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Bayesian Estimation of Parameter For Different Loss Functions Using Progressive Type-II Censored Data

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In this present work, we are going to show the various useful properties of the existing distribution known as $MG_{Exp}(\epsilon)$-distribution which have not quoted by the host authors like moments, mean deviation about mean, mean deviation about median, order statistics, count of uncertainty. Estimation procedures have been adopted under Bayesian estimation for progressive Type-II censored case. Simulation study has also been carried out to judge the behavior of the Bayes estimator at the long-run. Performance of the Bayes estimators and their posterior risks of the considered loss functions have been obtained, reported and compared for the considered values of sample size, effective sample size, parameter and removals. The comparison of Bayes estimators of all 6 chosen loss functions have been done on the ground of lowest posterior risks.

keywords: Bayesian estimation, $MG_{Exp}(\epsilon)$-distribution, loss function, posterior risk, censoring scheme.

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1 Introduction

The history of insertion of various distribution and transformation techniques is on full swing from few decades. Many authors and academicians working in the direction of proposition of new transformation and distribution. Various distributions have successfully added to statistical literature and have own merits and demerits. The exponential distribution has been extensively used in lifetime data analysis, but it is suitable for those situations where hazard rate is constant. Generally, it is not possible for real phenomena. For monotonic hazard rate, a number of distributions have been proposed and most widely used among these are Weibull and gamma distributions which are generalization of exponential distribution. Both of these distributions have increasing/decreasing hazard rate depending on their shape parameters. However, gamma distribution's distribution function and survival function, in particular, cannot be written in good closed forms, especially when the shape parameter is not an integer. Several authors did generalized exponential distribution by power transformation method given by Gupta et al. (1998) and DUS transformation method by Kumar et al. (15 a), SS transformation by Kumar et al. (15 b), PCM transformation by Kumar et al. (2021), MORKi distribution proposed by Afify et al. (2022) etc. Kumar et al. (2017) have been proposed a new transformation known as Minimum Guarantee Transformation which is given below

$$F(x) = \exp\left\{1 - \frac{1}{G(x)}\right\} \tag{1}$$

Where, $G(x)$ is baseline cumulative distribution function (CDF), and corresponding probability density function (PDF) is given below

$$f(x) = \exp\left\{1 - \frac{1}{G(x)}\right\}[G(x)]^{-2} \times g(x) \tag{2}$$

They used baseline distribution as exponential distribution and called it as Minimum Guarantee exponential symbolically $MG_{Exp}(\epsilon)$-distribution having the PDF as

$$f(x;\epsilon) = \epsilon e^{-\epsilon x}(1-e^{-\epsilon x})^{-2}\exp\left(-\frac{e^{-\epsilon x}}{1-e^{-\epsilon x}}\right) \;;\; x > 0,\ \epsilon > 0 \tag{3}$$

and associated CDF is

$$F(x;\epsilon) = \exp\left(-\frac{e^{-\epsilon x}}{1-e^{-\epsilon x}}\right) \;;\; x > 0,\ \epsilon > 0 \tag{4}$$

The reliability and hazard rate functions are

$$R(x;\epsilon) = 1 - \exp\left(-\frac{e^{-\epsilon x}}{1-e^{-\epsilon x}}\right) \tag{5}$$

and

$$h(x;\epsilon) = \epsilon e^{-\epsilon x}(1-e^{-\epsilon x})^{-2}\exp\left(-\frac{e^{-\epsilon x}}{1-e^{-\epsilon x}}\right)\left(1-\exp\left\{-\frac{e^{-\epsilon x}}{1-e^{-\epsilon x}}\right\}\right)^{-1} \tag{6}$$

The shapes presented in Figures 1, 2 and 3 are shapes of PDF, CDF and hazard function of $MG_{Exp}(\epsilon)$-distribution respectively.

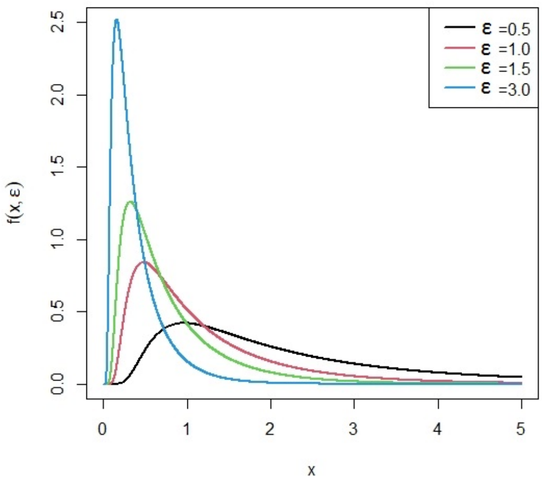

Figure 1: The plot of PDF for the various choices of value of the parameter ϵ of $MG_{Exp}(\epsilon)$-distribution.

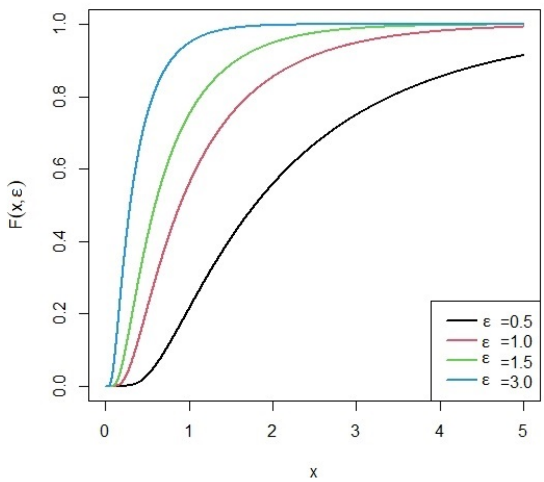

Figure 2: The plot of CDF for the various choices of value of the parameter ϵ of $MG_{Exp}(\epsilon)$-distribution.

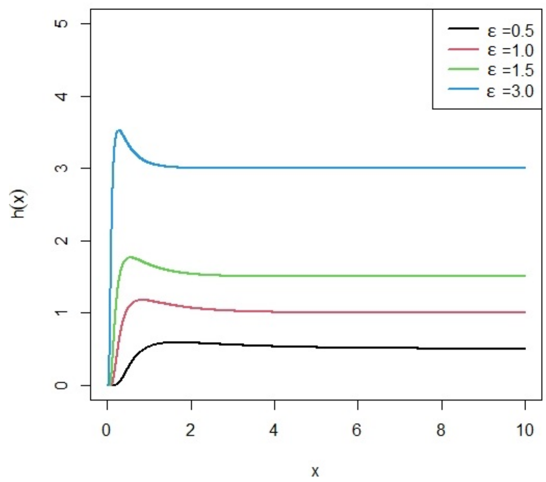

Figure 3: The plot of hazard rate function for the various choices of value of the parameter ϵ $MG_{Exp}(\epsilon)$-distribution.

Figure 3 explores the nature of hazard function of the $MG_{Exp}(\epsilon)$-distribution is increasing and inverted bathtub hazard rate function while the baseline (exponential distribution) has constant hazard rate function this is the delineation and parsimony of proposed distribution in comparison to baseline distribution.

In summary, the novelty of this work lies in its investigation of how parameter estimates for the $MG_{Exp}(\epsilon)$-distribution behave when applied to progressively Type-II censored samples for long-term use. The study likely involves statistical analysis, modeling, and possibly simulations to gain insights into the behavior of these parameter estimates in practical scenarios.

2 Statistical Properties

In this section, we discussed some statistical properties of $MG_{Exp}(\epsilon)$-distribution which have not derived yet. First we discuss about two lemma which are given below

Lemma 1
Let

$$\begin{aligned}\xi_1(\epsilon, r, \delta) &= \int_0^\infty x^r e^{-\delta x} \times \frac{\exp\left(-\frac{e^{-\epsilon x}}{1-e^{-\epsilon x}}\right)}{(1-e^{-\epsilon x})^2} dx \\ &= \sum_{i=0}^{\infty}\sum_{l=0}^{\infty} \frac{(-1)^i}{i!}\binom{-i-2}{l} * \frac{r!}{\epsilon^{r+1}} * \frac{1}{(i+l+1)^{r+1}}\end{aligned}$$

Proof:

$$
\begin{aligned}
\xi_1(\epsilon,r,\delta) &= \int_0^\infty x^r e^{-\delta x}(1-e^{-\epsilon x})^{-2} \times \exp\left(-\frac{e^{-\epsilon x}}{1-e^{-\epsilon x}}\right)dx \\
&= \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \int_0^\infty x^r e^{-\epsilon(1+i)x} \times (1-e^{-\epsilon x})^{-(i+2)}dx \\
&= \sum_{i=0}^{\infty}\sum_{l=0}^{\infty} \frac{(-1)^i}{i!} \binom{-i-2}{l} \int_0^\infty x^r \exp\left\{-\epsilon(i+l+1)x\right\} dx \\
\implies \xi_1(\epsilon,r,\delta) &= \sum_{i=0}^{\infty}\sum_{l=0}^{\infty} \frac{(-1)^i}{i!} \binom{-i-2}{l} \times \frac{r!}{\epsilon^{r+1}} \times \frac{1}{(i+l+1)^{r+1}}
\end{aligned}
$$

The r^{th} moment about point 0 of $MG_{Exp}(\epsilon)$-distribution is

$$
E(X^r) = \epsilon \times \xi_1(\epsilon,r,\epsilon) \tag{7}
$$

we obtain the first four moments about 0 on putting $r = 1,2,3,4$ of $MG_{Exp}(\epsilon)$-distribution are

$$
\begin{aligned}
E(X) &= \epsilon * \xi_1(\epsilon,1,\epsilon) \\
E(X^2) &= \epsilon * \xi_1(\epsilon,2,\epsilon) \\
E(X^3) &= \epsilon * \xi_1(\epsilon,3,\epsilon) \\
E(X^4) &= \epsilon * \xi_1(\epsilon,4,\epsilon)
\end{aligned}
$$

Lemma 2
Let

$$
\begin{aligned}
\xi_2(\epsilon,r,\delta,t) &= \int_t^\infty x^r e^{-\delta x}(1-e^{-\epsilon x})^{-2} * \exp\left(-\frac{e^{-\epsilon x}}{1-e^{-\epsilon x}}\right)dx \\
&= \sum_{i=0}^{\infty}\sum_{l=0}^{\infty}\sum_{p=0}^{r+1} \frac{r!}{p!\left[\epsilon(i+l+1)\right]^{r+1}} * e^{-(\epsilon(i+l+1)t)} \left\{\epsilon(i+l+1)t\right\}^p
\end{aligned}
$$

Proof:

$$
\begin{aligned}
\xi_2(\epsilon,r,\delta,t) &= \int_t^\infty x^r e^{-\delta x}(1-e^{-\epsilon x})^{-2} * \exp\left(-\frac{e^{-\epsilon x}}{1-e^{-\epsilon x}}\right)dx \\
&= \sum_{i=0}^{\infty} \int_t^\infty \frac{(-1)^i}{i!} x^r e^{-\epsilon i x} e^{-\epsilon x}(1-e^{-\epsilon x})^{-(i+2)}dx \\
&= \sum_{i=0}^{\infty}\sum_{l=0}^{\infty} \frac{(-1)^i}{i!} \binom{-i-2}{l} \int_t^\infty x^r e^{(-\epsilon(i+l+1)x)}dx \\
&= \sum_{i=0}^{\infty}\sum_{l=0}^{\infty}\sum_{p=0}^{r+1} \frac{(-1)^i}{i!} \binom{-i-2}{l} \frac{r!}{p!(\epsilon(i+l+1))^{r+1}} * e^{-(\epsilon(i+l+1)t)} \left\{\epsilon(i+l+1)t\right\}^p
\end{aligned}
$$

2.1 Conditional Moment

Conditional moment of r^{th} order is denoted by $E(X^r|X > r)$ and obtained by using lemma 2,

$$E(X^r|X > x) = \epsilon * \xi_2(\epsilon, r, \epsilon, x) \tag{8}$$

2.2 Median

Let M be the median of X and is obtained by solving the followings

$$\int_0^M f(x)dx = \int_M^\infty f(x)dx = \frac{1}{2}$$

then by equation (3), we get

$$\begin{aligned}
&\int_0^M f(x)dx = \frac{1}{2} \\
\implies &\exp\left[-\frac{e^{-\epsilon M}}{1-e^{-\epsilon M}}\right] = \frac{1}{2} \\
\implies &\frac{e^{-\epsilon M}}{1-e^{-\epsilon M}} = ln2 \\
\implies &(1-ln2)e^{-\epsilon x} = ln2 \\
\implies &e^{-\epsilon x} = \frac{ln2}{1-ln2} \\
\implies &M = -\frac{1}{\epsilon}ln\left\{\frac{ln2}{1+ln2}\right\}
\end{aligned}$$

2.3 Mean deviation about mean and median

The mean deviation about mean is defined as,

$$\eta_1(x) = \int_0^\infty |x-\mu|f(x)dx$$

where μ is mean of the random variable of X of $MG_{Exp}(\epsilon)$-distribution.

$$\begin{aligned}
&= \int_0^\mu (\mu - x)f(x)dx + \int_\mu^\infty (x-\mu)f(x)dx \\
&= 2\mu * F(\mu) - 2\mu + 2\int_\mu^\infty xf(x)dx
\end{aligned}$$

where F(.) be the CDF of $MG_{Exp}(\epsilon)$-distribution, then from lemma 2

$$\int_\mu^\infty xf(x)dx = \epsilon * \xi_2(\epsilon, 1, \epsilon, \mu)$$

Thus,

$$\eta_1(x) = 2\mu * F(\mu) - 2\mu + 2\epsilon * \xi_2(\epsilon, 1, \epsilon, \mu) \tag{9}$$

And mean deviation about median is defined as,

$$\begin{aligned}\eta_2(x) &= \int_0^{\infty} |x - M| f(x)dx \\ &= \int_0^{M} (M - x)f(x)dx + \int_M^{\infty} (x - M)f(x)dx \\ &= -\mu + 2\int_M^{\infty} xf(x)dx\end{aligned}$$

using lemma 2,

$$\int_M^{\infty} xf(x)dx = \epsilon * \xi_2(\epsilon, 1, \epsilon, M)$$

Thus,

$$\eta_2(x) = -\mu + 2\epsilon * \xi_2(\epsilon, 1, \epsilon, M) \tag{10}$$

2.4 Moment generating function and Characteristic function

The $MG_{Exp}(\epsilon)$-distribution's moment generating function (MGF) is

$$M_X(s) = E(e^{sX})$$

then by lemma 1, we get

$$M_X(s) = \epsilon * \xi_1(\epsilon, 0, \epsilon - s) \qquad ; s < \epsilon$$

and characteristic function of $MG_{Exp}(\epsilon)$-distribution can be obtained in a similar way

$$\phi_X(s) = \epsilon * \xi_1(\epsilon, 0, \epsilon - is)$$

3 Order Statistics

Let us choose random sample of size n from the $MG_{Exp}(\epsilon)$-distribution and corresponding order statistics is $X_{(1)} < X_{(2)} < \ldots < X_{(r)}$, then PDF of r^{th} order statistics is

$$\begin{aligned}f_r(x) &= \frac{n!}{(r-1)!(n-r)!} * F^{r-1}(x) * [1 - F(x)]^{n-r} * f(x) \\ &= \frac{n!}{(r-1)!(n-r)!} * \sum_{i=0}^{n-r} (-1)^i \binom{n-r}{i} F^{r+i+1}(x)f(x)\end{aligned} \tag{11}$$

Now using (3) & (4) in (11) we have,

$$f_r(x) = \frac{n!}{(r-1)!(n-r)!} * \epsilon \sum_{i=0}^{n-r} (-1)^i \binom{n-r}{i} \left\{ \exp\left(-\frac{e^{-\epsilon x}}{1 - e^{-\epsilon x}} \right) \right\}^{r+i+1} \left[\frac{e^{-\epsilon x} e^{-\frac{e^{-\epsilon x}}{1 - e^{-\epsilon x}}}}{(1 - e^{-\epsilon x})^2} \right]$$

$$\implies f_r(x) = \frac{n!}{(r-1)!(n-r)!}\epsilon \sum_{i=0}^{n-r}(-1)^i \binom{n-r}{i} * \frac{\exp\left\{-(r+i)\frac{e^{-\epsilon x}}{1-e^{-\epsilon x}} - \epsilon x\right\}}{(1-e^{-\epsilon x})^2} \tag{12}$$

and corresponding CDF of r^{th} order statistics is

$$F_r(x) = \sum_{i=r}^{n}\binom{n}{i}F^i(x) * [1-F(x)]^{n-i} = \sum_{i=r}^{n}\sum_{j=0}^{n-i}\binom{n}{i}\binom{n-i}{j}(-1)^j F^{i+j}(x) \tag{13}$$

Using (4) in (13), we get CDF of r^{th} order statistic

$$F_r(x) = \sum_{i=r}^{n}\sum_{j=0}^{n-i}\binom{n}{i}\binom{n-i}{j}(-1)^j e\left\{(i+j)\frac{e^{-\epsilon x}}{1-e^{-\epsilon x}}\right\} \tag{14}$$

4 Entropy count

Entropy is a count of average amount of information vested in r.v. X also it is the count of uncertainty of the distribution. Proposition of Renyi entropy by Rényi (1961) having the form,

$$R_\gamma = \frac{1}{1-\gamma}\ln\left(\int f^\gamma(x)dx\right) \qquad ;\gamma \neq 1 \tag{15}$$

Now

$$\begin{aligned}\int_0^\infty f^\gamma(x)dx &= \int_0^\infty \left[\epsilon e^{-\epsilon x}(1-e^{-\epsilon x})^{-2} * \exp\left(-\frac{e^{-\epsilon x}}{1-e^{-\epsilon x}}\right)\right]^\gamma dx \\ &= \epsilon^\gamma \sum_{i=0}^{\infty}\frac{(-1)^i\gamma^i}{i!} * \int_0^\infty e^{(-\epsilon(\gamma+i)x)} * (1-e^{-\epsilon x})^{-(2\gamma+1)}dx \\ &= \epsilon^\gamma \sum_{i=0}^{\infty}\sum_{l=0}^{\infty}\frac{(-1)^i\gamma^i}{i!}\binom{-2\gamma-i}{l}\int_0^\infty e^{(-\epsilon(\gamma+i+l)x)}dx \\ &= \epsilon^\gamma \sum_{i=0}^{\infty}\sum_{l=0}^{\infty}\frac{(-1)^i\gamma^i}{i!}\binom{-2\gamma-i}{l} * \frac{1}{\epsilon(\gamma+i+l)}\end{aligned}$$

using (15), we get required expression of Renyi entropy is

$$R_\gamma = \left(\frac{\gamma}{1-\gamma}\right)\ln\epsilon + \left(\frac{1}{1-\gamma}\right)\ln\left\{\sum_{i=0}^{\infty}\sum_{l=0}^{\infty}\frac{(-\gamma)^i}{i!}\binom{-i-2\gamma}{-l} * \frac{1}{\epsilon(\gamma+l+i)}\right\} \tag{16}$$

5 Progressive Type-II Censoring

In survival analysis, estimation of the unknown characteristics of any underlying phenomenon using complete observations is required. It is, however, very tedious to obtain the complete information associated with any life testing experiments due to time and cost constraints. Therefore, the problem of censored observations might be highly thought of which results to saving time and cost of the experiments. In statistical literature, variety of censoring schemes has been introduced for getting the inferences from the ongoing life testing experiments to specific probability models. The famous censoring are Type-I and Type-II censoring for more detailed about these see Balakrishnan (2007), Kumar et al. (2022). Here we are discussing progressive Type-II censoring for the considered lifetime distribution.

Let n identical items placed on a life testing experiments at time 0 with corresponding lifetimes $X_1, X_2, \ldots, X_n$ being independently and identically distributed with PDF (3). Further, suppose that m (non-negative integer) such that $m < n$ is fixed at the beginning of the experiment (where m is the number of units to be observed completely until failure) with $R_i(\geq 0)$ removal(s) and $\sum_{i=1}^{m} R_i + m = n$. This implies that progressive censoring will occur in m failure stage as failures. When the first failure occurred, At the time of the first failure X_1, R_1 surviving units are randomly removed from $n-1$ units; at the time of the second failure X_2, R_2 surviving units are randomly removed from $n-2-R_1$ units; similarly, $(m-1)^{th}$ failure units are randomly removed from $n-2-R_1$ units. monetary failure X_{m-1}, R_{m-1} units are removed at random from $n-(m-1)-\sum_{i=1}^{m-2} X_i$ and finally m^{th} unit fails X_m, followed by all remaining $(R_m = n-m-R_1-R_2-\ldots-R_{m-1})$ units. The set of an observed lifetime data $\underline{X} = (X_1, X_2, \ldots, X_m)$ is a progressive Type-II censored sample. According to Balakrishnan and Aggarwala (2000) a progressive Type-II censoring scheme consists of m failure stages and $R_1, R_2, \ldots, R_m$ random samples such that $n-m = \sum_{i=1}^{m} R_i$ with $R_i s$ fixed before the study, where R_i denotes the i^{th} censored sample. Progressive type-II censoring reduces to complete sample if $R_1 = R_2 = \ldots, R_m = 0$ and $m = n$.
The likelihood function based on the progressive Type-II censored sample
$\underline{X} = (X_1, X_2, \ldots, X_m)$ is given as

$$L(\underline{X}, \epsilon) = C \prod_{i=1}^{m} f(x_i, \epsilon) * [1 - F(x_i, \epsilon)]^{R_i} \tag{17}$$

where $C = n(n-1-R_1)(n-2-R_2)\ldots(n-m+1-\sum_{i=1}^{m-1} R_i)$.

Using (4) and (3) in (17), we get the likelihood function under progressive Type-II censored sample of $MG_{Exp}(\epsilon)$-distribution is

$$\begin{aligned} L(\underline{X}, \epsilon) &= \\ &= C \prod_{i=1}^{m} \epsilon e^{-\epsilon x_i} (1 - e^{-\epsilon x_i})^{-2} \exp\left(-\frac{e^{-\epsilon x_i}}{1 - e^{-\epsilon x_i}}\right) * \left[1 - \exp\left(-\frac{e^{-\epsilon x_i}}{1 - e^{-\epsilon x_i}}\right)\right]^{R_i} \\ &= C\epsilon^m * \exp\left[-\sum_{i=1}^{m}\left(\epsilon x_i + \frac{e^{-\epsilon x_i}}{1 - e^{-\epsilon x_i}}\right)\right] \prod_{i=1}^{m} (1 - e^{-\epsilon x_i})^{-2} * \left[1 - \exp\left(-\frac{e^{-\epsilon x_i}}{1 - e^{-\epsilon x_i}}\right)\right]^{R_i} \end{aligned} \tag{18}$$

Different methods of parametric inferences using progressive Type-II censoring are available in statistical literature. Lin et al. (2006) introduced the inferential procedure for log-gamma distribution using progressive Type-II censored observations. Balakrishnan and Hossain (2007) proposed the inference for the Type-II generalized logistic distribution under progressive Type-II censoring. The two parameters bathtub shaped lifetime distribution has been discussed using progressive Type-II censored data by Wu (2008). Kundu and Pradhan (2009) have been described the inferences for generalized exponential distribution using progressive Type-II censoring scheme. Krishna and Malik (2012) investigated the reliability estimation for Maxwell distributions with progressively Type-II censored data. Reliability estimation for Lindley and generalized exponential distributions has been discussed by Krishna and Kumar (2013). Rastogi et al. (2012) described the classical and Bayesian inference for a bathtub shaped distribution under progressive Type-II censoring. Recently, Sen et al. (2018) proposed the estimation procedures for xgamma distribution based on progressive Type-II censoring scheme. Also, Bayesian reliability estimation for Topp-Leone distribution under progressively Type-II censored data discussed by Feroze et al. (2021), Statistical analysis of Gompertz distribution based on progressively type-II censored competing risk model with binomial removals by Boulkeroua et al. (2022) and many more.

6 Bayesian Estimation

Here, we have considered estimation of the parameter ϵ of $MG_{Exp}(\epsilon)$-distribution in Bayesian paradigm only. In Bayesian paradigm, posterior probability is an effect of two components with a prior probability and n likelihood function, calculated from the statistical model for the observed data. The prior distribution of the parameters is assumed before the data observed. There is different kind of the prior distribution of parameters defined as proper and improper prior. Another way to define the priors are based on available advanced information and known as informative and non-informative prior. Here, we use an informative prior as a $G(a, b)$ prior for ϵ of the parameter ϵ of $MG_{Exp}(\epsilon)$-distribution is

$$\pi(\epsilon, a, b) = \frac{b^a}{\Gamma a}\epsilon^{a-1}e^{-b\epsilon} \quad ; a, b > 0, \epsilon > 0 \tag{19}$$

where, a, b representing hyper-parameters.

These can be obtained, if any two separate not associated information on ϵ are available, say prior mean & prior variance are known for more details see Singh et al. (2013). The mean & variance of the prior distribution (19) are $M = \frac{a}{b}$ & $V = \frac{a}{b^2}$ respectively. Thus, we take $M = \frac{a}{b}$ and $V = \frac{a}{b^2}$ giving $b = \frac{M}{V}$ and $a = \frac{M^2}{V}$. The informative gamma prior behaves like non-informative prior if the hyper-parameters are changes i.e. we fixed prior mean M and large prior variance V then gamma prior works as non-informative prior. For more applications for the use of gamma prior see Singh et al. (2013), Kumar et al. (15 a), Kumar et al. (15 b), Kumar et al. (2019), Kumar et al. (2020) and Kumar et al. (2021)

Observing the progressively Type-II censored sample data and using the likelihood function(18) and prior distribution (19) then the posterior distribution is given by

$$h(\epsilon|\underline{X}) = \frac{L(\underline{X},\epsilon)\pi(\epsilon,a,b)}{\int_0^\infty L(\underline{X},\epsilon)\pi(\epsilon,a,b)d\epsilon}$$

$$= \frac{\epsilon^{m+a-1}e^{-(b+\sum_{i=1}^m x_i)\epsilon}e^{-\sum_{i=1}^m\left(\frac{e^{-\epsilon x_i}}{1-e^{-\epsilon x_i}}\right)}\prod_{i=1}^m \frac{\left(1-e^{-\frac{e^{-\epsilon x_i}}{1-e^{-\epsilon x_i}}}\right)^{R_i}}{(1-e^{-\epsilon x_i})^2}}{\int_0^\infty \epsilon^{m+a-1}e^{-(b+\sum_{i=1}^m x_i)\epsilon}e^{-\sum_{i=1}^m\frac{e^{-\epsilon x_i}}{1-e^{-\epsilon x_i}}}\prod_{i=1}^m \frac{\left(1-e^{-\frac{e^{-\epsilon x_i}}{1-e^{-\epsilon x_i}}}\right)^{R_i}}{(1-e^{-\epsilon x_i})^2}d\epsilon}$$

$$\implies h(\epsilon|\underline{X}) = \frac{A(\epsilon)\phi(\epsilon,\underline{X})\xi(\epsilon,\underline{X},R)}{\int_0^\infty A(\epsilon)\phi(\epsilon,\underline{X})\xi(\epsilon,\underline{X},R)d\epsilon} \tag{20}$$

where $A(\epsilon) = \epsilon^{m+a-1}$

$\phi(\epsilon,\underline{X}) = e^{-(b+\sum_{i=1}^m x_i)\epsilon}e^{-\sum_{i=1}^m\frac{e^{-\epsilon x_i}}{1-e^{-\epsilon x_i}}}$

$\xi(\epsilon,\underline{X},R) = \prod_{i=1}^m \frac{\left(1-e^{-\frac{e^{-\epsilon x_i}}{1-e^{-\epsilon x_i}}}\right)^{R_i}}{(1-e^{-\epsilon x_i})^2}$; $R = (R_1, R_2, ..., R_m)$.

7 Bayes Estimators and Posterior Risks under different Loss Functions

In decision theory, the loss criterion is specified in order to obtain the best estimator for which Bayes risk is minimum or minimum posterior risks corresponding to respective loss functions. Several authors have been used this criterion to know the best Bayes estimator of the parameters corresponding to the loss function which have minimum posterior risks see Rahman et al. (2013), Ali Kazmi et al. (2012), Ali et al. (2013) and Kumar et al. (2020). The simplest form of loss function is squared error loss function (SELF), which is suitable when over estimation and under estimation are of same magnitudes has equal importance. However, in most of the real situations, this assumption is not possible. Sometimes, over estimation is more serious than under estimation and vice-versa. Here, we have consider six loss functions weighted square error loss function (WSELF), square error loss function (SELF), precautionary loss function (PLF), modified (quadratic) squared error loss function (M/Q SELF), logarithmic loss function (LLF) and exponentiated square error loss function (ESELF). The first five loss functions have been considered by Ali et al. (2013) and they have checked the performance of Bayes estimators (having smallest posterior risks of Bayes estimators of respective loss functions) of ϵ of Lindley distribution and ESELF introduced by Kumar et al. (2020) which is asymmetric loss function (over estimation is more serious than under estimation).

• `Square error loss function (SELF):`

The SELF was proposed by Legendre (1805) in the development of least-square theory and is defined as

$$L_S(\hat{\epsilon}, \epsilon) = (\hat{\epsilon} - \epsilon)^2 \tag{21}$$

corresponding Bayes estimator is posterior mean and is

$$\begin{aligned}\hat{\epsilon}_S &= E_\epsilon(\epsilon|\underline{X}) \\ &= \frac{\int_0^\infty \epsilon A(\epsilon)\phi(\epsilon, \underline{X})\xi(\epsilon, \underline{X}, R)d\epsilon}{\int_0^\infty A(\epsilon)\phi(\epsilon, \underline{X})\xi(\epsilon, \underline{X}, R)d\epsilon}\end{aligned} \tag{22}$$

Posterior risk of $\hat{\epsilon}_S$ is

$$\begin{aligned}R_S(\hat{\epsilon}_S, \epsilon) &= E(\epsilon^2|\underline{X}) - [E(\epsilon|\underline{X})]^2 \\ &= \frac{\int_0^\infty \epsilon^2 A(\epsilon)\phi(\epsilon, \underline{X})\xi(\epsilon, \underline{X}, R)d\epsilon}{\int_0^\infty A(\epsilon)\phi(\epsilon, \underline{X})\xi(\epsilon, \underline{X}, R)d\epsilon} - \left(\frac{\int_0^\infty \epsilon^2 A(\epsilon)\phi(\epsilon, \underline{X})\xi(\epsilon, \underline{X}, R)d\epsilon}{\int_0^\infty A(\epsilon)\phi(\epsilon, \underline{X})\xi(\epsilon, \underline{X}, R)d\epsilon}\right)^2\end{aligned} \tag{23}$$

• `Weighted SELF:`

The weighted loss function is defined as

$$L_W(\hat{\epsilon}, \epsilon) = \frac{(\hat{\epsilon} - \epsilon)^2}{\epsilon} \tag{24}$$

Bayes estimator of parameter ϵ is harmonic mean of the posterior density and is

$$\hat{\epsilon}_W = \left(E_\epsilon(\epsilon^{-1}|\underline{X})\right)^{-1} = \left[\frac{\int_0^\infty \epsilon^{-1} A(\epsilon)\phi(\epsilon, \underline{X})\xi(\epsilon, \underline{X}, R)d\epsilon}{\int_0^\infty A(\epsilon)\phi(\epsilon, \underline{X})\xi(\epsilon, \underline{X}, R)d\epsilon}\right]^{-1} \tag{25}$$

Posterior risk of $\hat{\epsilon}_W$ of the parameter ϵ is the difference of mean and harmonic mean of the posterior density and is

$$\begin{aligned}R_W(\hat{\epsilon}_W, \epsilon) &= E_\epsilon(\epsilon|\underline{X}) - \left(E_\epsilon(\epsilon^{-1}|\underline{X})\right)^{-1} \\ &= \frac{\int_0^\infty \epsilon A(\epsilon)\phi(\epsilon, \underline{X})\xi(\epsilon, \underline{X}, R)d\epsilon}{\int_0^\infty A(\epsilon)\phi(\epsilon, \underline{X})\xi(\epsilon, \underline{X}, R)d\epsilon} - \left[\frac{\int_0^\infty \epsilon^{-1} A(\epsilon)\phi(\epsilon, \underline{X})\xi(\epsilon, \underline{X}, R)d\epsilon}{\int_0^\infty A(\epsilon)\phi(\epsilon, \underline{X})\xi(\epsilon, \underline{X}, R)d\epsilon}\right]^{-1}\end{aligned} \tag{26}$$

• `Modified/Quadratic square error loss function (M/Q SELF):`

The M/Q SELF is defined as

$$L_W(\hat{\epsilon}, \epsilon) = \left(\frac{\hat{\epsilon}}{\epsilon} - 1\right)^2 \tag{27}$$

Bayes estimator of ϵ is

$$\hat{\epsilon}_M = \frac{E(\epsilon^{-1}|\underline{X})}{E(\epsilon^{-2}|\underline{X})} = \frac{\int_0^\infty \epsilon^{-1} A(\epsilon)\phi(\epsilon, \underline{X})\xi(\epsilon, \underline{X}, R)d\epsilon}{\int_0^\infty \epsilon^{-2} A(\epsilon)\phi(\epsilon, \underline{X})\xi(\epsilon, \underline{X}, R)d\epsilon} \tag{28}$$

posterior risk of the Bayes estimator of ϵ is

$$\begin{aligned} R_M(\hat{\epsilon}_M, \epsilon) &= 1 - \frac{[E(\epsilon^{-1}|\underline{X})]^2}{E(\epsilon^{-2}|\underline{X})} \\ &= 1 - \frac{\left(\int_0^\infty \epsilon^{-1} A(\epsilon)\phi(\epsilon, \underline{X})\xi(\epsilon, \underline{X}, R)d\epsilon\right)^2 \int_0^\infty \epsilon^{-2} A(\epsilon)\phi(\epsilon, \underline{X})\xi(\epsilon, \underline{X}, R)d\epsilon}{\int_0^\infty A(\epsilon)\phi(\epsilon, \underline{X})\xi(\epsilon, \underline{X}, R)d\epsilon} \end{aligned} \quad (29)$$

• `Precautionary loss function (PLF):`

Precautionary loss function (PLF) introduced by Norstrom (1996) and is given by

$$L_P(\hat{\epsilon}, \epsilon) = \frac{(\hat{\epsilon} - \epsilon)^2}{\hat{\epsilon}} \quad (30)$$

Bayes estimator of ϵ is

$$\hat{\epsilon}_P = \sqrt{E_\epsilon(\epsilon^2|\underline{X})} = \sqrt{\frac{\int_0^\infty \epsilon^2 A(\epsilon)\phi(\epsilon, \underline{X})\xi(\epsilon, \underline{X}, R)d\epsilon}{\int_0^\infty A(\epsilon)\phi(\epsilon, \underline{X})\xi(\epsilon, \underline{X}, R)d\epsilon}} \quad (31)$$

posterior risk of the Bayes estimator $\hat{\epsilon}_P$ of the parameter ϵ is

$$\begin{aligned} R_P(\hat{\epsilon}_P, \epsilon) &= 2\left[\sqrt{E_\epsilon(\epsilon^2|\underline{X})} - E_\epsilon(\epsilon|\underline{X})\right] \\ &= 2\left[\sqrt{\frac{\int_0^\infty \epsilon^2 A(\epsilon)\phi(\epsilon, \underline{X})\xi(\epsilon, \underline{X}, R)d\epsilon}{\int_0^\infty A(\epsilon)\phi(\epsilon, \underline{X})\xi(\epsilon, \underline{X}, R)d\epsilon}} - \frac{\int_0^\infty \epsilon A(\epsilon)\phi(\epsilon, \underline{X})\xi(\epsilon, \underline{X}, R)d\epsilon}{\int_0^\infty A(\epsilon)\phi(\epsilon, \underline{X})\xi(\epsilon, \underline{X}, R)d\epsilon}\right] \end{aligned} \quad (32)$$

• `Logarithmic loss function (LLF):`

Logarithmic loss function (LLF) is defined as

$$L_L(\hat{\epsilon}, \epsilon) = (\ln \hat{\epsilon} - \ln \epsilon)^2 \quad (33)$$

The Bayes estimator of ϵ is the geometric mean of posterior density and is

$$\hat{\epsilon}_L = e^{E(\ln \epsilon|\underline{X})} = \exp\left(\frac{\int_0^\infty \ln \epsilon A(\epsilon)\phi(\epsilon, \underline{X})\xi(\epsilon, \underline{X}, R)d\epsilon}{\int_0^\infty A(\epsilon)\phi(\epsilon, \underline{X})\xi(\epsilon, \underline{X}, R)d\epsilon}\right) \quad (34)$$

posterior risk of the Bayes estimator $\hat{\epsilon}_L$ of the parameter ϵ is

$$\begin{aligned} R_L(\hat{\epsilon}_L, \epsilon) &= Var(\ln \epsilon) \\ &= \frac{\int_0^\infty (\ln \epsilon)^2 A(\epsilon)\phi(\epsilon, \underline{X})\xi(\epsilon, \underline{X}, R)d\epsilon}{\int_0^\infty A(\epsilon)\phi(\epsilon, \underline{X})\xi(\epsilon, \underline{X}, R)d\epsilon} - \left(\frac{\int_0^\infty \ln \epsilon A(\epsilon)\phi(\epsilon, \underline{X})\xi(\epsilon, \underline{X}, R)d\epsilon}{\int_0^\infty A(\epsilon)\phi(\epsilon, \underline{X})\xi(\epsilon, \underline{X}, R)d\epsilon}\right)^2 \end{aligned} \quad (35)$$

• `Exponentiated Square error loss function (ESELF):`

The ESELF proposed by Kumar et al. (2020) and is defined as

$$L_E(\hat{\epsilon}, \epsilon) = \left(e^{-\hat{\epsilon}} - e^{-\epsilon}\right)^2 \tag{36}$$

Bayes estimator of parameter ϵ is given as

$$\hat{\epsilon}_E = -\ln\left(E(e^{-\epsilon}|\underline{X})\right) = -\ln\left(\frac{\int_0^\infty e^{-\epsilon} A(\epsilon)\phi(\epsilon, \underline{X})\xi(\epsilon, \underline{X}, R)d\epsilon}{\int_0^\infty A(\epsilon)\phi(\epsilon, \underline{X})\xi(\epsilon, \underline{X}, R)d\epsilon}\right) \tag{37}$$

and corresponding posterior risk is

$$\begin{aligned} R_E(\hat{\epsilon}_E, \epsilon) &= Var\left(e^{-\epsilon}|\underline{X}\right) \\ &= \frac{\int_0^\infty e^{-2\epsilon} A(\epsilon)\phi(\epsilon, \underline{X})\xi(\epsilon, \underline{X}, R)d\epsilon}{\int_0^\infty A(\epsilon)\phi(\epsilon, \underline{X})\xi(\epsilon, \underline{X}, R)d\epsilon} - \left(\frac{\int_0^\infty e^{-\epsilon} A(\epsilon)\phi(\epsilon, \underline{X})\xi(\epsilon, \underline{X}, R)d\epsilon}{\int_0^\infty A(\epsilon)\phi(\epsilon, \underline{X})\xi(\epsilon, \underline{X}, R)d\epsilon}\right)^2 \end{aligned} \tag{38}$$

8 Simulation Study

In this section, we have compared the considered Bayes estimators $\hat{\delta}_S$, $\hat{\delta}_W$, $\hat{\delta}_M$, $\hat{\delta}_P$, $\hat{\delta}_L$, $\hat{\delta}_E$ of the parameter ϵ of $MG_{Exp}(\epsilon)$-distribution in terms of posterior risks. It is clear that, the posterior risks are the function of m, the effective sample size, number of removals R_i, hyper-parameters a and b of the gamma prior. We have generated 10,000 sample from $MG_{Exp}(\epsilon)$-distribution for convergence of the results. The simulations were carried out for sample sizes $n = 15, 20, 30, 50$ for different choices of the effective sample size m with $m = \frac{60}{100}n = 60\%$ of n, $m = \frac{80}{100}n = 80\%$ of n.
We consider the removals with following five different schemes.

`Scheme 1:`
$R_1 = n - m$, $R_i = 0$; for $i \neq 1$.
`Scheme 2:`
$R_{\frac{m+1}{2}} = n - m$, $R_i = 0; i \neq \frac{m+1}{2}$, if m is odd and $R_{\frac{m}{2}} = n - m$, $R_i = 0; i \neq \frac{m}{2}$ if m is even.
`Scheme 3:`
$R_i = 0$; for $i \neq m$, $R_m = n - m$ this reduces to Type-II censoring.
`Scheme 4:`
$R_1 = \frac{n-m+1}{2}$, $R_m = \frac{n-m+1}{2}$ and $R_i = 0$ for $i \neq 1, m$ if $n - m$ is odd
and $R_1 = \frac{n-m}{2}$ $R_m = \frac{n-m}{2}$ and $R_i = 0$ for $i \neq 1, m$ if $n - m$ is even.
`Scheme 5:`
$R_1 = 1$, $R_{\frac{m+1}{2}} = n - m - 2$, $R_m = 1$ and $R_i = 0$ for $i \neq 1, \frac{m+1}{2}, m$ if m is odd
and $R_1 = 1$, $R_{\frac{m}{2}} = n - m - 2$, $R_m = 1$ and $R_i = 0$ for $i \neq 1, \frac{m}{2}, m$ if m is even.

The choice of hyper-parameters a and b is obtained by the relation $a = \frac{M^2}{V}$ and $b = \frac{M}{V}$

where M and V are the mean and variance of the prior distribution of ϵ. For detailed discussion see Singh et al. (2013), Kumar et al. (15 a), Kumar et al. (2019), Kumar et al. (2021). Here, we have considered the prior mean $M(M = 1.5, 2)$, prior variance $V(V = 0.1, 0.5, 2, 5)$ and the true values of the parameter ϵ are taken as 0.5, 1.5, 2, 5.

Table 1 represents the posterior risks of the Bayes estimators under considered loss functions for prior mean $M = 1.5$ and prior variance $V = 0.5$ with true value $\epsilon = 0.5$ for different sample sizes n and effective sample sizes m. We found that, the Bayes estimator $\hat{\epsilon}_E$ having the posterior risk $R_E(\hat{\epsilon}_E, \epsilon)$ minimum in all other Bayes estimators of considered loss functions. It is also noted that, as sample information increases posterior risks decreases. Similar patterns are also found in Table 2 and 3 for the true values of parameter $\epsilon = 1.5$ and 5 respectively.

Table 4 represents the posterior risks of the Bayes estimators under considered loss functions for $n = 20$ and effective sample size $m = 12$ for varying confidence level. We observed that as confidence level decreases posterior risks increases. And the Bayes estimator $\hat{\epsilon}_E$ under ESELF performs better than other Bayes estimators $\hat{\epsilon}_S$, $\hat{\epsilon}_W$, $\hat{\epsilon}_M$, $\hat{\epsilon}_P$, $\hat{\epsilon}_L$ under considered loss functions (SELF, WSELF, MSELF, PLF and LLF) in the sense of having smallest posterior risks. Similar patterns are also obtained for $m = 16$ and reported in Table 5.

9 Conclusion

In this chapter, we have considered a lifetime distribution and naming as minimum guarantee exponential distribution. We have derived some important statistical properties of this distribution which are not obtained by the host authors and any other authors yet like r^{th} moments about origin, first four raw moments, r^{th} conditional moments, median, mean deviation about mean and median, order statistics and Renyi entropy. We have given the procedure for estimation of parameter of $MG_{EXP}(\epsilon)$-distribution under Bayesian setup for progressive Type-II censored sample. We have also done simulation study to experience the nature of the estimators at long run. The works of the Bayes estimators for respective loss functions of the parameter ϵ in terms of minimum posterior risks of the Bayes estimators of ϵ. We found that, as prior variance increases posterior risks also increases. In over all considered situations, scheme 1 is better than other considered schemes and the Bayes estimator under ESELF performs better than other Bayes estimators under considered loss functions (SELF, WSELF, MSELF, PLF and LLF) in terms of having smallest posterior risks. Also, as effective sample information increases posterior risks decreases.
Lastly, this present work supports the theoretical approaches and is a good contribution in research in the hope of strengthening the considered distribution by the insertion of various useful properties.

Table 1: Posterior risks of the Bayes estimators under different loss functions for $M = 1.5,\ V = 0.5$ and $\epsilon = 0.5$ with varying n and m.

n	m	Scheme	$R_S(\hat{\epsilon}_S, \epsilon)$	$R_W(\hat{\epsilon}_W, \epsilon)$	$R_M(\hat{\epsilon}_M, \epsilon)$	$R_P(\hat{\epsilon}_P, \epsilon)$	$R_L(\hat{\epsilon}_L, \epsilon)$	$R_E(\hat{\epsilon}_E, \epsilon)$
15	9	I	0.01814	0.02765	0.04593	0.02841	0.04677	0.00478
		II	0.01825	0.02635	0.04117	0.02705	0.04178	0.00449
		III	0.01803	0.02541	0.03845	0.02603	0.03904	0.00430
		IV	0.01804	0.02593	0.04095	0.02662	0.04155	0.00443
		V	0.01808	0.02600	0.04075	0.02669	0.04135	0.00444
	12	I	0.01328	0.02139	0.03721	0.02187	0.03768	0.00375
		II	0.01334	0.02081	0.03502	0.02130	0.03538	0.00363
		III	0.01305	0.02004	0.03314	0.02057	0.03358	0.00349
		IV	0.01308	0.02074	0.03569	0.02118	0.03603	0.00363
		V	0.01308	0.02054	0.03480	0.02104	0.03521	0.00359
30	18	I	0.00758	0.01267	0.02367	0.01277	0.02349	0.00224
		II	0.00757	0.01178	0.01907	0.01226	0.01957	0.00209
		III	0.00732	0.01103	0.01820	0.01160	0.01907	0.00213
		IV	0.00733	0.01145	0.01895	0.01185	0.01930	0.00203
		V	0.00749	0.01153	0.01875	0.01201	0.01924	0.00205
	24	I	0.00502	0.00920	0.01802	0.00913	0.01762	0.00162
		II	0.00505	0.00844	0.01508	0.00864	0.01510	0.00150
		III	0.00490	0.00810	0.01395	0.00837	0.01414	0.00144
		IV	0.00509	0.00841	0.01557	0.00851	0.01545	0.00149
		V	0.00491	0.00832	0.01509	0.00847	0.01503	0.00148
50	30	I	0.00316	0.00630	0.01319	0.00603	0.01254	0.00120
		II	0.00334	0.00502	0.00787	0.00546	0.00843	0.00119
		III	0.00314	0.00591	0.00709	0.00532	0.00843	0.00110
		IV	0.00320	0.00501	0.00828	0.00535	0.00858	0.00120
		V	0.00332	0.00504	0.00801	0.00541	0.00846	0.00121
	40	I	0.00202	0.00429	0.00939	0.00397	0.00870	0.00074
		II	0.00163	0.00284	0.00537	0.00285	0.00519	0.00050
		III	0.00157	0.00270	0.00470	0.00283	0.00477	0.00048
		IV	0.00163	0.00311	0.00634	0.00298	0.00594	0.00054
		V	0.00158	0.00283	0.00549	0.00286	0.00525	0.00050

Table 2: Posterior risks of the Bayes estimators under different loss functions for $M = 1.5,\ V = 0.5$ and $\epsilon = 1.5$ with varying n and m.

n	m	Scheme	$R_S(\hat{\epsilon}_S, \epsilon)$	$R_W(\hat{\epsilon}_W, \epsilon)$	$R_M(\hat{\epsilon}_M, \epsilon)$	$R_P(\hat{\epsilon}_P, \epsilon)$	$R_L(\hat{\epsilon}_L, \epsilon)$	$R_E(\hat{\epsilon}_E, \epsilon)$
15	9	I	0.10333	0.06197	0.03981	0.06375	0.04062	0.00424
		II	0.10671	0.06034	0.03627	0.06189	0.03693	0.00361
		III	0.10223	0.05886	0.03425	0.06026	0.03482	0.00328
		IV	0.10447	0.05951	0.03609	0.06103	0.03674	0.00362
		V	0.10512	0.05959	0.03591	0.06111	0.03656	0.00358
	12	I	0.08298	0.05080	0.03300	0.05210	0.03361	0.00356
		II	0.08454	0.05018	0.03151	0.05139	0.03206	0.00329
		III	0.08235	0.04887	0.03023	0.04999	0.03072	0.00311
		IV	0.08248	0.04974	0.03180	0.05097	0.03236	0.00338
		V	0.08300	0.04954	0.03131	0.05074	0.03185	0.00329
30	18	I	0.05713	0.03454	0.02254	0.03517	0.02284	0.00245
		II	0.05781	0.03350	0.02014	0.03405	0.02037	0.00201
		III	0.05565	0.03266	0.01881	0.03314	0.01901	0.00178
		IV	0.05656	0.03305	0.02005	0.03358	0.02027	0.00202
		V	0.05708	0.03315	0.01997	0.03368	0.02020	0.00199
	24	I	0.04341	0.02751	0.01803	0.02791	0.01822	0.00197
		II	0.04470	0.02715	0.01703	0.02754	0.01720	0.00178
		III	0.04304	0.02633	0.01626	0.02669	0.01640	0.00167
		IV	0.04328	0.02665	0.01696	0.02701	0.01711	0.00180
		V	0.04388	0.02682	0.01694	0.02720	0.01710	0.00178
50	30	I	0.03341	0.02153	0.01428	0.02177	0.01438	0.00158
		II	0.03611	0.02111	0.01260	0.02139	0.01275	0.00125
		III	0.03304	0.02067	0.01189	0.02082	0.01197	0.00112
		IV	0.03499	0.02073	0.01256	0.02099	0.01269	0.00127
		V	0.03575	0.02096	0.01255	0.02123	0.01269	0.00125
	40	I	0.02564	0.01654	0.01096	0.01672	0.01102	0.00121
		II	0.02711	0.01652	0.01028	0.01680	0.01042	0.00108
		III	0.02551	0.01626	0.00992	0.01653	0.01008	0.00102
		IV	0.02597	0.01631	0.01023	0.01636	0.01035	0.00109
		V	0.02665	0.01631	0.01020	0.01658	0.01033	0.00107

Table 3: Posterior risks of the Bayes estimators under different loss functions for $M = 1.5,\ V = 0.5$ and $\epsilon = 5$ with varying n and m.

n	m	Scheme	$R_S(\hat{\epsilon}_S, \epsilon)$	$R_W(\hat{\epsilon}_W, \epsilon)$	$R_M(\hat{\epsilon}_M, \epsilon)$	$R_P(\hat{\epsilon}_P, \epsilon)$	$R_L(\hat{\epsilon}_L, \epsilon)$	$R_E(\hat{\epsilon}_E, \epsilon)$
15	9	I	0.42824	0.10929	0.02929	0.11180	0.02978	0.00038
		II	0.45414	0.10811	0.02689	0.11037	0.02730	0.00024
		III	0.42607	0.10686	0.02557	0.10896	0.02593	0.00019
		IV	0.44830	0.10729	0.02686	0.10953	0.02726	0.00025
		V	0.45130	0.10738	0.02670	0.10960	0.02709	0.00024
	12	I	0.39920	0.09701	0.02500	0.09900	0.02537	0.00026
		II	0.40501	0.09657	0.02398	0.09845	0.02432	0.00021
		III	0.39577	0.09525	0.02316	0.09704	0.02347	0.00018
		IV	0.39643	0.09594	0.02421	0.09784	0.02456	0.00023
		V	0.40057	0.09581	0.02388	0.09767	0.02421	0.00021
30	18	I	0.32639	0.07603	0.01829	0.07721	0.01850	0.00012
		II	0.35014	0.07496	0.01650	0.07599	0.01667	0.00006
		III	0.32035	0.07375	0.01550	0.07468	0.01564	0.00005
		IV	0.34381	0.07412	0.01645	0.07513	0.01661	0.00007
		V	0.34731	0.07437	0.01638	0.07538	0.01654	0.00006
	24	I	0.28948	0.06482	0.01507	0.06568	0.01522	0.00008
		II	0.29772	0.06445	0.01431	0.06524	0.01444	0.00006
		III	0.28781	0.06322	0.01376	0.06396	0.01388	0.00004
		IV	0.29046	0.06357	0.01428	0.06435	0.01441	0.00006
		V	0.29401	0.06390	0.01425	0.06469	0.01438	0.00006
50	30	I	0.25585	0.05446	0.01234	0.05506	0.01244	0.00005
		II	0.26513	0.05344	0.01098	0.05394	0.01105	0.00002
		III	0.25279	0.05249	0.01029	0.05294	0.01035	0.00002
		IV	0.25893	0.05279	0.01098	0.05328	0.01105	0.00003
		V	0.26308	0.05307	0.01091	0.05356	0.01099	0.00002
	40	I	0.21846	0.04518	0.00998	0.04559	0.01004	0.00003
		II	0.21726	0.04486	0.00942	0.04524	0.00948	0.00002
		III	0.21604	0.04379	0.00903	0.04413	0.00908	0.00002
		IV	0.21716	0.04404	0.00939	0.04441	0.00945	0.00003
		V	0.21484	0.04453	0.00939	0.04490	0.00945	0.00002

Table 4: Posterior risks of the Bayes estimators under different loss functions for $n = 20$, effective sample size $m = 12$ and $M = \epsilon = 2$ with varying V.

V	Scheme	$R_S(\hat{\epsilon}_S, \epsilon)$	$R_W(\hat{\epsilon}_W, \epsilon)$	$R_M(\hat{\epsilon}_M, \epsilon)$	$R_P(\hat{\epsilon}_P, \epsilon)$	$R_L(\hat{\epsilon}_L, \epsilon)$	$R_E(\hat{\epsilon}_E, \epsilon)$
0.1	I	0.06189	0.02987	0.01494	0.03005	0.01497	0.00110
	II	0.06176	0.02922	0.01407	0.02939	0.01412	0.00097
	III	0.06106	0.02866	0.01352	0.02878	0.01356	0.00089
	IV	0.06118	0.02902	0.01402	0.02918	0.01406	0.00097
	V	0.06138	0.02906	0.01400	0.02923	0.01405	0.00096
0.5	I	0.12594	0.05838	0.02862	0.05949	0.02898	0.00211
	II	0.12861	0.05662	0.02603	0.05758	0.02632	0.00169
	III	0.12556	0.05507	0.02439	0.05594	0.02465	0.00146
	IV	0.12624	0.05585	0.02582	0.05679	0.02610	0.00169
	V	0.12721	0.05604	0.02578	0.05700	0.02607	0.00167
2	I	0.16324	0.07187	0.03438	0.07400	0.03515	0.00249
	II	0.16563	0.06979	0.03113	0.07159	0.03172	0.00195
	III	0.16201	0.06808	0.02907	0.06966	0.02956	0.00165
	IV	0.16295	0.06884	0.03082	0.07059	0.03140	0.00195
	V	0.16401	0.06906	0.03078	0.07083	0.03136	0.00193
5	I	0.17110	0.07572	0.03590	0.07821	0.03680	0.00258
	II	0.17840	0.07390	0.03251	0.07599	0.03320	0.00200
	III	0.17024	0.07182	0.03028	0.07364	0.03085	0.00169
	IV	0.17405	0.07254	0.03214	0.07456	0.03281	0.00201
	V	0.17577	0.07291	0.03210	0.07496	0.03278	0.00198

Table 5: Posterior risks of the Bayes estimators under different loss functions for $n = 20$, effective sample size $m = 12$ and $M = \epsilon = 2$ with varying V.

V	Scheme	$R_S(\hat{\epsilon}_S, \epsilon)$	$R_W(\hat{\epsilon}_W, \epsilon)$	$R_M(\hat{\epsilon}_M, \epsilon)$	$R_P(\hat{\epsilon}_P, \epsilon)$	$R_L(\hat{\epsilon}_L, \epsilon)$	$R_E(\hat{\epsilon}_E, \epsilon)$
0.1	I	0.05498	0.02696	0.01349	0.02716	0.01353	0.00100
	II	0.05550	0.02665	0.01303	0.02686	0.01309	0.00093
	III	0.05403	0.02622	0.01271	0.02640	0.01277	0.00089
	IV	0.05467	0.02643	0.01302	0.02662	0.01307	0.00094
	V	0.05498	0.02650	0.01301	0.02671	0.01307	0.00093
0.5	I	0.10182	0.04809	0.02369	0.04891	0.02397	0.00175
	II	0.10364	0.04740	0.02255	0.04816	0.02280	0.00156
	III	0.10121	0.04622	0.02164	0.04693	0.02187	0.00145
	IV	0.10155	0.04680	0.02246	0.04756	0.02271	0.00158
	V	0.10208	0.04691	0.02244	0.04767	0.02268	0.00157
2	I	0.12428	0.05698	0.02754	0.05838	0.02804	0.00201
	II	0.12810	0.05650	0.02619	0.05779	0.02664	0.00176
	III	0.12345	0.05494	0.02506	0.05612	0.02545	0.00163
	IV	0.12413	0.05542	0.02602	0.05668	0.02646	0.00179
	V	0.12567	0.05578	0.02603	0.05705	0.02647	0.00177
5	I	0.13010	0.05920	0.02850	0.06078	0.02907	0.00207
	II	0.13389	0.05867	0.02709	0.06012	0.02759	0.00182
	III	0.13001	0.05697	0.02587	0.05829	0.02632	0.00167
	IV	0.13283	0.05753	0.02689	0.05895	0.02739	0.00185
	V	0.13108	0.05782	0.02689	0.05925	0.02739	0.00183

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Conflict of Interest

On behalf of all authors, the corresponding author states that there is no conflict of interest.

References

Afify, A. Z., Nassar, M., Kumar, D., and Cordeiro, G. M. (2022). A new unit distribution: Properties, inference, and applications. *Electronic Journal of Applied Statistical Analysis*, 15(2):438–462.

Ali, S., Aslam, M., and Kazmi, S. M. A. (2013). A study of the effect of the loss function on bayes estimate, posterior risk and hazard function for Lindley distribution. *Applied Mathematical Modelling*, 37(8):6068–6078.

Ali Kazmi, S. M., Aslam, M., and Ali, S. (2012). Preference of prior for the class of life-time distributions under different loss functions. *Pakistan Journal of Statistics*, 28(4).

Balakrishnan, N. (2007). Progressive censoring methodology: an appraisal. *Test*, 16(2):211–259.

Balakrishnan, N. and Aggarwala, R. (2000). *Progressive censoring: Theory, Methods, and Applications*. Springer Science & Business Media.

Balakrishnan, N. and Hossain, A. (2007). Inference for the Type II generalized logistic distribution under progressive Type II censoring. *Journal of Statistical Computation and Simulation*, 77(12):1013–1031.

Boulkeroua, F. B., Al-Jarallah, R. A., and Raqab, M. Z. (2022). Statistical analysis of Gompertz distribution based on progressively type-ii censored competing risk model with binomial removals. *Electronic Journal of Applied Statistical Analysis*, 15(2):367–398.

Feroze, N., Aslam, M., Khan, I. H., and Khan, M. H. (2021). Bayesian reliability estimation for the Topp–Leone distribution under progressively Type-II censored samples. *Soft Computing*, 25(3):2131–2152.

Gupta, R. C., Gupta, P. L., and Gupta, R. D. (1998). Modeling failure time data by Lehman alternatives. *Communications in Statistics-Theory and methods*, 27(4):887–904.

Krishna, H. and Kumar, K. (2013). Reliability estimation in generalized inverted exponential distribution with progressively Type II censored sample. *Journal of Statistical Computation and Simulation*, 83(6):1007–1019.

Krishna, H. and Malik, M. (2012). Reliability estimation in maxwell distribution with progressively Type-II censored data. *Journal of Statistical Computation and Simulation*, 82(4):623–641.

Kumar, D., Kumar, P., Kumar, P., Singh, S. K., and Singh, U. (2021). PCM Transformation: Properties and their estimation. *Journal of Reliability and Statistical Studies*, 14(2):373–392.

Kumar, D., Kumar, P., Kumar, P., Singh, U., and Chaurasia, P. K. (2020). A new asymmetric loss function for estimation of any parameter. *International Journal of Agricultural and Statistical Sciences*, 16(2):489–501.

Kumar, D., Kumar, P., Singh, S. K., and Singh, U. (2019). A new asymmetric loss function: Estimation of parameter of exponential distribution. *Journal of Statistics and Applied Probability Letters*, 6(1):37–50.

Kumar, D., Singh, U., and Singh, S. (2017). Life time distribution: Derived from some minimum guarantee distribution. *Sohag Journal of Mathematics*, 4(1):7–11.

Kumar, D., Singh, U., and Singh, S. K. (2015, a). A method of proposing new distribution and its application to bladder cancer patients data. *Journal of Statistics and Applied Probability Letters*, 2(3):235–245.

Kumar, D., Singh, U., and Singh, S. K. (2015, b). A new distribution using sine function-its application to bladder cancer patients data. *Journal of Statistics Applications & Probability*, 4(3):417.

Kumar, P., Kumar, D., and Singh, U. (2022). Classical and bayesian estimation of parameter of $SS_E(\epsilon)$-distribution under type-ii censored data. *Reliability: Theory & Applications*, 17(4 (71)):297–310.

Kundu, D. and Pradhan, B. (2009). Estimating the parameters of the generalized exponential distribution in presence of hybrid censoring. *Communications in Statistics—Theory and Methods*, 38(12):2030–2041.

Legendre, A. (1805). New methods for the determination of orbits of comets. *Courcier, Paris.*

Lin, C.-T., Wu, S. J., and Balakrishnan, N. (2006). Inference for log-gamma distribution based on progressively Type-II censored data. *Communications in Statistics-Theory and Methods*, 35(7):1271–1292.

Norstrom, J. G. (1996). The use of precautionary loss functions in risk analysis. *IEEE Transactions on Reliability*, 45(3):400–403.

Rahman, J., Aslam, M., and Ali, S. (2013). Estimation and prediction of inverse Lomax model via Bayesian approach. *Caspian Journal of Applied Sciences Research*, 2(3):43–56.

Rastogi, M. K., Tripathi, Y. M., and Wu, S.-J. (2012). Estimating the parameters of a bathtub-shaped distribution under progressive Type-II censoring. *Journal of Applied Statistics*, 39(11):2389–2411.

Rényi, A. (1961). On measures of entropy and information. In *Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability: Contributions to the Theory of Statistics*, volume 1, pages 547–561. University of California Press.

Sen, S., Chandra, N., and Maiti, S. S. (2018). Survival estimation in xgamma distribution under progressively Type-II right censored scheme. *Model Assisted Statistics and Applications*, 13(2):107–121.

Singh, S. K., Singh, U., and Kumar, D. (2013). Bayesian estimation of parameters of inverse Weibull distribution. *Journal of Applied Statistics*, 40(7):1597–1607.

Wu, S.-J. (2008). Estimation of the two-parameter bathtub-shaped lifetime distribution with progressive censoring. *Journal of Applied Statistics*, 35(10):1139–1150.