Note di Matematica Vol. 19 - n. 1, 1-6 (1999)

THE FLOCK DERIVATION IN $T_2(C)$

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Abstract. *When q is an odd prime power, in the generalized quadrangle $T_2(C)$there are two distinguished sets of points associated with a flock of the quadratic cone: the BLT-set and the dual flock. We prove here that the union of the dual flock with a special point of $T_2(C)$is isomorphic to the BLT-set of the same flock.*

1 Flock and the dual setting

Throughout this paper q will be an odd prime power.

Let $\mathbf{K}$ be a quadratic cone in $PG(3,q)$ with vertex v. A *flock* $\mathcal{F}$ of $\mathbf{K}$ is a partition of $\mathbf{K}\backslash\{v\}$ into q disjoint irreducible conics C_1, C_2, ..., C_q. The flock $\mathcal{F}$ is called linear if all the q planes of the conics C_i contain a common line.

Suppose the quadratic cone $\mathbf{K}$ is represented by the equation $x_0x_1 - x_2^2 = 0$, so $v = (0,0,0,1)$; if π_i is the plane with equation $a_ix_0 + b_ix_1 + c_ix_2 + x_3 = 0$, and C_i is the intersection conic of π_i with $\mathbf{K}$, then, by Thas [4], the set $\mathcal{F} = \{C_i/i = 1,\ldots,q\}$ is a flock of $\mathbf{K}$ if and only if $(c_i - c_j)^2$ $-4(a_i - a_j)$ $(b_i - b_j)$ is a non-square for all $i, j \in \{1,2,\ldots q\}$, with $i \neq j$.

Let $[a,b,c,d]$ be the plane with equation $ax_0 + bx_1cx_2 + dx_3 = 0$, and denote by δ the polarity of $PG(3,q)$ defined by $(a,b,c,d) \longleftrightarrow [a,b,c,d]$. The vertex $(0,0,0,1)$ of $\mathbf{K}$ is mapped by δ to the plane π with equation $x_3 = 0$. The lines of $\mathbf{K}$ are mapped to the tangent lines to the conic C of π with equation $x_2^2 - 4x_0x_1 = x_3 = 0$. The plane π_i is mapped to the point $(a_i,b_i,c_i,1)$ which does not belong to π. The set $\mathcal{D}(\mathcal{F}) = \{(a_i,b_i,c_i,1)/i = 1,2,\ldots,q\}$ is called the *dual flock* of $\mathcal{F}$. It easy to see that $\mathcal{F}$ is a flock if and only if any of the lines joining two points of $\mathcal{D}(\mathcal{F})$ intersects π in an interior point of C.

We notice that $\mathcal{F}$ is linear if and only if the points of $\mathcal{D}(\mathcal{F})$ are collinear.

Let $Q(4,q)$ be the non-singular quadric of the projective space $PG(4,q)$ defined by the equation $x_0x_1 - x_2^2 + x_3x_4 = 0$. A $BLT-set$ is a set $\mathcal{S}$ of $q+1$ pairwise non-collinear points of $Q(4,q)$ such that the polar line of the plane joining any three distinct points of $\mathcal{S}$ is exterior to $Q(4,q)$.

We want to recall here how to obtain a BLT-set starting with a flock (see [1]).

Fix the point $P_0 = (0,0,0,1,0) \in Q(4,q)$. The polar hyperplane $x_4 = 0$ of P_0 intersects $Q(4,q)$ in the cone $\mathbf{K}$ with equation $x_0x_1 - x_2^2 = x_4 = 0$.

Suppose the flock $\mathcal{F} = \{C_1,\ldots,C_q\}$ of $\mathbf{K}$ is determined by the planes π_i with equations $a_ix_0 + b_ix_1 + c_ix_2 + x_3 = x_4 = 0$; (recall that the coefficients verify the condition expressed before).

Let $\perp$ be the polarity defined by $Q(4,q)$. The line $\pi_i^{\perp}$ intersects $Q(4,q)$ in P_0 and in the point $P_i = (b_i, a_i, -\frac{1}{2}c_i, \frac{1}{4}c_i^2 - a_ib_i, 1)$. The set $\mathcal{S} = \{P_i/i = 0,1,\ldots,q\}$ is a BLT-set of the quadric $Q(4,q)$, and we denote it by $\mathcal{B}(\mathcal{F})$ to emphasize it is associated with the flock $\mathcal{F}$.

Remark the flock $\mathcal{F}$ is obtained from the BLT-set $\mathcal{B}(\mathcal{F})$ by putting $C_i = P_0^\perp \cap P_i^\perp \cap Q(4,q)$, for $i \in \{1,\dots,q\}$.

2 The generalized quadrangle $T_2(C)$

Let π be any plane of $PG(3,q)$ and C be an irreducible conic of π; construct the incidence structure $T_2(C)=(\mathcal{P},\mathcal{L},\mathbf{I})$ in the following way.
The elements of $\mathcal{P}$ (points) are:
1) the points of $PG(3,q)\backslash\pi$;
2) the planes of $PG(3,q)$ containing a tangent to C;
3) the symbol (∞).

The elements of $\mathcal{L}$ (lines) are:
a) the lines of $PG(3,q)$ not contained in π and incident with C;
b) the points of C.

Incidence: the points (∞) is incident with each line of type (b) and only with them; the other incidences derive from the usual incidence in $PG(3,q)$. The incidence structure $T_2(C)$is a generalized quadrangle of order q (see [3]).

Consider the classical generalized quadrangle associated with the non-singular quadratic $Q(4,q)$ of $PG(4,q)$ defined by the equation $x_0x_1 - x_2^2 + x_3x_4 = 0$, and the quadratic cone $\mathbf{K}$ with equation $x_0x_1 - x_2^2 = x_4 = 0$. The hyperplane $\Sigma : x_3 = 0$ of $PG(4,q)$ does not contain $P_0 = (0,0,0,1,0)$, so $\Sigma\cap\ \mathbf{K}$ is the conic C defined by $x_0x_1 - x_2^2 = x_3 = x_4 = 0$; let π be the plane of Σ containing C. Construct in $\Sigma \simeq PG(3,q)$ the generalized quadrangle $T_2(C)$as above. If l and m are lines of $Q(4,q)$ and y is a point of $Q(4,q)$, then the map θ defined by

$$\begin{aligned}
\theta &: P_0 = (0,0,0,1,0) \to (\infty),\\
\theta &: l \to l\cap\pi, \text{ for } P_0 \in l \subset P_0^\perp,\\
\theta &: y \in P_0^\perp\backslash\{P_0\} \to y^\perp \cap \Sigma,\\
\theta &: m \notin P_0^\perp \to < m,(0,0,0,1,0) > \cap\Sigma,\\
\theta &: (a,b,c,c^2-ab,1) \to (a,b,c,0,1),
\end{aligned}$$

is an isomorphism from $Q(4,q)$ to $T_2(C)$.

Let $\mathcal{B}(\mathcal{F})$ be the BLT-set associated with the flock $\mathcal{F}$ of $\mathbf{K}$ whose planes have equations $a_ix_0 + b_ix_1 + c_ix_2 + x_3 = x_4 = 0$ for $i = 1,\dots,q$; then

$$\mathcal{B}(\mathcal{F})^\theta = \overline{\mathcal{B}(\mathcal{F})} = \{(\infty)\} \cup \{\overline{P_i} = (b_i, a_i, -\frac{1}{2}c_i, 0, 1)/i = 1,\dots,q\}.$$

By the isomorphism θ we can read the properties of the BLT-set in $T_2(C)$, thus we can say that all points of $\overline{\mathcal{B}(\mathcal{F})} - \{(\infty)\}$ are of type (1) and all elements of $\overline{\mathcal{B}(\mathcal{F})}$ are pairwise not collinear; furthermore, if we put $\overline{P_0} = (\infty)$, every triad $(\overline{P_i},\overline{P_j},\overline{P_k})$ is acentric, thus there is no point in the quadrangle collinear with all them; then the line $l_{i,j}$ joining $\overline{P_i}$ and $\overline{P_j}$, for all $i,j \in \{1,2,\dots,q\}$ with $i \neq j$, intersects the plane π in a point interior to the conic C; in fact it is clear that $l_{i,j}$ intersects π in a point $Y \notin C$, because $\overline{P_i}$ and $\overline{P_j}$ are not collinear; on the other hand if Y is a point exterior to C, let t_Y be one of the tangent lines to C from Y, the plane $\alpha =< t_y, l_{i,j} >$ is a point of $T_2(C)$collinear with $\overline{P_i},\overline{P_j}$ and $\overline{P_0}$, but this is impossible; so the point Y is interior to C.

Lemma 1 *Let C be an irreducible conic in $PG(2,q)$. Let U_1, U_2 and U_3 be three points of C; let P and Q be points respectively on the lines $< U_1,U_3 >$ and $< U_2,U_3 >$, and let H be the common point of the lines $< P,Q >$ and $< U_1,U_2 >$; the following hold:*

(a) If P and Q are both interior points or both exterior points to C then H is exterior;

(b) If P interior and Q is exterior to C then H is interior.

Proof. Suppose P and Q are interior points of C, and let conic C have equation $x_1^2 - x_0x_2 = 0$. As $PGO(3,q)$ is 3-transitive on the points of C we can suppose $U_1 = (1,0,0)$, $U_2 = (0,0,1)$ and $U_3(1,1,1)$. The line $< U_1,U_3 >$ has equation $x_1 - x_2 = 0$ and the line $< U_2,U_3 >$ has equation $x_0 - x_1 = 0$. Let $P(y_0,y_1,y_2)$ and $Q = (z_0,z_1,z_2)$; then $y_1^2 - y_0y_2$ is a non-square in $GF(q)$ and $y_1 = y_2$, $z_1^2 - z_0z_2$ is a non-square in $GF(q)$ and $z_0 = z_1$.

The line $< U_1,U_2 >$ has equation $x_1 = 0$, while the line $< P,Q >$ has equation

$$(z_2y_1 - y_1z_0)x_0 - (z_2y_0 - z_0y_1)x_1 + (z_0y_0 - z_0y_1)x_2 = 0.$$

Then the point H has coordinates $(z_0y_0 - z_0y_1, 0, z_0y_1 - z_2y_1)$; so

$$h_1^2 - h_0h_2 = (-z_0y_0 + z_0y_1)(z_0y_1 - z_2y_1) = (z_1^2 - z_0z_2)(y_1^2 - y_0y_2)$$

is a square in $GF(q)$, and H is an exterior point to C. The other proofs can be worked out in a similar way. □

With the above notations we can now prove the following:

Proposition 2 *Let $S = \{P_0, P_1, \ldots, P_q\}$ be a set of $q+1$ points of $T_2(C)$, such that $P_0 = (\infty)$ and P_i is a point of type (1) for each $i \geq 1$. If for each pair (i,j), with $i \neq j$ and $i,j \in \{1,2,\ldots,q\}$, the line $< P_i,P_j >$ in Σ intersects π in an interior point to C, then each triple (P_i,P_j,P_k), with $i \neq j \neq k \neq i$, is an acentric triad in $T_2(C)$.*

Proof. Clearly each triple of type (P_0,P_i,P_j) is acentric, because all the points of $T_2(C)$collinear with P_0 are of type (2), and by assumption, any plane containing the line $l =< P_i,P_j >$ intersects π in a bisecant or exterior line to C.

Let (P_i,P_j,P_k) be a triple of points of $S - \{P_0\}$; there are no points of type (2) collinear with two points of the triple, so we can suppose there is a point of type (1) collinear with all three, say Y; remark that the lines $l_i =< P_i,Y >, l_j =< P_j,Y >$, $l_k =< P_k,Y >$ can not be coplanar, otherwise the points $A = l_i \cap C$, $B = l_j \cap C$ and $C = l_k \cap C$ are collinear or two lines between l_i, l_j, and l_k are coincident, both possibilities contraddicting the hypothesis; hence, in particular, we can remark that the points P_i,P_j,P_k are not collinear in Σ. Then the triangles $P_iP_jP_k$ and ABC are in a Desargues configuration with the point Y, so the points

$$< P_i,P_j > \quad \cap \quad < A,B >= M,$$

$$< P_i,P_k > \quad \cap \quad < A,C >= N,$$

$$< P_j,P_k > \quad \cap \quad < B,C >= H,$$

are collinear, all of them are interior points and $H =< M,N > \cap < B,C >$; this is impossible by Lemma 1. □

Then we can say the condition that the set $\mathcal{S} = \{(\infty)\} \cup \{P_i / i = 1,2,\ldots,q\} \subset T_2(\mathcal{C})$, with the points P_i of type (1), such that each triple of its points is an acentric triad is, in fact, equivalent to the property that each line $l_{i,j} =< P_i, P_j >$, with $i, j \in \{1,2,\ldots,q\}$ and $i \neq j$, intersects π in an interior point to $\mathcal{C}$.

Let Σ' be the tangent hyperplane to $Q(4,q)$ at P_0 with equation $x_4 = 0$, and let $\mathcal{F}$ be a flock of $\mathbf{K} = \Sigma' \cap Q(4,q)$ with the above notations; consider the set

$$\mathcal{D}(\mathcal{F}) \cup \{(\infty)\} = \{(\infty)\} \cup \{(a_i, b_i, c_i, 1, 0)/i = 1,2,\ldots,q\}$$

defined in Section 1 in the generalized quadrangle $T_2(\mathcal{C}')$ where $\mathcal{C}'$ has equation $4x_0x_1 - x_2^2 = x_3 = x_4 = 0$. On the other hand, we can consider the set

$$\overline{\mathcal{B}(\mathcal{F})} = \{(\infty)\} \cup \{(b_i, a_i, -\frac{1}{2}c_i, 0, 1)/i = 1,2,\ldots,q\}$$

in the generalized quadrangle $T_2(\mathcal{C})$, where $\mathcal{C} : x_0x_1 - x_2^2 = x_3 = x_4 = 0$ is in $\Sigma : x_3 = 0$. Consider the following isomorphism from Σ to Σ':

$$\phi : (x_0, x_1, x_2, 0, x_4) \to (x_1, x_0, -2x_2, x_4, 0);$$

induces an isomorphism between the generalized quadrangles $T_2(\mathcal{C})$and $T_2(\mathcal{C}')$ that maps the set $\overline{\mathcal{B}(\mathcal{F})}$ to the set $\mathcal{D}(\mathcal{F}) \cup \{(\infty)\}$; hence:

Theorem 3 *If $\mathcal{F}$ is a flock of a quadratic cone* $\mathbf{K}$ *in $PG(3,q)$, the set $\mathcal{D}(\mathcal{F}) \cup \{(\infty)\}$ and the BLT-set $\mathcal{B}(\mathcal{F})$ are isomorphic; more precisely there is an isomorphism ϕ from $T_2(\mathcal{C})$to $Q(4,q)$ such that: $(\infty)^\phi = (0,0,0,1,0)$ and $(\mathcal{D}(\mathcal{F}) \cup \{(\infty)\})^\phi = \mathcal{B}(\mathcal{F})$.*

3 The derivation in $T_2(\mathcal{C})$

Now we want to see the q derived flocks directly in the generalized quadrangle $T_2(\mathcal{C})$. As in Section 2 we denote by $\{\overline{P_0}, \ldots, \overline{P_q}\}$ the projection of the BLT-set $\mathcal{B}(\mathcal{F}) = \{P_0, P_1, \ldots, P_q\}$ in the $T_2(\mathcal{C})$; if K_i is the quadratic cone $P_i^\perp \cap Q(4,q)$, the corresponding element in $T_2(\mathcal{C})$is $\overline{K_i}$. We can see that, for $i = 1,2,\ldots,q$, $\overline{K_i}$ is the quadratic cone of Σ with vertex $\overline{P_i}$ and plane section $\mathcal{C}$; while the quadratic cone K_0 becomes the whole set of points of type (2) plus the point (∞).

Therefore each conic on $K_i, i \neq 0$, becomes a conic on $\overline{K_i}$, while a conic on K_0 becomes a set of $q+1$ planes of Σ, each one containing a tangent line to the conic $\mathcal{C}$ and forming a dual quadratic cone in Σ.

Proposition 4 *Let $i, j \in \{1,2,\ldots,q\}$, with $i \neq j$. The line $< P_0, P_i, P_j >^\perp$ is exterior to $Q(4,q)$ if and only if $\overline{K_i} \cap \overline{K_j}$ is the union of two disjoint irreducible conics, of which one is the conic $\mathcal{C}$.*

Proof. The line $< P_0, P_i, P_j >^\perp$ is exterior to $Q(4,q)$ if and only if the points are not pairwise collinear on the quadratic and $P_0^\perp \cap P_i^\perp \cap P_j^\perp$ is an exterior line to the quadratic. Let

$$C_{l,m} = P_l^\perp \cap P_m^\perp \cap Q(4,q) = K_l \cap K_m$$

for all $l, m \in \{0, 1, \dots, q\}$. The conics $C_{l,m}$ are pairwise disjoint. Moreover if

$$C_{h,k} \cap C_{r,s} = \emptyset \forall h, k, r, s \in \{0, i, j\}, \quad \text{and} \quad h \neq r \text{ or } k \neq s,$$

then

$$\overline{K_i} \cap \overline{K_j} = \overline{C}_{i,j} \cup C$$

and

$$\overline{C}_{i,j} \cap C = \emptyset;$$

On the other hand, since $K_i \cap K_0$ and $K_j \cap K_0$ are disjoint conics, the quadratic cones $\overline{K_i}$ and $\overline{K_j}$ have no common tangent planes.

Conversely, if we suppose $\overline{K_i} \cap \overline{K_j} = \overline{C}_{i,j} \cap C = \emptyset$, it is clear that the points P_i and P_j are not on $P_0^{\perp}$ and the conic $K_i \cap K_j = C_{i,j}$ is disjoint from K_0. Moreover, take $P \in C_{0,i} \cap C_{0,j}$, with $i \neq j$; then $\overline{P}$ is a point of type (2), thus it is a common tangent plane to the cones $\overline{K_i}$ and $\overline{K_j}$; so

$$P \in K_i \cap K_j \Rightarrow P \in C_{i,j} \Rightarrow \overline{C}_{i,j} \cap C \neq \emptyset,$$

a contradiction; so $C_{h,k} \cap C_{r,s} = \emptyset \forall h, k, r, s \in \{0, i, j\}$ with $h \neq r$ or $k \neq s$. □

Remark. In the preeceding proposition we have also proved that two quadratic cones intersect in two disjoint irreducible conics if and only if they have no common tangent planes.

At this point it is clear that it is possible to look at all derived flocks in the same 3-dimensional space of the original flock; it will be sufficient applying first the projection θ and then the isomorphism ϕ; now we want to compute explicitly this procedure.
The projection of K_i in $T_2(C)$is:

$$\overline{K_i} : x_0x_1 - a_ix_0x_3 - b_ix_1x_3 - x_2^2 - c_ix_2x_3 + d_ix_3^2 = 0$$

where $d_i = a_ib_i - \frac{1}{4}c_1^2$. So the conics of the flock $\mathcal{F}_i$ are given by $C_{i,j} = \overline{K_i} \cap \overline{K_j} \setminus C$ with $j = 1, \dots, q$, and $j \neq i$, plus the conic C. With a direct calculation we can see that the planes of the flock $\mathcal{F}_i$ are:
$\alpha \; : \; x_3 = 0$, and
$\alpha \; : \; (a_j - a_i)x_0 + (b_j - b_i)x_1 + (c_j - c_i)x_2 + (d_i - d_j)x_3 = 0$, with $j \neq i$.

Finally by ϕ the equation of the cone $\overline{K_i}$ becomes:

$$\overline{K_i'} : x_0x_1 - a_ix_1x_3 - \frac{1}{4}x_2^2 + \frac{1}{2}c_ix_2x_3 - b_ix_0x_3 + d_ix_3^2 = 0,$$

and the planes forming the derived flock $\mathcal{F}_i$ in Σ' are:
$\alpha_0 \; : \; 0$, and
$\alpha_j \; : \; (b_j - b_i)x_0 + (a_j - a_i)x_1 - \frac{1}{2}(c_j - c_i)x_2 + (d_i - d_j)x_3 = 0$, with $j \neq i$.

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Received March 25, 1998
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