T' = T - \{\kappa\} \in K_{A_j}$ for any $j < \lambda$, since this would immediately imply that $T \in K_{A_j}$, as well.

Let $\mathcal{A}_j = \{A_{j,\ell}\}_{\ell<\kappa}$ be defined as follows: If $|A_j| < \kappa$, let $A_{j,\ell} = A_j$, and if $|A_j| = \kappa$, let it be a filtration of A_j such that $T' \notin K_{A_j/A_{j,\ell}}$ for all $\ell < \kappa$. If for each $\ell < \kappa$, $B_\ell = \oplus_{j<\ell} A_{j,\ell}$, then $\mathcal{B} = \{B_\ell\}_{\ell<\kappa}$ is a filtration of A. Now, by induction on n, for all $j < \kappa$ we have

$$T' \notin K_{[\oplus_{j<\ell} A_j/A_{j,\ell}]\oplus[\oplus_{\ell\leq j<\kappa} A_j]} = K_{A/B_\ell},$$

however, this implies that $\Gamma_{T'}(A)$ is not stationary in κ, contrary to Lemma 2(b).

Turning to (b), it is easily checked that if G_i is cyclic, then $K_{G_i} = 0_{\mathcal{R}}$, so that the result follows from (a) $\boxed{QED}$

If $\mathcal{Q}$ is a class of ordinals, we now define a product on $\mathcal{Q}_K$: If K and L are $\mathcal{Q}_K$-invariants, let

$$KL = \{\, U \in \mathcal{Q}_f \mid U = S \cup T \ \text{ for some disjoint sets } S \in K, \ \text{ and } T \in L \,\}.$$

It is easy to verify that KL is another $\mathcal{Q}_K$-invariant. Equivalently, in this definition we can require that $S \in M_K$ and $T \in M_L$ are disjoint and $S \cup T \subseteq U$. It can readily be checked that the product is associative and commutative, and that for all $K \in \mathcal{Q}_K$, $0_{\mathcal{Q}}K = 0_{\mathcal{Q}}$ and $1_{\mathcal{Q}}K = K$.

4 Theorem. *If G and H are groups, then their $\mathcal{R}_K$-invariants satisfy*

$$K_{G \bigtriangledown H} = K_G K_H.$$

Proof. Let $T \in \mathcal{R}_f$; we show by induction on $n = |T|$ that $T \in K_{G\bigtriangledown H}$ iff $T \in K_G K_H$. Note first that if $n = 0$, then it follows trivially, since $G \bigtriangledown H$ has a non-zero element of infinite height iff both G and H have such elements (see, for example, 62.4 of [5]). So assume the result holds for all groups G and H and all finite sets of regular cardinals of size less than n. Let κ be the largest element of T and $T' = T - \{\kappa\}$.

Assume that $T \in K_{G\bigtriangledown H}$. Note that if $T'' \in K_{G\bigtriangledown H}$ were a proper subset of T, then by induction on n, $T'' \in K_G K_H$, so that $T \in K_G K_H$ as required; we may therefore assume $T \in M_{G\bigtriangledown H}$. By Lemmas 1(d) and 2(b) there is a subgroup A of G of cardinality κ which is T'-stationary. After possibly expanding A a bit, we may assume $A = B \bigtriangledown C$, where B and C are subgroups of G and H respectively, and $\max\{|B|, |C|\} = \kappa$. We will be done if we can show $T \in K_B K_C \subseteq K_G K_H$; in fact, after possibly replacing these groups by direct sums of copies of themselves (which by Theorem 3(a) does not affect their $\mathcal{R}_K$-invariants), we may assume that B and C have cardinality κ.

Let $\{B_i\}_{i<\kappa}$ and $\{C_i\}_{i<\kappa}$ be pure filtrations of B and C respectively. So for each $i < \kappa$, the kernel of the obvious map

$$B \bigtriangledown C \to [(B/B_i) \bigtriangledown C] \oplus [B \bigtriangledown (C/C_i)]$$

is

$$(B_i \bigtriangledown C) \cap (B \bigtriangledown C_i) = B_i \bigtriangledown C_i,$$

(see, for example, Lemma 7 of [18]) so that there is an embedding

$$(B \bigtriangledown C)/(B_i \bigtriangledown C_i) \to [(B/B_i) \bigtriangledown C] \oplus [B \bigtriangledown (C/C_i)]$$

It follows that either $T' \in K_{(B/B_i)\bigtriangledown C}$ for all i in some stationary set $S_0 \subseteq \Gamma_{T'}(B \bigtriangledown C)$ or $T' \in K_{B\bigtriangledown(C/C_i)}$ for all i in some stationary set $S_1 \subseteq \Gamma_{T'}(B \bigtriangledown C)$. Without loss of generality, assume that the former condition holds. Then by induction on n, for each $i \in S_0$, T' is the disjoint union of some $T_{i,B} \in K_{B/B_i}$ and $T_{i,C} \in K_C$. Since there are only a finite number of ways to so represent T', it follows that there is a stationary subset $S_2 \subseteq S_0$ such that for all $i, j \in S_2$, $T_{i,B} = T_{j,B}$ and $T_{i,C} = T_{j,C}$. Let T'_B be the former set and T_C be the latter. It follows from Lemma 2(a) that if we let $T_B = T'_B \cup \{\kappa\}$, then $T_B \in K_B$ and $T_C \in K_C$ are disjoint and $T = T_B \cup T_C \in K_B K_C \subseteq K_G K_H$, as required.

Conversely, suppose T is the disjoint union of $T_0 \in K_G$ and $T_1 \in K_H$. Without loss of generality, assume $\kappa \in T_0$. Note that induction on n again implies that there is no loss of generality in assuming $T_0 \in M_G$. Let $T'_0 = \{\, t \in$

$T_0 \mid t < \kappa\}$; so again by Lemmas 1(d) and 2(b) there is a subgroup B of G with $|B| = \kappa$ which is T_0'-stationary; let $\mathcal{B} = \{B_i\}_{i<\kappa}$ be a pure filtration of B. By (†), there is a subgroup C of H such that $|C| < \kappa$ and $T_1 \in K_C$. Note that $\{B_i \triangledown C\}_{i<\kappa}$ is a filtration of $B \triangledown C$ and for all $i \in \Gamma_{T_0'}(B)$ we have a pure exact sequence:

$$0 \to B_i \triangledown C \to B \triangledown C \to (B/B_i) \triangledown C \to 0$$

(see, for example, 63.2 of [5]). By induction,

$$T' = T_0' \cup T_1 \in K_{[(B/B_i)\triangledown C]} = K_{[(B\triangledown C)/(B_i \triangledown C)]}.$$

However, using Lemma 2(a) again, this means that $T \in K_{B\triangledown C} \subseteq K_{G\triangledown H}$, as required. $\boxed{QED}$

We pause to recall a few more standard definitions: If κ is an uncountable cardinal, then the group G is *κ-Σ-cyclic* if every subgroup A of G with $|A| < \kappa$ is Σ-cyclic. The subgroup A of G is said to be *κ-pure* if it is a summand of every subgroup C of G for which $A \subseteq C$ and $|C/A| < \kappa$. Finally, the κ-Σ-cyclic group G is *strongly κ-Σ-cyclic* if every subgroup $B \subseteq G$ with $|B| < \kappa$ is contained in a κ-pure subgroup A with $|A| < \kappa$. We will also adopt the convention that any group is strongly $\aleph_0$-Σ-cyclic.

5 Lemma. *Suppose G is a group and κ is an uncountable cardinal.*

(a) If κ is singular, then G is κ-Σ-cyclic iff it is κ^+-Σ-cyclic (where κ^+ is the next largest cardinal);

(b) If κ is regular and G is κ-Σ-cyclic, then $K_G \cap \kappa_f = \emptyset$ (i.e., no element of K_G consists entirely of regular cardinals smaller than κ).

Proof. Clearly (a) is a consequence of Shelah's Singular Compactness Theorem (see, for example, [3]). Regarding (b), if $T \in K_G$ and $T \subseteq \kappa$, then there is a subgroup A of G such that $|A| < \kappa$ and $T \in K_A$. It follows that A is not Σ-cyclic and hence G is not κ-Σ-cyclic. $\boxed{QED}$

A regular cardinal κ is *weakly Mahlo* if $\{\tau < \kappa \mid \tau \in \mathcal{R}\}$ is stationary in κ. Let $\mathcal{M}$ denote the class of all weakly Mahlo cardinals; if $\mathcal{M}$ is non-empty, let δ_m be its smallest element, and otherwise, let $\delta_m = \infty$. We say G is a *δ_m-group* if its final rank is less than δ_m. Note that since any group G is isomorphic to a direct sum $B \oplus G'$, where B is bounded and the rank and final rank of G' agree, the terms "cardinality" and "final rank" are often interchangeable.

6 Theorem. *A δ_m-group G is Σ-cyclic iff $K_G = 0_{\mathcal{R}}$.*

Proof. By Theorem 3(b), if G is Σ-cyclic, then $K_G = 0_{\mathcal{R}}$, so we concentrate on the converse, inducting on $\kappa = |G|$. Note that if $\kappa = \aleph_0$, the result is well known; so assume the result is valid for all groups of smaller cardinality than

κ. Next, if κ is singular, then for all subgroups A of G with $|A| < \kappa$, we have $K_A \subseteq K_G = 0_{\mathcal{R}}$, so that A is Σ-cyclic by induction. It follows from the Singular Compactness Theorem that G is Σ-cyclic, as well.

Therefore, we may assume that κ is regular. Consider first the case where κ is isolated, i.e., $\kappa = \aleph_{\beta+1}$ for some ordinal β. Note that by induction, G is κ-Σ-cyclic. In addition, for any finite subset T of $\mathcal{R} \cap \kappa = \{\, \tau \in \mathcal{R} \mid \tau < \kappa \,\}$, since $T \cup \{\kappa\}$ is not in K_G, there is a filtration $\mathcal{G}_T = \{A_{T,i}\}_{i<\kappa}$ of G consisting of pure subgroups such that $T \notin K_{G/A_{T,i}}$ for all $i < \kappa$. Since the number of such finite subsets is at most $\aleph_\beta$, it follows that $\mathcal{G} = \cap_T \mathcal{G}_T$ will also be a filtration of G. If we index $\mathcal{G} = \{A_i\}_{i<\kappa}$, and we let $A_0 = \{0\}$, then for every $i < \kappa$, $T \notin K_{G/A_i}$ for all finite $T \subseteq \mathcal{R} \cap \kappa$. However, this implies that $T \notin K_{A_{i+1}/A_i}$ for all finite $T \subseteq \mathcal{R} \cap \kappa$. Since $|A_{i+1}/A_i| < \kappa$, this means that K_{A_{i+1}/A_i} must be empty, so that A_{i+1}/A_i is Σ-cyclic. This, however, implies that $G \cong \oplus_{i<\kappa}[A_{i+1}/A_i]$ is Σ-cyclic as required.

Suppose next that κ is weakly inaccessible. Let $G = \{g_i\}_{i<\kappa}$ be an enumeration of G [in fact, the reader may wish to simply identify G with κ]. For each cardinal $\gamma < \kappa$, there are at most γ finite subsets of γ. It follows from the argument above that there is a filtration $\mathcal{G}_\gamma$ of G consisting of pure subgroups A such that for each finite set $T \subseteq \mathcal{R} \cap \gamma$, $T \notin K_{G/A}$. It follows by induction, then, that for $A \in \mathcal{G}_\gamma$, that G/A is γ-Σ-cyclic. Now, for $i < \kappa$, let $\mathcal{G}_i = \mathcal{G}_{|i|}$. Consider the diagonal intersection of the $\mathcal{G}_i$

$$\mathcal{D} = \{\, A \mid (\forall g_i \in A) A \in \mathcal{G}_i \,\}$$

Note that $\mathcal{D}$ will also be a filtration of G (see, for example, Proposition II.4.10 of [4]). Since κ is smaller than the first weakly Mahlo cardinal, there is a CUB subset of κ consisting of singular cardinals; after intersecting $\mathcal{D}$ with that CUB, we may assume that for every $A \in \mathcal{D}$, $\tau_A = \{\, i < \kappa \mid g_i \in A \,\}$ is a singular cardinal. We may also assume that $\{0\} \in \mathcal{D}$. If $A \in \mathcal{D}$, then for every cardinal $\gamma < \tau_A$, $A \in \mathcal{G}_\gamma$, so that G/A is γ-Σ-cyclic. However, since τ_A is singular, this implies that G/A is τ_A^+-Σ-cyclic.

Let A' be the next largest element of $\mathcal{D}$. Since A has cardinality τ_A, we can conclude that $A' \cong A_0 \oplus A_1$, where $A \subseteq A_0$ and A_0 and A_1 are Σ-cyclic and A_0 also has cardinality τ_A. Since G/A is τ_A^+-Σ-cyclic, A_0/A is also Σ-cyclic; and since A is pure in A_0, there is a spitting $A_0 = A \oplus A_0'$. It follows that each A is a summand of each A', whose complementary summand $C_A \cong A_0' \oplus A_1$ will be Σ-cyclic. Therefore, $G \cong \oplus_{A\in\mathcal{D}} C_A$ is Σ-cyclic, as required. $\boxed{QED}$

7 Corollary. *Suppose G is a group, $\kappa \in \mathcal{R}$ and $\kappa \le \delta_m$. Then G is κ-Σ-cyclic iff $K_G \cap \kappa_f = \emptyset$.*

Proof. One direction following from Lemma 5(b), assume A is a subgroup of G of cardinality less than κ; then $M_A \subseteq K_G \cap \kappa_f$, so if the latter is empty then M_A must always be empty, so that A must always be Σ-cyclic. $\boxed{QED}$

If $\mathcal{Q}$ is a class of ordinals and $K \in \mathcal{Q}_K$, then let $K^\perp$ denote the collection of all $S \in \mathcal{Q}_f$ such that S is not disjoint from any element of K, or equivalently, S is not disjoint from any element of M_K. We will refer to $K^\perp$ as the K-*complement* of K. Note that for $J, K \in \mathcal{Q}_K$, we have $JK = 0_{\mathcal{Q}}$ iff $J \subseteq K^\perp$ iff $K \subseteq J^\perp$. The following expresses the above results in this new notation:

8 Corollary. *Suppose G and H are groups.*

(a) If $G \bigtriangledown H$ is Σ-cyclic, then $K_G K_H = 0_{\mathcal{R}}$ (or equivalently, $K_G \subseteq {K_H}^\perp$, or $K_H \subseteq {K_G}^\perp$).

(b) Conversely, if G and H are δ_m-groups and $K_G K_H = 0_{\mathcal{R}}$ (or equivalently, $K_G \subseteq {K_H}^\perp$, or $K_H \subseteq {K_G}^\perp$), then $G \bigtriangledown H$ is Σ-cyclic.

We now prove a significant realization theorem. Before doing so, recall that a stationary subset $E \subseteq \kappa$ is *non-reflecting* if

$$\{\, \gamma \in E \mid \ \mathrm{cf}(\gamma) > \aleph_0 \text{ and } E \cap \gamma \text{ is stationary in } \gamma \,\} = \emptyset.$$

If $T \in \mathcal{R}_f$, we say a group G is T-*principal* if $M_G = \{T\}$.

9 Lemma. *(V=L) Assuming the axiom of constructibility, suppose T is a finite subset of $\mathcal{R} - \mathcal{M}$. If $T = \emptyset$, let $\kappa = \aleph_0$, and otherwise, let κ be the greatest element of T; then there is a strongly κ-Σ-cyclic, T-principal group G of cardinality κ.*

Proof. As usual, we induct on $n = |T|$. If $n = 0$, then we can simply let G be any countable, non-separable group. Next, suppose the result is valid for finite subsets of $\mathcal{R} - \mathcal{M}$ of size less than $n = |T| > 0$.

Let $T' = T - \{\kappa\}$ and κ' be defined from T' as was κ from T. By our induction hypothesis, there is a group G' which is a strongly κ'-Σ-cyclic T'-principal group of cardinality κ'. Let $D = \{\, i < \kappa \mid i > \kappa' \text{ is a limit ordinal of cofinality } \kappa' \,\}$. Clearly, D is a stationary subset of κ (since the κ'-th element of any CUB is a member of D). It follows from Theorem VI.3.13 of [4] that D has a non-reflecting stationary subset $E \subseteq D$. There is a strongly κ-Σ-cyclic group G, with a filtration $\mathcal{A} = \{A_i\}_{i<\kappa}$ such that:

(a) If $i \notin E$, then A_i is κ-pure in G;

(b) If $i \in E$, then $A_{i+1}/A_i \cong G'$.

(The verification of this claim closely mimics the, by now, standard construction of strongly κ-free groups contained in Theorem VII.2.3 of [4], and will be omitted.)

We need to show that this G is T-principal; since G and T clearly satisfy (†), we can conclude that $T \in K_G$. Consequently, we need to verify that if T_0 is a finite subset of $\mathcal{R}$ which does not contain T, then $T_0 \notin K_G$. Suppose, therefore, that $T \not\subseteq T_0$ and $T_0 \in K_G$. Choose $\kappa_0 \in K_0$ as in (†); note that if $\kappa_0 < \kappa$, then it would follow that $T_0 \in K_{G_0}$ for some subgroup G_0 of G of smaller cardinality. However, G is κ-Σ-cyclic, so this is impossible. Therefore, we must have $\kappa_0 = \kappa$.

If $T_0' = T_0 - \{\kappa\}$, then $T' \not\subseteq T_0'$, and so $T_0' \notin K_{G'}$. If for some $i \in E$ we had $T_0' \in K_{G/A_i}$, then there would be a $j > i$ such that $T_0' \in K_{A_j/A_i}$. But A_{i+1} is κ-pure in G, so that $A_j/A_i \cong (A_j/A_{i+1}) \oplus G'$, and in this sum, the first term is Σ-cyclic. This implies that T_0' cannot be an element of K_{G/A_i}.

On the other hand, if $i \notin E$, then G/A_i is κ-Σ-cyclic, so that it follows that $T_0' \notin K_{G/A_i}$. We can conclude that $T_0 \notin K_G$, and this contradiction proves the result. $\boxed{QED}$

10 Theorem. *(V=L) Assuming the axiom of constructibility, if M is an $\mathcal{R}_f$-antichain such that $(\cup M) \cap \mathcal{M} = \emptyset$, then there is a group G such that $M_G = M$.*

Proof. For each $T \in M$, let G_T be T-principal group, and let $G = \oplus_{T\in M} G_T$. Using Theorem 3(a), it is easily checked that $M_G = M$. $\boxed{QED}$

We now present a natural way to "measure" how close a group G is to being Σ-cyclic. If A is a non-empty collection of finite sets, let

$$\|A\| = \min\{\, |S| \mid S \in A\,\}.$$

If $K \in \mathcal{Q}_K$, we define

$$c(K) = \begin{cases} 2^{-\|K\|}, & \text{if } K \neq 0_{\mathcal{Q}}; \\ 0, & \text{if } K = 0_{\mathcal{Q}}. \end{cases} \tag{1.1}$$

(we could clearly have used M_K in this definition instead of K); and if G is a group, we let $c(G) = c(K_G)$. We summarize a few elementary properties of these definitions in the following:

11 Theorem. *The following hold:*

(a) If G is a group, then $c(G) = 1$ iff G has elements of infinite height;

(b) If G is Σ-cyclic, then $c(G) = 0$;

(c) If G is a δ_m-group and $c(G) = 0$, then G is Σ-cyclic;

(d) If H is a subgroup of G, then $c(H) \leq c(G)$;

(e) If $\{G_i\}_{i\in I}$ is a collection of groups, then $c(\oplus_{i\in I} G_i) = \sup\{\, c(G_i) \mid i \in I\,\} = c(G_i)$ for some $i \in I$;

(f) If G and H are groups, then $c(G \bigtriangledown H) \leq c(G)c(H)$.

Proof. These follow either directly from the definitions, or from Lemma 1, and Theorems 3, 4 and 6. $\boxed{QED}$

The last result of this section describes how the results of [15] can be reduced to the techniques presented above. Since our approach in this paper is not only more general than that of [15], but is also self-contained, we will omit its proof. In that work, for every ordinal α, a partially ordered set P_α was defined, and for every group G, an invariant $\mu_\alpha(G) \in P_\alpha$ was constructed. Since an ordinal α can be identified with the set of all smaller ordinals, we can speak of α_f, α_f-antichains, α_K-invariants and α_K. Let $\theta_\alpha : \alpha \to \mathcal{R}$ be defined by $\theta(\beta) = \aleph_{\beta+1}$, so that θ_α enumerates all the isolated (and hence regular) cardinals less than $\aleph_{\alpha+1}$. A natural induction can be used to prove the following:

12 Theorem. *If α is an ordinal, then there is a natural order and product preserving bijection $\phi_\alpha : P_\alpha \to \alpha_K$ such that if G is a group, then*

$$\phi_\alpha(\mu_\alpha(G)) = \{\, S \in \alpha_f \mid \theta_\alpha(S) \in K_G \,\}.$$

Theorem 12 implies that the results of [15] can be obtained simply by restricting our attention in this paper to the class of isolated (and hence regular) cardinals. The current approach, therefore, allows us to extend these notions to regular limit cardinals, though some of the same kind of limitations that occurred in [15] at weakly inaccessible cardinals reoccur in the present context at those cardinals that are weakly Mahlo.

3 Applications to groups

We begin this section by observing that our results really only depend upon the behavior of the socles of groups (i.e., $G[p] = \{\, x \in G \mid px = 0 \,\}$). To that end, suppose $H_{\omega+1}$ is the "generalized Prüfer group" of length $\omega + 1$. For a group G, let $G' = G \bigtriangledown H_{\omega+1}$. Note that G' will be $p^{\omega+1}$*-projective* (i.e., $p^{\omega+1}\mathrm{Ext}(G, X) = \{0\}$ for all X) and two such groups are isomorphic iff there is an *isometry* (i.e., an isomorphism that preserves heights) between their respective socles (see, for example, [6]).

The following is a generalization of the classical result that when G and H are both p^ω-high subgroups of A (i.e., maximal with respect to the property $p^\omega A \cap G = \{0\}$), and G is Σ-cyclic, then so is H (see, for example, [10]); and if A is a δ_m-group, it actually implies that result:

13 Theorem. *Suppose G and H are groups. The following hold:*

(a) $K_G = K_{G'}$;

(b) *If there is an isometry $f : G[p] \to H[p]$, then $K_G = K_H$;*

(c) *If G and H are both p^ω-high subgroups of A (i.e., maximal with respect to the property $p^\omega A \cap G = \{0\}$), then $K_G = K_H$.*

Proof. Regarding (a), by Theorem 4, $K_{G'} = K_G K_{H_{\omega+1}} = K_G 1_{\mathcal{R}} = K_G$. Now, (b) then follows, since the isometry f induces an isometry $f' : G'[p] \to H'[p]$; but since G' and H' are $p^{\omega+1}$-projective, it follows that $G' \cong H'$, so that $K_G = K_{G'} = K_{H'} = K_H$. Finally, (c) follows from (b) and the fact that when G and H are p^ω-high subgroups of A, then there is an isometry $f : G[p] \to H[p]$. [This last fact follows from the observation that $G[p]$ and $H[p]$ map to the same subgroup under the homomorphism $A \to A/p^\omega A$.] $\boxed{QED}$

It is a classical result, due to Hill (see [8]), that if G is the ascending union of a sequence of pure subgroups, $\{G_i\}_{i<\omega}$, such that each G_i is Σ-cyclic, then G itself is Σ-cyclic. The following, then, can be viewed as a generalization of that result (especially for δ_m-groups).

14 Theorem. *Suppose κ is a regular cardinal and G is a group which is the smoothly ascending union of pure subgroups $\{G_i\}_{i<\kappa}$. Then*

$$\{\, T \in K_G \mid \kappa \notin T \,\} = \cup_{i<\kappa} \{\, T \in K_{G_i} \mid \kappa \notin T \,\}.$$

Before we begin, note that if $\kappa = \aleph_0$, what we are asserting is that $K_G = \cup_{i<\omega} K_{G_i}$. In particular, if each $K_{G_i} = 0_{\mathcal{R}}$, then $K_G = 0_{\mathcal{R}}$. Therefore, if each G_i is Σ-cyclic and G is a δ_m-group, then it follows that G is also Σ-cyclic.

Proof. Since the containment $\supseteq$ is routine, we consider the inclusion $\subseteq$. We prove by induction on $n = |T|$ that if $T \in K_G$ and $\kappa \notin T$ then there is a $j < \kappa$ such that $T \in K_{G_j}$.

Note that if $n = 0$, then $T = \emptyset$. So if $T \in K_G$, then G has a non-zero element of infinite height. Since all the G_j are pure, it would follow that for some $j < \kappa$, G_j would have a non-zero element of infinite height, which is just what is being asserted.

Suppose, therefore, that we have verified the result for all $T_0 \in K_G$ with $\kappa \notin T_0$ and $0 < |T_0| < n$, and $T \in K_G$, with $\kappa \notin T$ and $|T| = n$. Let γ be the largest element of T, so that $\gamma \neq \kappa$. Note first that if T is not in M_G, then it has a proper subset $T_1 \in M_G$. By induction, then, there is a $j < \kappa$ such that $T_1 \in K_{G_j}$, so that $T \in K_{G_j}$, as required. These remarks, therefore, justify the assumption that $T \in M_G$.

Find a subgroup H of G such that $\gamma = |H|$ and $T \in K_H$. Since T is minimal in K_G, it follows that $T \in M_H$. Consider first the case where $\gamma < \kappa$: it follows that there is a $j < \kappa$ such that $H \subseteq G_j$, and it immediately follows that $T \in K_H \subseteq K_{G_i}$, as required.

Assume next that $\gamma > \kappa$. Since γ is regular, there is a $\lambda < \kappa$ such that $|H \cup G_j| = \gamma$ for all $\gamma \geq \lambda$. Without loss of generality, assume this holds for all $j < \kappa$. After possibly expanding H a bit (without changing its cardinality), we can also assume that

(a) For all $j < \kappa$, $H \cap G_j$ is pure in G_j, and hence in G, and hence in H, and $|H \cap G_j| = \gamma$;

If $T' = T - \{\gamma\}$, then H is T'-stationary, so let $\mathcal{H} = \{H_i\}_{i<\gamma}$ be a filtration of H. Note that $[H_i + (H \cap G_j)]/(H \cap G_j)$ will be a smoothly ascending chain with union $H/(H \cap G_j)$, so by restricting to a CUB subset, we may assume that for all $i < \gamma$ and $j < \kappa$,

(b) $[H_i + (H \cap G_j)]/(H \cap G_j)$ is pure in $H/(H \cap G_j)$, and so $H_i + (H \cap G_j)$ will be pure in H.

Note that this implies that $[H_i + (H \cap G_j)]/H_i$ is a pure subgroup of H/H_i, and their union over $j < \kappa$ will be H/H_i.

Let $S = \Gamma_{T'}(H) \subseteq \gamma$. Then for all $i \in S$, by induction on n, we can conclude that there is a $j_i < \kappa$ such that T' is in the $\mathcal{R}_K$-invariant corresponding to

$$[H_i + (H \cap G_{j_i})]/H_i \cong (H \cap G_{j_i})/(H_i \cap G_{j_i})$$

It follows from Fodor's Lemma (see, for example, Corollary II.4.11 of [4]) that there is a fixed $j_0 < \kappa$ such that $S_0 = \{\, i \in J \mid j_i = j_0 \,\}$ is stationary in γ.

Since $\{H_i \cap G_{j_0}\}_{i<\gamma}$ is a filtration of $H \cap G_{j_0}$, it follows that $H \cap G_{j_0}$ is T'-stationary, so by Lemma 2(a), $T \in K_{H \cap G_{j_0}} \subseteq K_{G_{j_0}}$, as required. $\boxed{QED}$

15 Corollary. *Suppose G is a group and $\kappa \in \mathcal{R}$.*

(a) If G can be expressed as the smoothly ascending union of the pure subgroups $\{G_i\}_{i<\kappa}$ where each G_i is Σ-cyclic, then $\kappa \in \cap M_G$;

(b) Conversely, in the constructible universe (V=L), if M is an $\mathcal{R}_f$-antichain, $(\cup M) \cap \mathcal{M} = \emptyset$ and $\kappa \in \cap M$, then there is a group G, which can be expressed as a smoothly ascending chain indexed by κ consisting of pure Σ-cyclic subgroups, such that $M_G = M$.

Proof. Regarding (a), since $K_{G_i} = 0_{\mathcal{R}}$, it follows that $\cup_{i<\kappa} K_{G_i} = 0_{\mathcal{R}}$, so that $\kappa \in T$ for all $T \in K_G$, and hence $\kappa \in T$ for all $T \in M_G$.

Turning to (b), if $T \in M$, let B_T be a group that is $T - \{\kappa\}$-principal. Suppose A is a $\{\kappa\}$-principal group of cardinality κ that is κ-Σ-cyclic. Note that A can clearly be expressed as the smoothly ascending union of pure Σ-cyclic subgroups $\{X_i\}_{i<\kappa}$. It follows that $G_T = A \bigtriangledown B_T$ is the smoothly ascending union of the pure subgroups $\{X_i \bigtriangledown B_T\}_{i<\kappa}$, each of which is again Σ-cyclic. Note that G_T is T-principal, and it follows that

$$G = \oplus_{T \in M} G_T$$

has all the required properties. QED

Recall that if κ is an infinite cardinal, then a homomorphism $g : A \to G$ is called κ-bijective if its kernel and cokernel have cardinality less than κ. It is a routine exercise that if A and G are separable groups and $g : A \to G$ is an ω_1-bijective homomorphism, then A is Σ-cyclic iff G is Σ-cyclic (see, for example, [1]). We generalize this observation in the following:

16 Theorem. *Suppose A and G are separable groups, $\kappa \in \mathcal{R}$ and $g : A \to G$ is a κ-bijective homomorphism. Then*

$$\{\, T \in M_G \mid T \not\subseteq \kappa \,\} \subseteq K_A \text{ and } \{\, T \in M_A \mid T \not\subseteq \kappa \,\} \subseteq K_G.$$

Proof. Suppose first that $T \in M_A$ and $T \not\subseteq \kappa$. Since A is separable, T is non-empty; let τ be the largest element of T, so that $\tau \geq \kappa$. If X be the kernel of g, let $\gamma = \max\{|X|, |G/g(A)|\} < \kappa$. Let H be a subgroup of A of cardinality τ such that $T \in K_H$. Note first that by possibly expanding H a bit, we may assume $X \subseteq H$; observe further that $T \in M_A$ implies that $T \in M_H$, so by Lemma 2(b), H is $T' = T - \{\tau\}$-stationary, and we let $\{H_i\}_{i<\tau}$ be a filtration of H. Note that for some $i_0 < \tau$, $X \subseteq H_{i_0}$. It follows that for $i \geq i_0$, we have

$$H/H_i \cong g(H)/g(H_i),$$

and since $T' \in K_{H/H_i}$ for i ranging over a stationary set, it follows from Lemma 2(a) that $T \in K_{g(H)}$, as well, so that it is also in K_G, as desired.

Similarly, suppose $T \in M_G$, $T \not\subseteq \kappa$ and τ is the greatest element of T. Again, let H be a subgroup of G of cardinality τ such that $T \in M_H$. Let Y be a subgroup of G of cardinality less than κ such that $g(A) + Y = G$. Note that replacing H by $H + Y$ does not alter its cardinality or the fact that $T \in M_H$. Again, if $T' = T - \{\tau\}$, then H is T'-stationary; let $\{H_i\}_{i<\tau}$ be a filtration of H. It follows that there is an i_0 such that $Y \subseteq H_{i_0}$. If we let $H' = g^{-1}(H)$ and $H'_i = g^{-1}(H_i)$, then $\{H'_i\}_{i<\tau}$ is a filtration of H'. Now, for every $i \geq i_0$ we have $(H \cap g(A)) + H_i = H$ [since if $h \in H$, then $h = x + y$, where $x \in g(A)$ and $y \in Y \subseteq H_i$, and it follows that $x = h - y \in H \cap g(A)$, as required]. This implies that there are isomorphisms

$$H'/H'_i \cong (H \cap g(A))/(H_i \cap g(A)) \cong [(H \cap g(A)) + H_i]/H_i = H/H_i.$$

Therefore, H' is T'-stationary, so that $T \in K_{H'} \subseteq K_A$, as required. QED

17 Corollary. *Suppose A and G are separable groups and $g : A \to G$ is a κ-bijective homomorphism. Then*

$$\{\, S \in M_A \mid S \cap \kappa = \emptyset \,\} = \{\, S \in M_G \mid S \cap \kappa = \emptyset \,\}.$$

The following observation clearly implies the aforementioned result of [1], at least for δ_m-groups, by considering when these $\mathcal{R}_K$-invariants equal $0_{\mathcal{R}}$:

18 Corollary. *If A and G are separable groups and $g : A \to G$ is an ω_1-bijection, then $K_A = K_G$.*

19 Corollary. *Suppose A, B, G and H are separable δ_m-groups and $g : A \to G$ and $h : B \to H$ are ω_1-bijections. Then $A \bigtriangledown B$ is Σ-cyclic iff $G \bigtriangledown H$ is Σ-cyclic.*

Proof. Since $K_{A\bigtriangledown B} = K_A K_B = K_G K_H = K_{G\bigtriangledown H}$, we have $K_{A\bigtriangledown B} = 0_{\mathcal{R}}$ iff $K_{G\bigtriangledown H} = 0_{\mathcal{R}}$. $\boxed{QED}$

We now introduce some notation, originally due to Hill (see, e.g., [9]). If A and B are subgroups of G, we will write $A\|B$ if for all $a \in A$ and $b \in B$, if $n < \omega$ and $n \leq ht_G(a+b)$, then there is an $x \in A \cap B$ such that $n \leq ht_G(a+x)$. Note that this will imply that $n \leq ht_G(b - x)$, as well, so that the relation is symmetric. In Hill's original definition, $n < \omega$ was allowed to be any (possibly infinite) ordinal, but since we are primarily concerned with separable groups, this restriction will be appropriate for our uses. We note two easily verified properties of this definition:

(A) If A and B are infinite subgroups of G, then there is a subgroup A' containing A such that $|A| = |A'|$ and $A'\|B$.

(B) If B is a subgroup of G and $\{A_i\}_{i<\lambda}$ is an ascending chain of subgroups of G with union A, then if $A_i\|B$ for all $i < \lambda$, then $A\|B$.

A classical result, due to Dieudonne ([2]), can be slightly, but equivalently, reformulated thus: When A is a subgroup of G where $C = G/A$ is Σ-cyclic, then G is Σ-cyclic iff A is contained in a pure subgroup B of G which is also Σ-cyclic. The following can be viewed as a generalization of this result, at least for δ_m-groups, by letting $K = 0_{\mathcal{R}}$:

20 Theorem. *Suppose $K \in \mathcal{R}_K$, A is a subgroup of G and $C = G/A$. If $K_C \subseteq K$, then $K_G \subseteq K$ iff A is contained in a pure subgroup B of G such that $K_B \subseteq K$.*

Before beginning, note that we can restate the theorem in the following way: If A is a subgroup of G and $C = G/A$, then whenever B is a pure subgroup of G containing A, we have

$$K_G \cup K_C = K_B \cup K_C.$$

Again, another way to state this is that with the above notation, if $T \in K_G - K_C$ and B is a pure subgroup of G containing A, then $T \in K_B$.

Proof. We verify this last way of expressing the result by inducting on $n = |T|$. If $n = 0$, then if $T \in K_G - K_C$ and B is a pure subgroup of G containing A, then there is a non-zero $x \in G$ of infinite height. Since C is separable, it follows that $x + A = 0 + A \in C$, so that $x \in A \subseteq B$. It therefore follows that x has infinite height in B, so that $T \in K_B$, as required.

We may therefore assume that $n > 0$, so that $T \in K_G - K_C$ is non-empty, and let B be some pure subgroup of G containing A. In fact, the induction hypothesis clearly implies that we may assume that $T \in M_G$ (since if $T_0 \in K_G$ is a proper subset of T, then $T_0 \notin K_C$, so that by induction $T_0 \in K_B$, which gives $T \in K_B$). Let κ be the largest element of T, and $T' = T - \{\kappa\}$ and X be a T'-stationary subgroup of G of cardinality κ. Note that after expanding X while not altering its cardinality, we may assume that $B_X = B \cap X$ is pure in X. Let $A_X = A \cap X \subseteq A$ and $C_X = [X + A]/A \subseteq C$. Suppose first that $|A_X| < \kappa$: It follows that $X \to C_X$ is a κ-bijection. Since $T \in M_X$, we could conclude from Theorem 16 that $T \in K_{C_X} \subseteq K_C$, which is not true. We can therefore conclude that $|A_X| = \kappa$. Suppose now that $|C_X| < \kappa$: It follows that $A_X \to X$ is a κ-bijection, and by Theorem 16, we again have that $T \in K_{A_X} \subseteq K_A \subseteq K_B$, as required. We may therefore also assume that $|C_X| = \kappa$.

By a standard argument using (A) and (B) above, we can construct a filtration $\{Y_i\}_{i<\kappa}$ of X such that for all $i < \kappa$, $Y_i \| B_X$.

For $i < \kappa$, let $G_i = X/Y_i$ and

$$
\begin{aligned}
A_i &= [A_X + Y_i]/Y_i \cong A_X/(A \cap Y_i),\\
B_i &= [B_X + Y_i]/Y_i \cong B_X/(B \cap Y_i),\\
C_i &= G_i/A_i = (X/Y_i)/([A_X + Y_i]/Y_i) \cong X/[A_X + Y_i].
\end{aligned}
$$

Since $A \cap ((A \cap X) + Y_i) = (A \cap X) + (A \cap Y_i) = A \cap X$, we have isomorphisms

$$
\begin{aligned}
C_i &\cong [X/A_X]/[(A_X + Y_i)/A_X]\\
&= [X/(A \cap X)]/[((A \cap X) + Y_i)/(A \cap X)]\\
&= [X/(A \cap X)]/[((A \cap X) + Y_i)/(A \cap ((A \cap X) + Y_i))]\\
&\cong [(X + A)/A]/[((A \cap X) + Y_i) + A)/A]\\
&= [(X + A)/A]/[(A + Y_i)/A] = C_X/[(A + Y_i)/A].
\end{aligned}
$$

We now verify that our construction guarantees that B_i is pure in G_i: Let $b + Y_i$ be an element of B_i (where $b \in B_X$) whose height in G_i is at least $n < \omega$. It follows that there is a $y \in Y_i$ such that $b + y$ has height at least n in X. Therefore, since $Y_i \| B_X$, there is a $z \in B_X \cap Y_i$ such that $b + z$ has height at least n. However, since B_X is pure in X, this implies that $b + z = p^n b_0$ for some $b_0 \in B_X$, so that $p^n(b_0 + Y_i) = b + Y_i$, as required.

Now observe that $\{(A+Y_i)/A\}_{i<\kappa}$ is a filtration of C_X. Since $T \notin K_C$, it follows that $T \notin K_{C_X}$, so there is a CUB subset $\mathcal{C} \subseteq \kappa$ such that $T' \notin K_{C_X/[(A+Y_i)/A]} = K_{C_i}$ for all $i \in \mathcal{C}$. Since $\{Y_i\}_{i\in\mathcal{C}}$ is a filtration of X, for all i in some stationary set $S \subseteq \mathcal{C}$, we have $T' \in K_{G_i}$. It follows from induction that for every $i \in S$, $T' \in K_{B_i} = K_{B_X/(B\cap Y_i)}$. However, since $\{B \cap Y_i\}_{i<\kappa}$ is a filtration of B_X, it follows that B_X is T'-stationary, so that $T \in K_{B_X} \subseteq K_B$, completing the proof. $\boxed{QED}$

21 Corollary. *If A is a pure subgroup of G and $C = G/A$, then $K_G \cup K_C = K_A \cup K_C$.*

Proof. This follows from the discussion immediately following Theorem 20 by letting $B = A$. $\boxed{QED}$

We noted early on that if A is a subgroup of G, that $K_A \subseteq K_G$. As to quotients, we have the following interesting special case of this last result:

22 Corollary. *If A is a pure Σ-cyclic subgroup of G and $C = G/A$, then $K_G \subseteq K_C$.*

For example, when A is a basic subgroup of G, then C is divisible and the last result generalizes the observation that $K_G \subseteq 1_{\mathcal{R}} = K_C$.

We now relate $\mathcal{R}_K$-invariants to a class of groups defined by Hill in [9]. The separable group G is said to be *almost Σ-cyclic* if it has a collection of subgroups $\mathcal{C}$ such that:

(1) For all $C \in \mathcal{C}$, C is closed in G (i.e., G/C is separable);

(2) If λ is an ordinal and $\{C_i\}_{i<\lambda} \subseteq \mathcal{C}$ is an ascending chain in $\mathcal{C}$, then $\cup_{i<\lambda} C_i \in \mathcal{C}$;

(3) If $X \subseteq G$ is countable, then there is a countable $C \in \mathcal{C}$ such that $X \subseteq C$.

We begin our discussion of this class with the following elementary observation:

23 Lemma. *Suppose G is almost Σ-cyclic using the family $\mathcal{C}$ and $X \subseteq G$ is infinite. Then there is an $A \in \mathcal{C}$ such that $|X| = |A|$ and $X \subseteq A$.*

Proof. We prove this by induction on $|X|$, it being part of the definition if this is countable. Assume, therefore, that it works for all subsets of smaller cardinality, and let $\lambda = |X|$, $X = \{x_i\}_{i<\lambda}$ and $X_\alpha = \{x_i\}_{i<\alpha}$. We inductively choose $A_\alpha \in \mathcal{C}$ such that

(a) $X_\alpha \subseteq A_\alpha$;

(b) If $\beta < \alpha < \lambda$ then $A_\beta \subseteq A_\alpha$;

(c) $|A_\alpha| = |\alpha| + \aleph_0$.

If α is a limit, we just take unions, and if $\alpha = \beta+1$ is isolated, then by induction there is an $A_{\alpha+1} \in \mathcal{C}$ such that $A_\alpha \cup \{x_\alpha\} \subseteq A_{\alpha+1}$ and $|A_\alpha| = |A_{\alpha+1}|$, completing the proof. $\boxed{QED}$

24 Lemma. *If G is the smoothly ascending union of closed pure subgroups $\{G_i\}_{i<\lambda}$, and for each $i < \lambda$, G_i is almost Σ-cyclic, then G is almost Σ-cyclic.*

Proof. Clearly, we may assume that λ is a limit ordinal. If $x \in G$, let $s(x)$ be the first $i < \lambda$ such that $x \in G_i$, and if $X \subseteq G$, let $s(X) = \{\, s(x) \mid x \in X \,\}$. Suppose for each $i < \lambda$ that $\mathcal{C}_i$ is a collection of subgroups of G_i which show that it is almost Σ-cyclic. Let $\mathcal{C}$ be the collection of subgroups A of G satisfying the following:

(a) For each $i \in s(A)$, $A \cap G_i \in \mathcal{C}_i$;

(b) For each $i \in s(A)$, $A \| G_i$.

It is fairly clear that the ascending union of groups in $\mathcal{C}$ will once again be in $\mathcal{C}$, and that any countable subset of G can be embedded in a countable member of $\mathcal{C}$. The crucial point, however, is to verify that every $A \in \mathcal{C}$ is actually closed. Let x_n be a sequence in A converging (in the p-adic topology) to $y \in G$; we may assume $ht(y - x_n) \geq n$. Note that if there is $i \in s(A)$ such that $s(x_n) \leq i$ for all n, then each $x_n \in A \cap G_i$, and since this group is closed in G_i, which is closed in G, it follows that $y \in A \cap G_i \subseteq A$, as required. It follows, therefore, that we may assume $\gamma = \sup\{\, s(x_n) \mid n < \omega \,\}$ is a limit ordinal. Since G_γ is closed, it follows that $y \in G_\gamma$, so that $s(y) < \gamma$. Choose $\ell < \omega$ such that $s(y) < s(x_\ell) < \gamma$. Now, if $i = s(x_\ell)$, then by condition (b), for each n, since $ht(x_n - y) \geq n$, we can write $x_n - w_n = z_n$, where $w_n \in A \cap G_i$ and $ht(z_n) \geq n$. It follows that $y - w_n = (y - x_n) + (x_n - w_n) = (y - x_n) + z_n$ also has height at least n, so that w_n converges to y in the p-adic topology. It follows that $y \in A \cap G_i \subseteq A$, as required. $\boxed{QED}$

25 Lemma. *Suppose G is a separable group and $\kappa \in \mathcal{R}$. Then $T = \{\kappa\} \in M_G$ iff G has a subgroup H of cardinality κ that is $\emptyset$-stationary, i.e., for some (and hence every) filtration $\{H_i\}_{i<\kappa}$ of H,*

$$\Gamma_\emptyset(H) = \{\, i < \kappa \mid H_i \text{ is not closed in } H \text{ (in the } p\text{-adic topology)} \,\}$$

is stationary in κ.

Proof. Suppose $T \in M_G$. By Lemmas 1(d) and 2(b), G has an $\emptyset$-stationary subgroup of cardinality κ. Conversely, if such an H exists, then by Lemma 2(a), $T \in K_H \subseteq K_G$. Since G is separable, $\emptyset$ is not in K_G, so that we can conclude that T is actually a minimal set in K_G, as required. $\boxed{QED}$

We have now done all the "heavy lifting" for our characterization of almost Σ-cyclic groups using $\mathcal{R}_K$-invariants. We will adopt the convention that a countable group G is $\emptyset$-stationary iff it is not separable.

26 Theorem. *If G is a group, then the following are equivalent:*

(a) G is almost Σ-cyclic;

(b) $c(G) \leq 1/4$;

(c) Every $T \in K_G$ has at least two elements;

(d) G has no subgroups of regular cardinality which are $\emptyset$-stationary.

Proof. Note that (b) and (c) are clearly equivalent and by Lemma 25, these are equivalent to (d). Suppose next that (a) holds, so that G is almost Σ-cyclic using the collection of subgroups $\mathcal{C}$, and H is an arbitrary subgroup of G with $|H| = \kappa \in \mathcal{R}$; in fact, let $H = \{x_i\}_{i<\kappa}$ be an enumeration of H. Using Lemma 23, we inductively construct a smoothly ascending chain of groups $\{A_i\}_{i<\kappa} \subseteq \mathcal{C}$ such that for all $\beta < \kappa$

(a) $\{x_i\}_{i<\beta} \subseteq A_\beta$;

(b) $|A_\beta| = |\beta| + \aleph_0 < \kappa$.

Once again, at limit ordinals we just take unions, and at isolated ordinals $\beta = \gamma + 1$, we employ Lemma 23 to construct A_β containing A_γ and x_γ.

Note that each A_i is closed in G, so that $H_i = A_i \cap H$ is closed in H. It follows that $\Gamma_\emptyset(H)$ is empty, and in particular, non-stationary. Since this applies to all subgroups of G of regular cardinality, we can conclude that (d) holds.

We prove that (b-d) implies (a) by induction on $|G| = \tau$, so assume $|G| = \tau$ and it holds for all groups of strictly smaller cardinality. Let κ be the cofinality of τ. We first verify that G is the union of a smoothly ascending chain $\{G_i\}_{i<\kappa}$ such that each G_i is pure and closed in G, and $|G_i| < \tau$: Note that this is trivial if τ is regular (since otherwise G would be an $\emptyset$-stationary subgroup of itself), so assume $\kappa < \tau$. Suppose G is the smoothly ascending union of the pure subgroups $\{G_i\}_{i<\kappa}$. If $S = \{\, j < \kappa \mid G_j \text{ is not closed in } G\,\}$ is stationary in κ, then, for every $j \in S$, let X_j be a countable subgroup of G such that $[X_j + G_j]/G_j$ has elements of infinite height. Consider $H = \langle X_j : j \in S\rangle$; clearly H has cardinality κ. For each $i < \kappa$, let $H_i = H \cap G_i$, so that $\{H_i\}_{i<\kappa}$ is a filtration of H. Since for every $j \in S$, $[X_j + G_j]/G_j \cong X_j/[X_j \cap G_j]$ embeds in H/H_j, it follows that $S \subseteq \Gamma_\emptyset(H)$; therefore, H is $\emptyset$-stationary, contrary to our assumption on G.

Restricting to a CUB subset, we may assume that each G_i is closed in G. Since G has no subgroup of regular cardinality that is $\emptyset$-stationary, it follows that the same can be said of G_i for all $i < \kappa$. By induction, therefore, each G_i is almost Σ-cyclic, so by Lemma 24, G is almost Σ-cyclic, as required. $\boxed{QED}$

Note that if G is almost Σ-cyclic and $|G| = \aleph_1$, then every $T \in M_G$ must be a subset of $\{\aleph_1\}$, and hence $K_G = 0_{\mathcal{R}}$, and G is Σ-cyclic. This observation is Theorem 2 of [9]. On the other hand, it can easily be verified that the groups G constructed in Theorem 4 of [9] have cardinality $\aleph_2$ and $M_G = \{\{\aleph_1, \aleph_2\}\}$. These G are, therefore, the smallest possible examples of groups that are almost Σ-cyclic which fail to actually be Σ-cyclic.

In [9] it was asked whether a summand of a group that is almost Σ-cyclic is also almost Σ-cyclic. The following observation provides an even more general result:

27 Corollary. *Suppose G is almost Σ-cyclic and A is an arbitrary subgroup of G. Then A is also almost Σ-cyclic.*

Proof. By Theorem 11(d) and Theorem 26 we have $c(A) \leq c(G) \leq 1/4$. Applying Theorem 26 again, we have that A is also almost Σ-cyclic. $\boxed{QED}$

We now generalize the three results on Σ-cyclic groups mentioned in the introduction to the class of almost Σ-cyclic groups. The following is a version of the aforementioned theorem of Hill from [8]:

28 Corollary. *Suppose G is a group which is the union of an ascending sequence of pure subgroups $\{G_i\}_{i<\omega}$. If each G_i is almost Σ-cyclic, then so is G.*

Proof. By Theorem 14, $K_G = \cup_{i<\omega} K_{G_i}$, so in particular, since every element of each K_{G_i} has at least two elements, the same can be said of G. $\boxed{QED}$

In [9] it was observed (using slightly different language) that if A and G are separable groups and $f : A \to G$ is an ω_1-bijective homomorphism, then G is almost Σ-cyclic if A is almost Σ-cyclic. We now generalize that result to larger cardinalities. If $\kappa \in \mathcal{R}$, we will say a group G is *κ-almost Σ-cyclic* if every subgroup H of G of cardinality less than κ is almost Σ-cyclic. It easily follows that a group is $\aleph_1$-almost Σ-cyclic iff it is separable and $\aleph_2$-almost Σ-cyclic iff it is $\aleph_2$-Σ-cyclic.

29 Corollary. *Suppose $\kappa \in \mathcal{R}$, A and G are κ-almost Σ-cyclic groups and $g : A \to G$ is a κ-bijective homomorphism. Then A is almost Σ-cyclic iff G is almost Σ-cyclic.*

Proof. Suppose G is almost Σ-cyclic and $T \in M_A$. Since A is separable, we can conclude T is non-empty, so let τ be its largest element. If $\tau < \kappa$, then A has a subgroup H of cardinality at most τ with $T \in K_H$; since A is κ-almost Σ-cyclic and $|H| \leq \tau < \kappa$, H is almost Σ-cyclic, so T has at least two elements by Theorem 26. If $\tau \geq \kappa$, then by Theorem 16, $T \in K_G$, and since G is almost

Σ -cyclic, Theorem 26 implies that T has at least two elements. However, by Theorem 26 once more, this implies that A is almost Σ-cyclic, as required.

Conversely, suppose A is almost Σ-cyclic and $T \in M_G$. Since G is separable, we can conclude T is non-empty, so let τ be its largest element. If $\tau < \kappa$, then G has a subgroup H of cardinality at most τ with $T \in K_H$; since G is κ-almost Σ-cyclic and $|H| \leq \tau < \kappa$, H is almost Σ-cyclic, so T has at least two elements. If $\tau \geq \kappa$, then again by Theorem 16, $T \in K_A$, and since A is almost Σ-cyclic, T has at least two elements. However, by a final application of Theorem 26, this implies that G is almost Σ-cyclic, as required. $\boxed{QED}$

The interested reader can verify that the last result is also true for Σ-cyclic groups (as opposed to almost Σ-cyclics).

Next we show that Dieudonne's result for Σ-cyclic groups generalizes to almost Σ-cyclic groups:

30 Corollary. *Suppose A is a subgroup of G and $C = G/A$ is almost Σ-cyclic. Then G is almost Σ-cyclic iff A is contained in a pure subgroup $B \subseteq G$ which is almost Σ-cyclic.*

Proof. If G is almost Σ-cyclic, the result follows from Corollary 27, so assume we can find the pure subgroup B. Since $K_G \subseteq K_B \cup K_C$, every element of K_G is in either K_B or K_C, and in particular, it must have at least two elements. The result, therefore, follows from Theorem 26. $\boxed{QED}$

Again we can be a bit more specific in the case of pure subgroups:

31 Corollary. *Suppose A is a pure subgroup of G and $C = G/A$ is almost Σ-cyclic. Then G is almost Σ-cyclic iff A is almost Σ-cyclic.*

Our next observation, which follows directly from Theorems 13(c) and 26, continues the parallel between Σ-cyclic and almost Σ-cyclic groups:

32 Corollary. *Suppose G and H are p^ω-high subgroups of a group A. If G is almost Σ-cyclic, then so is H.*

We now turn to characterizing the elements of $\cap M_G$, at least in some set theoretic environments:

33 Theorem. *Suppose G is a δ_m-group, $\kappa \in \mathcal{R}$, $\kappa < \delta_m$ and H is some $\{\kappa\}$-principal δ_m-group of cardinality κ. Then the following are equivalent:*

(a) $\kappa \in \cap M_G$;

(b) $c(G \bigtriangledown H) = 0$;

(c) G is the union of a smoothly ascending chain of pure subgroups $\{G_i\}_{i\leq\lambda}$ starting at $G_0 = \{0\}$ such that for all $i < \lambda$, G_{i+1}/G_i is a κ-Σ-cyclic group of cardinality at most κ.

Proof. We first observe that if A is a group with $|A| \leq \kappa$, then by Corollary 7, $A \bigtriangledown H$ is Σ-cyclic iff A is a κ-Σ-cyclic group.

Note that (a) is equivalent to (b), since $\kappa \in \cap M_G$ iff $K_G K_H = 0_{\mathcal{R}}$ iff $c(G \bigtriangledown H) = 0$.

Assume next that (c) is valid and we are given $\{G_i\}_{i \leq \lambda}$ with the described properties; note that for each i, the pure exact sequence

$$0 \to G_i \to G_{i+1} \to (G_{i+1}/G_i) \to 0,$$

determines another pure-exact sequence,

$$0 \to G_i \bigtriangledown H \to G_{i+1} \bigtriangledown H \to (G_{i+1}/G_i) \bigtriangledown H \to 0.$$

Now, our hypotheses imply each $(G_{i+1}/G_i) \bigtriangledown H$ is Σ-cyclic, so that the latter exact sequences all must necessarily split. It follows that

$$G \bigtriangledown H \cong \oplus_{i<\lambda}[(G_{i+1}/G_i) \bigtriangledown H],$$

so that $c(G \bigtriangledown H) = 0$.

Conversely, suppose $c(G \bigtriangledown H) = 0$ for some κ-principal H. Fix a decomposition $G \bigtriangledown H \cong \oplus_{j \in J} C_j$, where each C_j is cyclic. If $\lambda = |G|$, we can inductively construct a smoothly ascending chain of pure subgroups $\{G_i\}_{i<\lambda}$ of G such that

(0) $G_0 = \{0\}$;

(1) $G_i \bigtriangledown H = \oplus_{j \in J_i} C_j$ for some subset $J_i \subseteq J$;

(2) $|G_{i+1}/G_i| \leq \kappa$.

These conditions imply that $c((G_{i+1}/G_i) \bigtriangledown H) = 0$, so that $K_{G_{i+1}/G_i} K_H = 0_{\mathcal{R}}$, but this is equivalent to the statement that G_{i+1}/G_i is a κ-Σ-cyclic group, as required. $\boxed{QED}$

We conclude this section with a new characterization of the class of Σ-cyclic δ_m-groups, at least in V=L:

34 Corollary. *(V=L) Assuming the axiom of constructibility, a δ_m-group G is Σ-cyclic iff for an infinite collection of cardinals $\kappa_n \in \mathcal{R}$ with $\kappa_n < \delta_m$, we can write G as the smoothly ascending union of pure subgroups $\{G_{n,i}\}_{i<\lambda_n}$ starting at $G_{n,0} = \{0\}$ such that for all $i < \lambda_n$, $G_{n,i+1}/G_{n,i}$ is κ_n-Σ-cyclic of cardinality at most κ_n.*

Proof. If $c(G) = 0$, then it can be written as a smoothly ascending union of pure subgroups whose corresponding factors are, in fact, cyclic, and such an expression satisfies the hypotheses for any infinite cardinal κ.

Conversely, if G satisfies this property, then for all $n < \omega$, $\kappa_n \in \cap M_G$. The only way this is possible is for $M_G = \emptyset$, but this implies that G is Σ-cyclic, as required. $\boxed{QED}$

4 Applications to Nunke's Problem

We begin by observing that the obvious translation of Nunke's problem to the class of almost Σ-cyclic groups has an essentially trivial solution:

35 Corollary. *If G and H are separable groups, then $G \bigtriangledown H$ is almost Σ-cyclic.*

Proof. Note $c(G) = 2^{-j}$ and $c(H) = 2^{-k}$, where k, j are non-negative integers, and by Theorem 11(a), since G and H are separable, neither j and k can equal 0. Using Theorem 11(f), since $c(G \bigtriangledown H) \leq c(G)c(H) \leq (1/2)(1/2) = 1/4$, the result is an immediate consequence of Theorem 26. $\boxed{QED}$

We collect some properties of K-complements in the following:

36 Lemma. *If $\mathcal{Q}$ is a class of ordinals and $J, K \in \mathcal{Q}_K$, then*

(a) $JK = \emptyset$ iff $K \subseteq J^\perp$ iff $J \subseteq K^\perp$;

(b) $J \subseteq K$ implies $K^\perp \subseteq J^\perp$;

(c) $K \subseteq K^{\perp\perp}$;

(d) $K^\perp = K^{\perp\perp\perp}$;

(e) $K = K^{\perp\perp}$ iff $K = J^\perp$ for some $J \in \mathcal{Q}_K$;

(f) $(J \cap K)^\perp = J^\perp \cup K^\perp$;

(g) If $\{K_i\}_{i \in I} \subseteq \mathcal{Q}_K$, then $(\cup_{i\in I} K_i)^\perp = \cap_{i \in I}(K_i{}^\perp)$.

Proof. (a), (b) and (c) follow immediately from the definitions. (d) and (e) follow from (b) and (c). For (f), since $J \cap K \subseteq J, K$, we have $J^\perp, K^\perp \subseteq (J \cap K)^\perp$, so $J^\perp \cup K^\perp \subseteq (J \cap K)^\perp$. Conversely, if $S \in (J \cap K)^\perp$ then $S \cap V \neq \emptyset$ for all $V \in J \cap K$. Now, if $S \notin J^\perp$, then there is an $T_0 \in J$ such that $S \cap T_0 = \emptyset$. Therefore, for all $U \in K$, $T_0 \cup U \in J \cap K$, which implies that $S \cap (T_0 \cup U) \neq \emptyset$. Since $S \cap T_0 = \emptyset$, we conclude that $S \cap U \neq \emptyset$, so $S \in K^\perp$, proving (f). Regarding (g), $S \in (\cup_{i \in I} K_i)^\perp$ iff $S \cap T \neq \emptyset$ for all $T \in \cup_{i\in I} K_i$ iff $S \cap T \neq \emptyset$ for all $i \in I$ and for all $T \in K_i$ iff $S \in \cap_{i\in I}(K_i{}^\perp)$. $\boxed{QED}$

We say $K \in \mathcal{Q}_K$ is *closed* if it satisfies (e). By (f) and (g), the complements of the closed elements of $\mathcal{Q}_K$ form a topology on $\mathcal{Q}_f$. For $K \in \mathcal{Q}_K$, $K^{\perp\perp}$ is the smallest closed set containing K, so it is the closure of K in the topology, and we denote $K^{\perp\perp}$ by $\overline{K}$. Naturally, K is *dense* iff $\overline{K} = 1_{\mathcal{Q}}$. We will see that when $\mathcal{Q}$ is finite, every element of $K_{\mathcal{Q}}$ is closed, but that this fails when $\mathcal{Q}$ is infinite.

If $\mathcal{V}$ is a subclass of $\mathcal{Q}$ there is a natural function $\psi^{\mathcal{Q}}_{\mathcal{V}} : \mathcal{Q}_K \to \mathcal{V}_K$, where for $K \in \mathcal{Q}_K$,

$$\psi^{\mathcal{Q}}_{\mathcal{V}}(K) = K \cap \mathcal{V}_f = \{\, S \in \mathcal{V}_f \mid S \in K \,\}.$$

In addition, whenever $K \in \mathcal{Q}_K$, then $\cup M_K$ is a set, and if we let $\mathcal{V} = \cup M_K$, then it is fairly clear for $X \in \mathcal{Q}_f$, that $X \in K$ iff $X \cap \mathcal{V} \in K$, and that we might as well assume all calculations regarding K take place in $\mathcal{V}$. We will utilize this observation often in the sequel without making specific mention of it.

We note in passing that if $K_1, \dots, K_n$ are in $\mathcal{Q}_K$, then the elements of the iterated product $K_1 \cdots K_n$ are the unions of pairwise disjoint sets $S_1 \cup \cdots \cup S_n$ such that each $S_i \in K_i$. Equivalently, $T \in K_1 \cdots K_n$ iff for $i = 1, \dots n$, there are pairwise disjoint sets $S_i \in M_{K_i}$ such that $S_1 \cup \cdots \cup S_n \subseteq T$. We use this observation in the following:

37 Proposition. *If $\mathcal{Q}$ is a finite set of ordinals with $|\mathcal{Q}| = n$, $n < m < \omega$ and $K_1 \dots, K_m \in \mathcal{Q}_K$ are proper $\mathcal{Q}_K$-invariants (i.e., each $K_i \neq 1_{\mathcal{Q}}$), then $K_1 \cdots K_m = 0_{\mathcal{Q}}$.*

Proof. Note that any pairwise disjoint union $S_1 \cup \cdots \cup S_m \in K_1 \cdots K_m$ must have at least m elements, which implies that no such things exist. $\boxed{QED}$

We can use K_G-invariants to give short and elementary proofs of nearly all of the previous work on Nunke's problem. For example, the next result is a restatement of Theorem 5 of [13] and follows directly from Proposition 37.

38 Corollary. *Suppose $n, m < \omega$, $n < m$ and $G_1, \dots, G_m$ are separable groups of final rank at most $\aleph_n$. Then $G_1 \bigtriangledown \cdots \bigtriangledown G_m$ is Σ-cyclic.*

Proof. Note that if we let $\mathcal{V} = \{\aleph_1, \dots, \aleph_n\}$, then $\cup_{i \leq m} M_{G_i} \subseteq \mathcal{V}$, so we can perform our computation in $\mathcal{V}$. The result therefore follows directly from Proposition 37. $\boxed{QED}$

On the other hand, the last result can be significantly strengthened for almost Σ-cyclic groups:

39 Corollary. *Suppose $n, m < \omega$, $n/2 < m$ and $G_1, \dots, G_m$ are almost Σ-cyclic groups of final rank at most $\aleph_n$. Then $G_1 \bigtriangledown \cdots \bigtriangledown G_m$ is Σ-cyclic.*

Proof. Note that every element of K_{G_i} has at least two elements, so if $S_1 \in K_{G_1}, \dots, S_m \in K_{G_m}$ are pairwise disjoint, it follows that $2m \leq n$. $\boxed{QED}$

Of course, a $\mathcal{Q}_K$-invariant K is *finitely generated* if M_K is finite. In this case, $\mathcal{V} = \cup M$ will also be finite. We next note that for finitely generated $\mathcal{Q}_f$ invariants, K-nilpotency is easily characterized:

40 Corollary. *If $K \in \mathcal{Q}_K$ is finitely generated, then K is K-nilpotent iff $K \neq 1_{\mathcal{Q}}$ (i.e., K is proper).*

Proof. Note $1_{\mathcal{Q}}$ is not K-nilpotent, but Proposition 37 implies that any proper $\mathcal{Q}_K$-invariant is. $\boxed{QED}$

Suppose K is a $\mathcal{Q}_K$-invariant and $X \in \mathcal{Q}_f$. Our next result characterizes when $X \in \overline{K}$. Recall that a collection of sets $D \subseteq \mathcal{Q}_f$ is a Δ-*system* if there is a fixed $X \subseteq \mathcal{Q}$ such that $S \cap T = X$ for all distinct pairs $S, T \in D$. The set X

is called the *root* of D. This notion has an important role in set theory (see, for example, section II.6 of [4]).

41 Lemma. *If $\mathcal{Q}$ is a class of ordinals, $X \in \mathcal{Q}_f$ and $K \in \mathcal{Q}_K$, then the following are equivalent:*

(a) $X \in \overline{K}$;

(b) For all $Z \in (\mathcal{Q}-X)_f$, there exists $S \in K$, such that $X \subseteq S$ and $Z \cap S = \emptyset$;

and if $\mathcal{Q}$ is infinite, these are equivalent to:

(c) There is an infinite subset $D \subseteq K$ which is a Δ-system with X as a root.

Proof. We first suppose that X satisfies (b) and verify that (a) holds. If $Z \in K^{\perp}$, we need to show that $Z \cap X \neq \emptyset$. Note that if this does not happen, then by hypothesis, there is an $S \in K$ such that $X \subseteq S$ and $S \cap Z = \emptyset$. But this would contradict that $Z \in K^{\perp}$.

We now show that (a) implies (b), so suppose $X \in \overline{K}$, and $Z \in (\mathcal{Q} - X)_f$. Since $Z \cap X = \emptyset$, we can conclude that $Z \notin \overline{K}^{\perp} = K^{\perp\perp\perp} = K^{\perp}$. It follows that there is an $S_0 \in K$ such that $Z \cap S_0 = \emptyset$. Letting $S = X \cup S_0$ gives the required set.

We now show the equivalence of (b) and (c) assuming that $\mathcal{Q}$ is infinite: Suppose first that we are given the infinite Δ-system, $D \subseteq K$, with X as a root. If Z is a finite subset of $\mathcal{Q} - X$, then since D is a Δ-system, every $z \in Z$ is an element of at most one element of D. Therefore, there is a $T \in D$ which contains no element of Z, as required. Conversely, suppose X satisfies (b); we will construct $D = \{T_n \mid n < \omega\}$ by induction. Choose $T_0 \in K$ such that $X \subseteq T_0$ (such a T_0 must exist by letting Z equal, say, $\emptyset$ in (b)). Since $\mathcal{Q}$ is infinite, we may, in fact, expand T_0 so that the containment $X \subset T_0$ is proper. Suppose we have constructed $T_0, \ldots, T_n$, all of which properly contain X, such that for all distinct $i, j \leq n$, $T_i \cap T_j = X$. Using (b), we can find $T_{n+1} \in K$ such that there is a proper containment $X \subset T_{n+1}$, and such that

$$T_{n+1} \cap ((T_0 \cup \cdots \cup T_n) - X) = \emptyset.$$

It follows that for all distinct $i, j \leq n+1$, $T_i \neq T_j$ and $T_i \cap T_j = X$, completing the argument. $\boxed{QED}$

42 Theorem. *If $\mathcal{Q}$ is a class of ordinals and $K \in \mathcal{Q}_K$, then K is closed iff M_K has no infinite subset that is a Δ-system.*

Proof. Let $C = \overline{K}$. Suppose first that $\mathcal{Q}$ is finite. If $X \in C$, then letting $Z = \mathcal{Q} - X \in (\mathcal{Q} - X)_f$ in Lemma 41(b), we can find an $S \in K$ such that

$X \subseteq S$ and $X \cap Z = \emptyset$. These conditions, however, imply that $X = S \in K$, so that K is closed, as required.

We may therefore assume that $\mathcal{Q}$ is infinite. Now, assume that $\{ T_i \mid i < \omega \} \subseteq M_K$ is a Δ-system having X as a root. By Lemma 41(c), we can conclude $X \in C$. Note that since X is a proper subset of each T_i, and these are all minimal sets in K, we can conclude that $X \notin K$, so that K is not closed.

Conversely, suppose $C \neq K$ and choose $X \in C - K$. By Lemma 41(c), there is a Δ-system, $\{T_i : i < \omega\} \subseteq K$ having X as a root. For each $i < \omega$, choose $S_i \in M_K$ such that $S_i \subseteq T_i$. Since $X \notin K$, we can conclude that for all $i < \omega$, S_i is not contained in X. However, since for all distinct $i, j < \omega$, $S_i \cap S_j \subseteq X$, we can conclude that $S_i \neq S_j$. Because X has only finitely many subsets, there is $X' \subseteq X$ such that $S_i \cap X = X'$ for infinitely many $i < \omega$, and it immediately follows that this infinite subset is a Δ-system with X' as a root. $\boxed{QED}$

43 Corollary. *If $\mathcal{Q}$ is a class of ordinals and $K \in \mathcal{Q}_K$ is finitely generated, then K is closed.*

Proof. Since M_K has no infinite subsets, it must be closed. $\boxed{QED}$

44 Corollary. *If $\mathcal{Q}$ is a class of ordinals and $K \in \mathcal{Q}_K$ is closed, then M_K is countable.*

Proof. It is a well known result that any uncountable collection of finite sets has a infinite subcollection which is a Δ-system. (See, for example Corollary II.6.2 of [4].) $\boxed{QED}$

45 Example. A specific closed $\mathcal{Q}$-invariant K such that M_K is (countably) infinite can be constructed as follows: Suppose $\mathcal{Q} = \omega$ and for every odd number $j < \omega$, $S_j = \{j\} \cup \{ k < j \mid k \text{ is even} \}$. It is obvious that the collection of all such S_j is an ω_f-antichain, and if $j_0 < j_1$ are odd numbers, then $S_{j_0} \cap S_{j_1} = \{ k < j_0 \mid k \text{ is even} \}$, from which it can readily be seen that $M_K = \{ S_j \mid j \text{ is odd} \}$ has no infinite subset that is a Δ-system.

46 Lemma. *If $\mathcal{Q}$ is a class of ordinals and $J, K \in \mathcal{Q}_K$ are closed, then so is their product JK.*

Proof. Suppose JK is not closed. Then by Theorem 42, we can find an infinite collection $\{S_n \cup T_n\}_{n<\omega} \subseteq JK$, where $S_n \in M_J$ and $T_n \in M_K$ are disjoint, which is a Δ-system with X as a root. Note that either $\{S_n\}_{n<\omega}$ or $\{T_n\}_{n<\omega}$ must be an infinite collection; without loss of generality, assume it is the former. Note that for all $n, m < \omega$, $S_n \cap S_m \subseteq X$. Since X has only a finite number of subsets, there is an infinite subset $L \subseteq \omega$ such that for all $n \in R$, $S_n \cap X$ always equals some fixed $X_0 \subseteq X$. It follows $\{ S_n \mid n \in L \}$ is an infinite Δ-system with X_0 as a root, so by Theorem 42 that J is not closed, contrary to hypothesis. $\boxed{QED}$

The above results have the following consequence regarding products.

47 Theorem. *If $\mathcal{Q}$ is a class of ordinals and $J, K \in \mathcal{Q}_K$, then $\overline{JK} = \overline{J}\,\overline{K}$.*

Proof. The result is trivial if $\mathcal{Q}$ is finite, so assume $\mathcal{Q}$ is infinite. Note first that $JK \subseteq \overline{J}\,\overline{K}$, and by Lemma 46, the latter is closed, so $\overline{JK} \subseteq \overline{J}\,\overline{K}$. Conversely, suppose $X \cup Y \in \overline{J}\,\overline{K}$, where $X \in \overline{J}$, $Y \in \overline{K}$ and $X \cap Y = \emptyset$. We use the characterization of the closure of a $\mathcal{Q}_K$-invariant given by Lemma 41(b). If $Z \in (\mathcal{Q} - (X \cup Y))_f$, then since $X \in \overline{J}$, there is an $S \in J$ such that

$$X \subseteq S \text{ and } S \cap (Z \cup Y) = \emptyset.$$

Now, since $Y \in \overline{K}$, there is a $T \in K$ such that

$$Y \subseteq T \text{ and } T \cap (Z \cup S) = \emptyset.$$

It follows that $S \cap T = \emptyset$, so that $S \cup T \in JK$. Also, since

$$(X \cup Y) \subseteq (S \cup T) \text{ and } (S \cup T) \cap Z = \emptyset,$$

it follows that $X \cup Y \in \overline{JK}$. This proves that $\overline{J}\,\overline{K} \subseteq \overline{JK}$, so that $\overline{JK} = \overline{J}\,\overline{K}$. $\boxed{QED}$

48 Example. It is usually not the case that $(JK)^{\perp} = J^{\perp} K^{\perp}$, even for finite sets $\mathcal{Q}$. If $\mathcal{Q} = 2 = \{0, 1\}$, $M_J = \{\{0\}\}$ and $M_K = \{\{1\}\}$, then $M_{JK} = \{\{0, 1\}\}$ and $M_{(JK)^{\perp}} = \{\{0\}, \{1\}\}$. On the other hand, $J^{\perp} = J$ and $K^{\perp} = K$, so $M_{J^{\perp} K^{\perp}} = \{\{0, 1\}\} \neq M_{(JK)^{\perp}}$.

Recall the *K-index* of a $\mathcal{Q}_K$-invariant K is the smallest positive integer n such that $K^n = 0_{\mathcal{Q}}$ if it exists, and otherwise, we say K has infinite K-index.

49 Corollary. *If $\mathcal{Q}$ is a class of ordinals and $K \in \mathcal{Q}_K$, then K and $\overline{K}$ have the same K-index.*

Proof. By Theorem 47, $\overline{K^n} = \overline{K}^{\,n}$, so $K^n = 0_{\mathcal{Q}}$ iff $\overline{K^n} = 0_{\mathcal{Q}}$ iff $\overline{K}^{\,n} = 0_{\mathcal{Q}}$. $\boxed{QED}$

The following reformulation of the above discussion gives a more complete characterization of K-nilpotency:

50 Theorem. *Suppose $\mathcal{Q}$ is a class of ordinals and $K \in \mathcal{Q}_K$. Then the following are equivalent:*

(a) K is K-nilpotent;

(b) K is not dense;

(c) $K^{\perp} \neq 0_{\mathcal{Q}}$;

(d) There is $J \in \mathcal{Q}_K$ such that $J \neq 0_{\mathcal{Q}}$ but $JK = 0_{\mathcal{Q}}$;

(e) $K \neq 1_{\mathcal{Q}}$ and M_K has no infinite pairwise disjoint subset.

Proof. (a) $\Rightarrow$ (b): If K is K-nilpotent, so is $\overline{K}$, which implies that it is necessarily proper, and hence K is not dense.

(b) $\Rightarrow$ (c): If $K^{\perp} = 0_{\mathcal{Q}}$, then $\overline{K} = 1_{\mathcal{Q}}$, so that K is dense.

(c) $\Rightarrow$ (d): Just let $J = K^{\perp}$.

(d) $\Rightarrow$ (a): Suppose $X \in J$ has n elements. If $S_1, \ldots, S_{n+1}$ are in K, then $S_1 \cap X \neq \emptyset, \ldots, S_{n+1} \cap X \neq \emptyset$. If follows that there are $i, j \in \{1, \ldots, n+1\}$ such that $S_i \cap S_j \neq \emptyset$. Therefore, $K^{n+1} = \emptyset$, as required.

(a) $\Rightarrow$ (e): Is clear.

(e) $\Rightarrow$ (b): If K is dense and proper, then $\mathcal{Q}$ must be infinite, and by Lemma 41(c), there is an infinite Δ-system $\{S_n\}_{n<\omega} \subseteq K$ with $\emptyset$ as a root. We may clearly assume that each $S_n \in M_K$, but then this would be an infinite, pairwise disjoint subset, as required. $\boxed{QED}$

As a consequence of the argument which showed (d) $\Rightarrow$ (a), we have the following bound on K-indices:

51 Corollary. *If $\mathcal{Q}$ is a class of ordinals, $K \in \mathcal{Q}_K$ is K-nilpotent and $n = \| K^{\perp} \|$, then $K^{n+1} = 0_{\mathcal{Q}}$.*

We next note that, even in the finite case, the bound to the K-index mentioned in Corollary 51 is, in some sense, the best possible.

52 Example. Suppose $\mathcal{Q} = n = \{0, \ldots, n-1\}$, M is the $\mathcal{Q}_f$-antichain $\{\, \{i\} \mid i < n \,\}$. It can be checked that ${K_M}^{\perp}$ has only one minimal set, namely n itself, and that the K-index of K_M is exactly $n + 1 = \| {K_M}^{\perp} \| + 1$.

On the other hand, this bound may be strict:

53 Example. Suppose $\mathcal{Q} = 3 = \{0, 1, 2\}$, $M = \{\{0, 1\}, \{0, 2\}, \{1, 2\}\}$. It can be checked that ${K_M}^{\perp} = K_M$, so that $2 = \| {K_M}^{\perp} \|$, but $K_M^2 = 0_{\mathcal{Q}}$, so that K_M has K-index $2 < 2 + 1$.

We now translate Theorem 50 into the language of groups. In keeping with the above terminology, recall that the *K-index* of a group G is the smallest positive integer n such that $G^n = G \bigtriangledown \cdots \bigtriangledown G$ (n-copies) is Σ-cyclic if it exists, and otherwise, we say it has infinite K-index. A group is *K-nilpotent* if it has finite K-index; let $\mathcal{K}$ denote the class of K-nilpotent groups. Note that if G is a separable group of cardinality $\aleph_n$, then by Corollary 38, the K-index of G is at most $n + 1$; and if G is almost Σ-cyclic of cardinality $\aleph_n$, then by Corollary 39, its K-index is at most $n/2 + 1$. The next result follows easily from the definition:

54 Proposition. *Suppose G is a group.*

(a) If $G \in \mathcal{K}$ and A is a subgroup of G, then $A \in \mathcal{K}$;

(b) If G is the direct sum of A and B, then $G \in \mathcal{K}$ iff A and $B \in \mathcal{K}$;

(c) If $G \in \mathcal{K}$ and H is any group, then $G \bigtriangledown H \in \mathcal{K}$.

The following gives a complete answer to the nilpotent version of Nunke's problem (at least for δ_m-groups):

55 Theorem. *If G is a δ_m-group, then the following are equivalent:*

(a) $G \in \mathcal{K}$;

(b) K_G is not dense;

(c) ${K_G}^{\perp} \neq 0_{\mathcal{R}}$;

(d) There is a group H of with $c(H) > 0$ but $c(G \bigtriangledown H) = 0$;

(e) G is separable and M_G has no infinite pairwise disjoint subset.

Proof. Each part corresponds to a statement from Theorem 50. The only small point to be considered is whether it is necessary to use the axiom of constructibility to produce the group H mentioned in (d). This is, however, unnecessary, since if G has K-index n, we need merely let $H = G^{n-1}$. $\boxed{QED}$

The following is a direct consequence of Corollary 51.

56 Corollary. *If G is a δ_m-group, $G \in \mathcal{K}$ and $n = \|\, {K_G}^{\perp} \,\|$, then the K-index of G does not exceed $n + 1$. Furthermore, in the constructible universe (V=L),*

$$2^{-n} = \max\{\, c(H) \mid G \bigtriangledown H \text{ is } \Sigma\text{-cyclic} \,\}.$$

Proof. Regarding the second statement, $G \bigtriangledown H$ is Σ-cyclic implies $K_H \subseteq {K_G}^{\perp}$, so that $c(H) \leq 2^{-n}$. On the other hand, in V=L, we can construct a group H such that $K_H = {K_G}^{\perp}$, and for this H, $c(H) = 2^{-n}$. $\boxed{QED}$

We conclude by showing that $\mathcal{K}$ is closed under other natural operations.

57 Corollary. *If $n < \omega$, A and G are separable δ_m-groups and $g : A \to G$ is an ω_n-bijection, then $A \in \mathcal{K}$ iff $G \in \mathcal{K}$.*

Proof. It follows from Corollary 17 that

$$\{\, S \in M_A \mid S \cap \aleph_n = \emptyset \,\} = \{\, S \in M_G \mid S \cap \aleph_n = \emptyset \,\}.$$

From this, it can be seen that M_A has an infinite pairwise disjoint subset iff M_G has an infinite pairwise-disjoint subset. It follows, then, that A is K-nilpotent iff G is K-nilpotent. $\boxed{QED}$

We have the following Dieudonne-type result for $\mathcal{K}$.

58 Corollary. *Suppose A is a subgroup of the δ_m-group G and $C = G/A \in \mathcal{K}$. Then $G \in \mathcal{K}$ iff A is contained in a pure subgroup B of G such that $B \in \mathcal{K}$. In fact, if $c(B^m) = 0$ and $c(C^n) = 0$, then $c(G^{m+n-1}) = 0$.*

PROOF. Clearly, if $G \in \mathcal{K}$, then we may let $B = G$. Conversely, if we are given the pure subgroup $B \in \mathcal{K}$, then by Theorem 20, $K_G \subseteq K_B \cup K_C$. Therefore, if $S_1, \ldots, S_{m+n-1} \in K_G$, then either m of these sets are in K_B or n of them are in K_C. In either case, this means that they cannot be pairwise disjoint, so that the result follows. $\boxed{QED}$

The last result can be sharpened if A itself is pure.

59 Corollary. *Suppose A is a pure subgroup of the δ_m-group G and $C = G/A \in \mathcal{K}$. Then $G \in \mathcal{K}$ iff $A \in \mathcal{K}$. In fact, if $c(A^m) = 0$ and $c(C^n) = 0$, then $c(G^{m+n-1}) = 0$.*

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