Electronic Journal of Applied Statistical Analysis
EJASA, Electron. J. App. Stat. Anal.
http://siba-ese.unisalento.it/index.php/ejasa/index
e-ISSN: 2070-5948
DOI: 10.1285/i20705948v15n1p145

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Published: 20 May 2022

Electronic Journal of Applied Statistical Analysis
Vol. 15, Issue 01, , 145-166
DOI: 10.1285/i20705948v15n1p145

Bayesian inference for the reliability of generalized inverted exponential distribution under progressive type-I interval censoring

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In this paper, we consider the maximum likelihood (ML) and the Bayesian estimators of the parameters, reliability and hazard functions for the generalized inverted exponential distribution under progressive type-I interval censoring. We propose EM algorithm to obtain the ML estimators. The asymptotic confidence intervals are constructed based on the ML estimators. In order to construct the asymptotic confidence intervals of the reliability and hazard functions, we compute variances of them by using delta method. It is observed that the closed-form expressions for the Bayesian estimates cannot be obtained. So we use Tierney-Kadane's approximation and Gibbs sampling method to obtain these estimates. We also derive the Bayesian credible intervals by using Gibbs sampling. Monte-Carlo simulation study is performed to compare the performances of the proposed methods concerning different sample sizes and censoring schemes. Finally, a real data set is analyzed for illustrative purposes.

keywords: Progressive interval type-I censoring, Generalized inverted exponential distribution, Bayes estimation, EM algorithm, Tierney-Kadane approximation, Gibbs sampling.

1 Introduction

In reliability and life testing experiments the failure times of test units may not be recorded exactly, since test units may be lost or removed from the test before failures. Therefore the problem of censored observations occurs quite commonly while observing

ISSN: 2070-5948
http://siba-ese.unisalento.it/index.php/ejasa/index

lifetime data. To overcome this problem, variety of censoring schemes have been introduced in the literature. The most common censoring schemes are type-I and type-II. However type-I and type-II censoring schemes do not allow to experimenter to remove surviving units at points before termination time, see Lawless (2011). For this reason, progressive censoring scheme in which the units are allowed to remove at some other times before the end of experiment got much attention, see Balakrishnan and Cramer (2014). See also Balakrishnan (2007) and Balakrishnan et al. (2000) for the further details on progressive censoring combining with type-I or type-II censoring. In practice, it is often impossible to observe the testing procedure continuously. Hence failure time can not be measured exactly. It can be only recorded whether a test units fails in an interval. In such cases interval censored data can be observed.

Aggarwala* (2001) introduced progressive type-I interval censoring which is a combination of progressive type-I censoring and interval censoring schemes. In this work, classical estimation for a one-parameter exponential distribution under progressive type-I interval censoring has been discussed.

Under progressive type-I interval censoring, observations are only known within two consecutively pre scheduled times and items could be allowed to withdraw at pre-scheduled time points. We know briefly describe this scheme. Suppose that n units be placed on a life test simultaneously at time $t_0 = 0$ and under this censoring test units are inspected only at m pre-specified times $t_1 < t_2 < ... < t_m$ where t_m is the scheduled time to terminate the test. The number X_i denote the number of observed failures in the interval $(t_{i-1}, t_i]$ and R_i denote the surviving items randomly removed from the life test at t_i, for $i = 1, 2, ..., m$. The values R_i are pre-specified by a prescribed percentage p_i of the remaining Y_i surviving units, $i = 1, 2, ..., m$ (with $p_m = 1$). In other words, R_i can be determined as $R_i = \lfloor p_i Y_i \rfloor$ at each inspection time t_i for $i = 1, 2, ...m$, where $\lfloor x \rfloor$ is the greatest integer less than or equal to x. Therefore $\{X_i, R_i, t_i\}$, $i = 1, 2, ..., m$ represents a progressively type-I interval censored sample with continuous distribution function $F(t, \theta)$, where θ is the parameter vector. Here, sample size is $n = \sum_{i=1}^{m}(X_i + R_i)$. Then, the associated likelihood function is given by Aggarwala* (2001) as

$$L(\theta) \propto \prod_{i=1}^{m} [F(t_i;\theta) - F(t_{i-1};\theta)]^{X_i} [1 - F(t_i;\theta)]^{R_i}, \tag{1}$$

where $t_0 = 0$. There exists several works on inference for different life time distributions based on progressive type-I interval censoring. See, for example Arabi Belaghi et al. (2017), Ashour and Afify (2007), Lin et al. (2009), Ng and Wang (2009), Chen et al. (2013), Ding-Geng and Tzong-Ru (2011), Xiuyun and Zaizai (2011), Teimouri and Gupta (2012), Singh and Tripathi (2018), Du et al. (2018), Teimouri (2020) considered the concept of progressive type-I interval censoring and compared different estimation methods when the underlying distributions are Burr XII, exponentiated Weibull, log-normal, Weibull, generalized exponential, generalized Rayleigh, gamma, Gombertz-Makeham, inverse Weibull, log-logistic and Chen, respectively. In this study, we consider estimation of parameters, reliability and hazard functions of generalized inverted exponential distribution (GIED) under type-I interval censoring.

The GIED was introduced by Abouammoh and Alshingiti (2009) generalizing inverted exponential distribution by using a shape parameter. According to complete samples, they study on the distributional properties and reliability characteristics of the GIED. They also observed that GIED is a special case of the exponentiated Frechet distribution and in many situations, this distribution may provide a better fit than gamma, Weibull, and generalized exponential distributions, see Abouammoh and Alshingiti (2009). The hazard function of GIED can be decreasing or increasing but not constant depending on a shape parameter. For this reason, GIED is flexible and appropriate for several application area in practice, for instance; in accelerated life testing, horse racing, supermarkets queue, sea currents, wind speeds, etc., see Kotz and Nadarajah (2000). Krishna and Kumar (2013) studied on reliability estimation in GIED under progressive type-II censoring. Dey and Pradhan (2014) took into account the statistical inference of the unknown parameter of GIED based on hybrid censored data. Dey and Dey (2014) obtained the ML and the Bayes estimates and the associated credible intervals for parameters of GIED under progressively type-II censoring. Dey et al. (2016) discussed non-Bayesian and Bayesian estimation methods for the parameters of GIED based on upper record values. Dube et al. (2016) obtained the ML and the Bayes estimators of the unknown parameters and reliability characteristics of GIED under progressive first-failure censored samples. Dey and Nassar (2020) consider estimation of the parameters of GIED under constant stress accelerated life test.
The probability density function (pdf) and cumulative distribution function (cdf) of GIED are defined as

$$f(x) = \frac{\alpha\lambda}{x^2} e^{-\lambda/x}(1 - e^{-\lambda/x})^{\alpha-1}, \quad x > 0, \quad \alpha, \lambda > 0 \tag{2}$$

and

$$F(x) = 1 - (1 - e^{-\lambda/x})^{\alpha}, \quad x > 0, \quad \alpha, \lambda > 0, \tag{3}$$

respectively. It is denoted by GIED(α, λ), where α and λ are the shape and scale parameters, respectively. The reliability and hazard functions of GIED are given by

$$S(x) = (1 - e^{-\lambda/x})^{\alpha}, \quad x > 0, \quad \alpha, \lambda > 0 \tag{4}$$

and

$$h(x) = \frac{\alpha\lambda}{x^2(e^{-\lambda/x} - 1)}, \quad x > 0, \quad \alpha, \lambda > 0, \tag{5}$$

respectively.
The objective of our study is to estimate the unknown parameters, reliability and hazard functions of GIED based on progressive type-I interval censoring. We obtain the ML estimates by using and expectation-maximization (EM) algorithm and compute observed Fisher information matrix. The asymptotic confidence intervals are also proposed based on the ML estimates. We further consider the Bayes estimates under squared error loss function. Since the Bayes estimates of the parameters do not have closed forms, we use Tierney-Kadane (TK) approximation and Gibbs sampling methods to compute the Bayes estimates. Moreover we constructed the corresponding Bayesian credible intervals

using Gibbs sampling.
To the best of our knowledge, this is the first study on estimation of parameters, reliability and hazard functions of GIED using EM algorithm and the Bayesian approach under progressive type-I interval censoring.
The rest of this paper is organized as follows. The ML estimates for parameters, reliability and hazard functions of GIED are obtained by using EM algorithm in Section 2. The observed Fisher information matrix and asymptotic confidence intervals are also discussed in this section. The Bayes estimates of parameters, reliability and hazard functions of GIED are derived in Section 3 using TK approximation and Gibbs sampling method. Furthermore, in this section the Bayesian credible intervals are obtained. In Section 4, Monte-Carlo simulation study is performed to compare performances of proposed methods. Real data set is analysed for illustration purposes in Section 5. Finally, concluding remarks are given in Section 6.

2 Maximum likelihood estimation

Let $\{X_i, R_i, t_i\}$, $i = 1, 2, ..., m$ be a progressively type-I interval censored sample from GIED, the likelihood function (1) can be specified as follows:

$$L(\alpha, \lambda) \propto \prod_{i=1}^{m}[(1-e^{-\lambda/t_{i-1}})^{\alpha} - (1-e^{-\lambda/t_i})^{\alpha}]^{X_i}[(1-e^{-\lambda/t_i})^{\alpha}]^{R_i}. \tag{6}$$

The corresponding log-likelihood function is

$$\ln L = \sum_{i=1}^{m} X_i \ln[(1-e^{-\lambda/t_{i-1}})^{\alpha} - (1-e^{-\lambda/t_i})^{\alpha}] + \sum_{i=1}^{m} R_i \ln[(1-e^{-\lambda/t_i})^{\alpha}] \tag{7}$$

By setting the derivatives of (7) with respect to parameters of GIED to zero, the ML estimators pf α and λ can be obtained by solving the following likelihood equations.

$$\begin{aligned}\frac{\partial \ln L}{\partial \alpha} &= \sum_{i=1}^{m} X_i \frac{(1-e^{-\lambda/t_{i-1}})^{\alpha}\log(1-e^{-\lambda/t_{i-1}}) - (1-e^{-\lambda/t_i})^{\alpha}\log(1-e^{-\lambda/t_i})}{(1-e^{-\lambda/t_{i-1}})^{\alpha} - (1-e^{-\lambda/t_i})^{\alpha}} \\ &+ \sum_{i=1}^{m} R_i \log(1-e^{-\lambda/t_i})\end{aligned} \tag{8}$$

$$\begin{aligned}\frac{\partial \ln L}{\partial \lambda} &= \sum_{i=1}^{m} X_i \frac{\frac{\alpha}{t_{i-1}}e^{-\lambda/t_{i-1}}(1-e^{-\lambda/t_{i-1}})^{\alpha-1} - \frac{\alpha}{t_i}e^{-\lambda/t_i}(1-e^{-\lambda/t_i})^{\alpha-1}}{(1-e^{-\lambda/t_{i-1}})^{\alpha} - (1-e^{-\lambda/t_i})^{\alpha}} \\ &+ \sum_{i=1}^{m} R_i \frac{\frac{\alpha}{t_i}(1-e^{-\lambda/t_{i-1}})^{\alpha-1}e^{-\lambda/t_i}}{(1-e^{-\lambda/t_i})^{\alpha}}.\end{aligned}$$

Since above likelihood equations cannot be solved analytically, iterative methods are required to solve nonlinear likelihood equations. In this study, we use EM algorithm introduced by Dempster et al. (1977) to compute the ML estimators of α and λ, since

EM algorithm is treated as a powerful tool for deriving the ML estimators where data are censored, see Ng et al. (2002). For more details on EM algorithm and its application to incomplete data, see Dempster et al. (1977), McLachlan and Krishnan (2007), Ng et al. (2002). EM algorithm for GIED under progressively type-I interval censoring is discussed in section (2.1).

2.1 EM Algorithm

Let suppose that $\tau_{ij},\ j = 1, 2, ..., X_i$ denote lifetimes of the observed failures in the interval $(t_{i-1}, t_i]$ and $\tau^*_{ij},\ j = 1, 2, ..., R_i$ denote lifetimes of the units censored at t_i for $i = 1, 2, ..., m$. Then the log-likelihood function for the complete sample can be written as

$$\begin{aligned}
\ln L^c &\propto \sum_{i=1}^{m}\left[\sum_{j=1}^{X_i}\log(f(\tau_{i,j},\theta)) + \sum_{j=1}^{R_i}\log(f(\tau^*_{ij},\theta))\right] \\
&= n\ln(\alpha) + n\ln(\lambda) + \sum_{i=1}^{m}\left[\sum_{j=1}^{X_i}\ln\Big(\frac{1}{\tau_i^2}\Big) + \sum_{j=1}^{R_i}\ln\Big(\frac{1}{\tau_i^{*2}}\Big)\right] \\
&- \lambda\sum_{i=1}^{m}\left[\sum_{j=1}^{X_i}\frac{1}{\tau_{ij}} + \sum_{j=1}^{R_i}\frac{1}{\tau^*_{ij}}\right] + (\alpha-1)\sum_{i=1}^{m}\left[\sum_{j=1}^{X_i}\ln[1-e^{-\lambda/\tau_{ij}}]\right. \\
&+ \left.\sum_{j=1}^{R_i}\ln[1-e^{-\lambda/\tau^*_{ij}}]\right], \qquad (9)
\end{aligned}$$

where $n = \sum_{i=1}^{m}(X_i + R_i)$. Then we take the derivative of the (9) with respect to α and λ and obtain following equations:

$$\begin{aligned}
\frac{\partial \ln L^c}{\partial\alpha} &= \frac{n}{\alpha} + \sum_{i=1}^{m}\left[\sum_{j=1}^{X_i}\ln[1-e^{-\lambda/\tau_{ij}}] + \sum_{j=1}^{R_i}\ln[1-e^{-\lambda/\tau^*_{ij}}]\right] \\
\frac{\partial \ln L^c}{\partial\lambda} &= \frac{n}{\lambda} - \sum_{i=1}^{m}\left[\sum_{j=1}^{X_i}\frac{1}{\tau_{ij}} + \sum_{i=1}^{R_i}\frac{1}{\tau^*_{ij}}\right] \\
&+ (\alpha-1)\sum_{i=1}^{m}\left[\sum_{j=1}^{X_i}\frac{1/\tau_{ij}e^{-\lambda/\tau_{ij}}}{[1-e^{-\lambda/\tau_{ij}}]} + \sum_{j=1}^{R_i}\frac{1/\tau^*_{ij}e^{-\lambda/\tau^*_{ij}}}{[1-e^{-\lambda/\tau^*_{ij}}]}\right]. \qquad (10)
\end{aligned}$$

The EM algorithm consists two steps, namely expectation step (E-step) and maximization step (M-step). The EM cycle starts with taking initial values of parameter, say $\alpha^{(0)}$ and $\lambda^{(0)}$. Next, in order to perform E-step, the following conditional expectations are computed using numerical integration methods.

$$\begin{aligned}
E_{1i}(\alpha,\lambda) &= E\left[\ln(1-e^{-\lambda/\tau_{ij}})|\tau_{ij} \in (t_{i-1}, t_i]\right] \\
&= \frac{1}{(1-e^{-\lambda/t_{i-1}})^\alpha - (1-e^{-\lambda/t_i})^\alpha}\int_{t_{i-1}}^{t_i}\ln[1-e^{-\lambda/y}]f(y)dy,
\end{aligned}$$

$$
\begin{aligned}
E_{2i}(\alpha,\lambda) &= E\Bigg[\ln(1-e^{-\lambda/\tau^*_{ij}})|\tau^*_{ij}\in[t_i,\infty)\Bigg]\\
&= \frac{1}{(1-e^{-\lambda/t_i})^{\alpha}}\int_{t_i}^{\infty}\ln[1-e^{-\lambda/y}]f(y)dy,\\
E_{3i}(\alpha,\lambda) &= E\Bigg[\frac{1}{\tau_{ij}}|\tau_{ij}\in(t_{i-1},t_i]\Bigg]\\
&= \frac{1}{(1-e^{-\lambda/t_{i-1}})^{\alpha}-(1-e^{-\lambda/t_i})^{\alpha}}\int_{t_{i-1}}^{t_i}\frac{1}{y}f(y)dy,\\
E_{4i}(\alpha,\lambda) &= E\Bigg[\frac{1}{\tau^*_{ij}}|\tau^*_{ij}\in[t_i,\infty)\Bigg]\\
&= \frac{1}{(1-e^{-\lambda/t_i})^{\alpha}}\int_{t_i}^{\infty}\frac{1}{y}f(y)dy,\\
E_{5i}(\alpha,\lambda) &= E\Bigg[\frac{1/\tau_{ij}e^{-\lambda/\tau_{ij}}}{[1-e^{-\lambda/\tau_{ij}}]}|\tau_{ij}\in(t_{i-1},t_i]\Bigg]\\
&= \frac{1}{(1-e^{-\lambda/t_{i-1}})^{\alpha}-(1-e^{-\lambda/t_i})^{\alpha}}\int_{t_{i-1}}^{t_i}\frac{\frac{1}{y}e^{-\lambda/y}}{1-e^{-\lambda/y}}f(y)dy,\\
E_{6i}(\alpha,\lambda) &= E\Bigg[\frac{1/\tau^*_{ij}e^{-\lambda/\tau^*_{ij}}}{[1-e^{-\lambda/\tau^*_{ij}}]}|\tau^*_{ij}\in(t_i,\infty]\Bigg]\\
&= \frac{1}{(1-e^{-\lambda/t_i})^{\alpha}}\int_{t_i}^{\infty}\frac{\frac{1}{y}e^{-\lambda/y}}{1-e^{-\lambda/y}}f(y)dy.
\end{aligned}
$$

Then, likelihood equations given in (8) are replaced by

$$
\begin{aligned}
\frac{\partial \ln L^c}{\partial\alpha} &= \frac{n}{\lambda}+\sum_{i=1}^{m}\Bigg[X_iE_{1i}+R_iE_{2i}\Bigg],\\
\frac{\partial \ln L^c}{\partial\lambda} &= \frac{n}{\lambda}-\sum_{i=1}^{m}\Bigg[X_iE_{3i}+R_iE_{4i}\Bigg]-(\alpha-1)\sum_{i=1}^{m}\Bigg[X_iE_{5i}+R_iE_{6i}\Bigg].
\end{aligned}
\tag{11}
$$

In M-step, equations in (11) are maximized with respect to α and λ to get the $(k+1)$th parameter estimates. Therefore, if the kth stage estimates of parameters are $\alpha^{(k)}$ and $\lambda^{(k)}$, the $(k+1)$th stage updated estimates are

$$
\begin{aligned}
\alpha^{(k+1)} &= -n\Bigg(\sum_{i=1}^{m}[X_iE_{1i}+R_iE_{2i}]\Bigg)^{-1}\\
\lambda^{(k+1)} &= n\Bigg(\sum_{i=1}^{m}[X_iE_{3i}+R_iE_{4i}]-(\alpha-1)\sum_{i=1}^{m}[X_iE_{5i}+R_iE_{6i}]\Bigg)^{-1}
\end{aligned}
\tag{12}
$$

The EM cycle is repeated till the defined convergence criteria is satisfied. Furthermore, the ML estimators of reliability and hazard functions are obtained by substituting the ML estimators of α and λ into (4) and (5) as follows:

$$S(t) = (1 - e^{-\hat{\lambda}_{ML}/t})^{\hat{\alpha}_{ML}}, \quad t > 0; \quad h(t) = \frac{\hat{\alpha}_{ML}\hat{\lambda}_{ML}}{t^2(e^{-\hat{\lambda}_{ML}/t} - 1)}, \quad t > 0 \tag{13}$$

where $\hat{\alpha}_{ML}$ and $\hat{\lambda}_{ML}$ are the ML estimators of α and λ, respectively.

2.2 Approximate confidence intervals

In this section,we obtain the asymptotic variances and covariances of the ML estimators of (α, λ) by computing the inverse of observed Fisher information matrix denoted by $I^{-1}(\alpha, \lambda)$. It is obtained as

$$\begin{aligned} I^{-1}(\alpha, \lambda) &= \begin{bmatrix} -\frac{\partial^2 \ln L(\alpha,\lambda)}{\partial \alpha^2} & -\frac{\partial^2 \ln L(\alpha,\lambda)}{\partial \alpha \partial \lambda} \\ -\frac{\partial^2 \ln L(\alpha,\lambda)}{\partial \lambda \partial \alpha} & -\frac{\partial^2 \ln L(\alpha,\lambda)}{\partial \lambda^2} \end{bmatrix}^{-1}_{(\alpha,\lambda)=(\hat{\alpha}_{ML},\hat{\lambda}_{ML})} \\ &= \begin{bmatrix} \mathrm{Var}(\hat{\alpha}) & \mathrm{cov}(\hat{\alpha}, \hat{\lambda}) \\ \mathrm{cov}(\hat{\lambda}, \hat{\alpha}) & \mathrm{Var}(\hat{\lambda}). \end{bmatrix} \end{aligned} \tag{14}$$

The derivatives in (14) are

$$\begin{aligned} \frac{\partial^2 \ln L(\alpha, \lambda)}{\partial \alpha^2} &= \sum_{i=1}^{m} \Bigg(\frac{(\delta'_{i-1,\lambda}\varphi_{i-1} + \delta_{i-1}\varphi'_{i-1,\lambda} - \delta'_{i,\lambda}\varphi_i - \delta_i\varphi'_{i,\lambda})\delta_{i-1}}{(\delta_{i-1} - \delta_i)^2} \\ &- \frac{(\delta'_{i-1,\lambda}\varphi_{i-1} + \delta_{i-1}\varphi'_{i-1,\lambda} - \delta'_{i,\lambda}\varphi_i - \delta_i\varphi'_{i,\lambda})\delta_i}{(\delta_{i-1} - \delta_i)^2} \\ &- \frac{(\delta_{i-1}\varphi_{i-1} - \delta_i\varphi_i)(\delta'_{i-1,\lambda}\varphi'_{i,\lambda})}{(\delta_{i-1} - \delta_i)^2} \Bigg), \\ \frac{\partial^2 \ln L(\alpha, \lambda)}{\partial \alpha \partial \lambda} &= \sum_{i=1}^{m} X_i \Bigg(\frac{(\delta'_{i-1,\lambda}\varphi_{i-1} + \delta_{i-1}\varphi'_{i-1,\lambda} - \delta'_{i,\lambda}\varphi_i - \delta_i\varphi'_{i,\lambda})\delta_{i-1}}{(\delta_{i-1} - \delta_i)^2} \\ &- \frac{(\delta'_{i-1,\lambda}\varphi_{i-1} + \delta_{i-1}\varphi'_{i-1,\lambda} - \delta'_{i,\lambda}\varphi_i - \delta_i\varphi'_{i,\lambda})\delta_i}{(\delta_{i-1} - \delta_i)^2} \\ &+ \sum_{i=1}^{m} R_i \frac{1/tB_i}{1 - B_i} \Bigg), \end{aligned}$$

$$\frac{\partial^2 \ln L(\alpha,\lambda)}{\partial\lambda\partial\alpha} = \sum_{i=1}^{m} X_i\Bigg[\frac{\left(\frac{B_{i-1}}{t_{i-1}}(A_{i-1}+\alpha A'_{i-1,\alpha})-\frac{B_i}{t_i}(A_i+A'_{i,\alpha})\right)}{(\delta_{i-1}-\delta_i)}$$

$$-\frac{\left(\frac{\alpha}{t_{i-1}}B_{i-1}A_{i-1}-\frac{\alpha}{t_i}B_iA_i)(\delta'_{i-1,\alpha}-\delta'_{i,\alpha})\right)}{(\delta_{i-1}-\delta_i)^2}\Bigg]$$

$$+\sum_{i=1}^{m} R_i\frac{\left(\frac{B_i}{t_i}A_i+\alpha\frac{B_i}{t_i}A'_{i,\alpha}\right)\delta_i-\alpha\frac{B_i}{t_i}A_i\delta'_{i,\alpha}}{\delta_i^2},$$

$$\frac{\partial^2 \ln L(\alpha,\lambda)}{\partial\lambda^2} = \sum_{i=1}^{m} X_i\frac{\Bigg[\Bigg(\frac{\alpha}{t_{i-1}}(B'_{i-1,\lambda}A_{i-1}+B_{i-1}A'_{i-1,\lambda})}{(\delta_{i-1}-\delta_i)}$$

$$-\frac{\frac{\alpha}{t_i}(B'_{i,\lambda}A_i+B_iA'_{i,\lambda})\Bigg)}{(\delta_{i-1}-\delta_i)}$$

$$-\frac{\left(\frac{\alpha}{t_{i-1}}A_{i-1}B_{i-1}-\frac{\alpha}{t_i}A_iB_i\right)(\delta'_{i-1,\lambda}-\delta'_{i,\lambda})}{(\delta_{i-1}-\delta_i)^2}\Bigg]$$

$$+\sum_{i=1}^{m} R_i\left[\frac{\frac{\alpha}{t_i}(A'_{i,\lambda}B_i-A_iB'_{i,\lambda})\delta_i}{\delta_i^2}\right].$$

Here,

$$\delta_i=(1-e^{-\lambda/t_i})^{\alpha},\ \delta_{i-1}=(1-e^{-\lambda/t_{i-1}})^{\alpha},\ \varphi=\log(1-e^{-\lambda/t_i}),\ \varphi=\log(1-e^{-\lambda/t_{i-1}}),$$

$$A_i=(1-e^{-\lambda/t_i})^{\alpha-1},\ A_i=(1-e^{-\lambda/t_{i-1}})^{\alpha-1},\ B_i=e^{-\lambda/t_i} \text{ and } B_{i-1}=e^{-\lambda/t_{i-1}}.$$

Let $\theta=\{\delta,\varphi,A,B\}$ denotes a set of above expressions. In this case $(\theta'_{i,\alpha},\theta'_{i-1,\alpha})$ and $(\theta'_{i,\lambda},\theta'_{i-1,\lambda})$ are the derivatives (θ_i,θ_{i-1}) with respect to α and λ, respectively.

The $100(1-\gamma)$ asymptotic confidence intervals of α and λ are constructed as

$$\hat{\alpha}\pm z_{\gamma/2}\sqrt{\mathrm{Var}(\hat{\alpha})} \quad \text{and} \quad \hat{\lambda}\pm z_{\gamma/2}\sqrt{\mathrm{Var}(\hat{\lambda})}, \tag{15}$$

where $z_{\gamma/2}$ is the upper $(\gamma/2)$th percentile of the standard normal distribution. Further, to construct the asymptotic confidence intervals of the reliability and hazard functions denoted by $S(t)$ and $h(t)$, respectively, we compute the approximate estimates of the variance for reliability and hazard functions using delta method. For the further details about the delta methods, see Greene (2003), Ahmed (2015) and EL-Sagheer (2018). The approximate variance of $S(t)$ and $h(t)$ computed as

$$\hat{\sigma}^2(S(t)) = [\nabla\hat{S}(t)]^T[I^{-1}(\hat{\alpha}_{ML}, \hat{\lambda}_{ML})][\nabla\hat{S}(t)] \quad (16)$$

and

$$\hat{\sigma}^2(h(t)) = [\nabla\hat{h}_{(t)}]^T[I^{-1}(\hat{\alpha}_{ML}, \hat{\lambda}_{ML})][\nabla\hat{h}_{(t)}], \quad (17)$$

where, $\nabla\hat{S}(t)$ and $\nabla\hat{h}(t)$ are the gradients of $S(t)$ and $h(t)$ with respect to α and λ, respectively, Mahmoud et al. (2020). Approximate $100(1-\gamma)\%$ CI of the $S(t)$ and $h(t)$ are constructed as

$$\hat{S}(t) \pm z_{\gamma/2}\sqrt{\hat{\sigma}^2_{S(t)}} \quad \text{and} \quad \hat{h}(t) \pm z_{\gamma/2}\sqrt{\hat{\sigma}^2_{h(t)}}. \quad (18)$$

3 Bayesian estimation

In this section, we derive the Bayes estimators for α, λ, $S(t)$ and $h(t)$ under the squared error loss (SEL) function defined as

$$L_{\text{SEL}}(g(\theta), \hat{g}(\theta)) = (g(\theta) - \hat{g}(\theta))^2, \quad (19)$$

where $\hat{g}(\theta)$ denotes an estimate of the future $g(\theta)$. The independent priors of the parameters α and λ are required to implement a Bayesian analysis. In this study, we assume that α and λ are a prior distributed as independent gamma distributions with respective densities

$$\pi_1(\alpha; a_1, b_1) \propto \alpha^{a_1-1}e^{-b_1\alpha} \quad (20)$$

$$\pi_1(\lambda; a_2, b_2) \propto \lambda^{a_2-1}e^{-b_2\lambda}, \quad (21)$$

and the corresponding joint prior density function can be written as

$$\pi(\alpha, \lambda) = \pi(\alpha)\pi(\lambda) \propto \alpha^{a_1-1}e^{-b_1\alpha}\lambda^{a_2-1}e^{-b_2\lambda}, \quad (22)$$

where hyper-parameters $(a_i, b_i) \quad i = 1, 2$ are assumed to be known positive constants. By combining the likelihood function (6) with the joint prior density function (22), the joint posterior density function of α and λ is obtained as

$$\pi^*(\alpha, \lambda|x) = \frac{L(x|\alpha, \lambda)\pi(\alpha, \lambda)}{\int_0^\infty \int_0^\infty L(x|\alpha, \lambda)\pi(\alpha, \lambda)}. \quad (23)$$

Then, the Bayes estimator of any parametric function $g(\alpha, \lambda)$ is derived by

$$\hat{g} = \int_0^\infty \int_0^\infty L(x|\alpha, \lambda)\pi(\alpha, \lambda). \quad (24)$$

It is clear that the multiple integrals in (24) cannot be solved analytically. Thus, different technique are used to derive explicit solutions for the Bayesian estimators. Therefore, in this study, we propose Tierney and Kadane (1986)'s approximation and Gibbs sampling method to obtain the Bayesian estimators of α, λ, $S(t)$ and $h(t)$. The details of these methods are given in the following subsections.

3.1 Tierney-Kadane approximation

Consider the following functions defined as

$$\begin{aligned}\delta(\alpha,\lambda) &= \frac{1}{n}\Big[\ln L(\alpha,\lambda)+\ln\pi(\alpha,\lambda)\Big],\\ \delta^*_\alpha(\alpha,\lambda) &= \frac{1}{n}\Big[\ln L(\alpha,\lambda)+\ln\pi(\alpha,\lambda)\alpha\Big],\\ \delta^*_\lambda(\alpha,\lambda) &= \frac{1}{n}\Big[\ln L(\alpha,\lambda)+\ln\pi(\alpha,\lambda)\lambda\Big]\end{aligned} \tag{25}$$

and assume that $(\hat{\alpha}_\delta,\hat{\lambda}_\delta)$, $(\hat{\alpha}_{\delta^*_\alpha},\hat{\lambda}_{\delta^*_\alpha})$ and $(\hat{\alpha}_{\delta^*_\lambda},\hat{\lambda}_{\delta^*_\lambda})$ maximize the functions in (26), respectively. The Bayes estimators of α and λ are respectively computed as

$$\hat{\alpha}_{TK}=\sqrt{\frac{|\Sigma^*_\alpha|}{|\Sigma|}}\exp\Big[n\Big\{\delta^*_\alpha(\hat{\alpha}_{\delta^*_\alpha},\hat{\lambda}_{\delta^*_\alpha})-\delta(\hat{\alpha}_\delta,\hat{\lambda}_\delta)\Big\}\Big] \tag{26}$$

and

$$\hat{\lambda}_{TK}=\sqrt{\frac{|\Sigma^*_\lambda|}{|\Sigma|}}\exp\Big[n\Big\{\delta^*_\lambda(\hat{\alpha}_{\delta^*_\lambda},\hat{\lambda}_{\delta^*_\lambda})-\delta(\hat{\alpha}_\delta,\hat{\lambda}_\delta)\Big\}\Big]. \tag{27}$$

Here, $|\Sigma|$, $|\Sigma_\alpha|$ and $|\Sigma_\lambda|$ denote the determinants of negative inverse Hessian of $\delta(\alpha,\lambda)$, $\delta^*_\alpha(\alpha,\lambda)$ and $\delta^*_\lambda(\alpha,\lambda)$, respectively and computed as

$$\begin{aligned}|\Sigma| &= \left[\frac{\partial^2\delta}{\partial\alpha^2}\times\frac{\partial^2\delta}{\partial\lambda^2}-\frac{\partial^2\delta}{\partial\alpha\partial\lambda}\times\frac{\partial^2\delta}{\partial\lambda\partial\alpha}\right]^{-1}_{\alpha=\hat{\alpha}_\delta,\lambda=\hat{\lambda}_\delta}\\ |\Sigma^*_\alpha| &= \left[\frac{\partial^2\delta^*_\alpha}{\partial\alpha^2}\times\frac{\partial^2\delta^*_\alpha}{\partial\lambda^2}-\frac{\partial^2\delta^*_\alpha}{\partial\alpha\partial\lambda}\times\frac{\partial^2\delta^*_\alpha}{\partial\lambda\partial\alpha}\right]^{-1}_{\alpha=\hat{\alpha}_{\delta^*_\alpha},\lambda=\hat{\lambda}_{\delta^*_\alpha}}\\ |\Sigma^*_\lambda| &= \left[\frac{\partial^2\delta^*_\lambda}{\partial\alpha^2}\times\frac{\partial^2\delta^*_\lambda}{\partial\lambda^2}-\frac{\partial^2\delta^*_\lambda}{\partial\alpha\partial\lambda}\times\frac{\partial^2\delta^*_\lambda}{\partial\lambda\partial\alpha}\right]^{-1}_{\alpha=\hat{\alpha}_{\delta^*_\alpha},\lambda=\hat{\lambda}_{\delta^*_\alpha}}.\end{aligned} \tag{28}$$

Here,

$$
\begin{aligned}
\frac{\partial^2 \ln\delta}{\partial\alpha^2} &= \frac{1}{n}\left[\frac{\partial^2 \ln L}{\partial\alpha^2} - \frac{a_1-1}{\alpha^2}\right], \quad \frac{\partial^2 \ln\delta}{\partial\lambda^2} = \frac{1}{n}\left[\frac{\partial^2 \ln L}{\partial\lambda^2} - \frac{a_2-1}{\lambda^2}\right],\\
\frac{\partial^2 \ln\delta}{\partial\alpha\partial\lambda} &= \frac{1}{n}\left[\frac{\partial^2 \ln L}{\partial\alpha\partial\lambda}\right], \quad \frac{\partial^2 \ln\delta}{\partial\lambda\partial\alpha} = \frac{1}{n}\left[\frac{\partial^2 \ln L}{\partial\lambda\partial\alpha}\right]\\
\frac{\partial^2 \ln\delta^*_\alpha}{\partial\alpha^2} &= \frac{1}{n}\left[\frac{\partial^2 \ln L}{\partial\alpha^2} - \frac{a_1-1}{\alpha^2} - \frac{1}{\alpha^2}\right],\\
\frac{\partial^2 \ln\delta^*_\alpha}{\partial\lambda^2} &= \frac{1}{n}\left[\frac{\partial^2 \ln L}{\partial\lambda^2} - \frac{a_2-1}{\lambda^2}\right]\\
\frac{\partial^2 \ln\delta^*_\alpha}{\partial\alpha\partial\lambda} &= \frac{1}{n}\left[\frac{\partial^2 \ln L}{\partial\alpha\partial\lambda}\right], \quad \frac{\partial^2 \ln\delta^*_\lambda}{\partial\lambda\partial\alpha} = \frac{1}{n}\left[\frac{\partial^2 \ln L}{\partial\lambda\partial\alpha}\right],\\
\frac{\partial \ln\delta^*_\lambda}{\partial\alpha^2} &= \frac{1}{n}\left[\frac{\partial^2 \ln L}{\partial\alpha^2} - \frac{a_1-1}{\alpha^2}\right],\\
\frac{\partial^2 \ln\delta^*_\lambda}{\partial\lambda^2} &= \frac{1}{n}\left[\frac{\partial^2 \ln L}{\partial\lambda^2} - \frac{a_2-1}{\lambda^2} - \frac{1}{\lambda^2}\right]\\
\frac{\partial^2 \ln\delta^*_\lambda}{\partial\alpha\partial\lambda} &= \frac{1}{n}\left[\frac{\partial^2 \ln L}{\partial\alpha\partial\lambda}\right], \quad \frac{\partial^2 \ln\delta^*_\lambda}{\partial\lambda\partial\alpha} = \frac{1}{n}\left[\frac{\partial^2 \ln L}{\partial\lambda\partial\alpha}\right].
\end{aligned} \tag{29}
$$

Second derivatives of log-likelihood function (7) with respect to α and λ are reported in subsection 2.2. It should be noted that a regularity condition is required for approximation given in (26) and (27). In the next section we discuss the Gibbs sampling method.

3.2 Gibbs sampling method

In this section, we adopt the Gibbs sampling method via Metropolis-Hasting (MH) algorithm proposed by Metropolis et al. (1953) to compute the Bayes estimates of α, λ, $S(t)$ and $h(t)$ and the corresponding credible intervals. Based on the independent gamma priors $\alpha \sim Gamma(a_1, b_1)$ and $\lambda \sim Gamma(a_2, b_2)$, the posterior pdf of α and λ can be given as

$$
\begin{aligned}
\pi_1^*(\alpha|x) &=\propto \alpha^{a_1-1} e^{-b_1\alpha} \prod_{i=1}^{m} [(1-e^{-\lambda/t_{i-1}})^\alpha - (1-e^{-\lambda/t_i})^\alpha]^{X_i} [(1-e^{-\lambda/t_i})^\alpha]^{R_i}\\
\pi_1^*(\lambda|x) &=\propto \lambda^{a_2-1} e^{-b_2\lambda} \prod_{i=1}^{m} [(1-e^{-\lambda/t_{i-1}})^\alpha - (1-e^{-\lambda/t_i})^\alpha]^{X_i} [(1-e^{-\lambda/t_i})^\alpha]^{R_i}.
\end{aligned} \tag{30}
$$

It should be noted that the density functions in (31) are not known. In this case, MH algorithm is used to generate samples from an arbitrary proposal distribution. It is as-

sumed that proposal distribution for α and λ is normal distribution. The steps of Gibbs sampling method via MH algorithm is as follows:
Step 1: Give the initial values $(\alpha^{(0)}, \lambda^{(0)})$ and set $j = 1$.
Step 2: Generate α' and λ' using the normal distributions $N(\alpha^{j-1}, \mathrm{Var}(\alpha))$ and $N(\lambda^{j-1}, \mathrm{Var}(\lambda))$, respectively. Here $\mathrm{Var}(\alpha)$ and $\mathrm{Var}(\lambda)$ can be obtained from the main diagonal in inverse Fisher information matrix (14).
Step 3: Compute

$$d_1 = \min\Big\{1, \pi_1(\alpha'|x)/\pi_1(\alpha^{j-1}|x)\Big\},$$
$$d_2 = \min\Big\{1, \pi_2(\lambda'|x)/\pi_2(\lambda^{j-1}|x)\Big\},$$

where π_1 and π_2 denotes the joint posterior distribution of α and λ, respectively.
Step 4: Generate samples u_1 and u_2 from a Uniform$(0, 1)$ distribution.
Step 5:

If $u_1 \leq d_1$ then set $\alpha^{(j)} = \alpha'$ otherwise $\alpha^{(j)} = \alpha^{(j-1)}$.

If $u_2 \leq d_2$ then set $\lambda^{(j)} = \lambda'$ otherwise $\lambda^{(j)} = \lambda^{(j-1)}$.
Step 6: Compute $S(t)$ and $h(t)$ at $(\alpha^{(j)}, \lambda^{(j)})$.
Step 7: Set $j = j + 1$.
Step 8: Repeat steps (2)-(7) N times to obtain required number of samples. By discarding initial N_0 number of burn-in samples, the Bayes estimates of α, λ, $S(t)$ and $h(t)$ can be obtained using $N - N_0$ samples as follows:

$$\hat{\alpha}_{Gibbs} = \frac{1}{N-N_0}\sum_{N}^{i=N_0+1}\alpha_i$$
$$\hat{\lambda}_{Gibbs} = \frac{1}{N-N_0}\sum_{N}^{i=N_0+1}\lambda$$
$$\hat{S(t)}_{Gibbs} = \frac{1}{N-N_0}\sum_{N}^{i=N_0+1}S(t) \tag{31}$$

and

$$\hat{h(t)}_{Gibbs} = \frac{1}{N-N_0}\sum_{N}^{i=N_0+1}h(t).$$

Burn-in is defined as the number of initial samples that are discarded since samples from the early iterations are not from the target posterior.
Furthermore, by ordering $\theta^{(j)}$, $j = N_0 + 1, ..., N$ in ascending orders, using the method of Chen and Shao (1999), the $100(1-\gamma)$ credible intervals for θ are given as

$$\Big(\theta_{([N\gamma/2])}, \theta_{([N(1-\gamma/2)])}\Big), \tag{32}$$

where θ is α, λ, $S(t)$, $h(t)$ and [.] represents the integer value function.

4 Monte Carlo simulation study

In this section, we perform a Monte Carlo simulation study to assess the performance of the ML and the Bayesian estimates of parameters as well as lifetime parameters reliability and hazard functions of GIED(α, λ) based on point and interval estimations. The progressive type-I interval censored sampling data $\{X_i, R_i, t_i\}$, $i = 1, 2, ..., m$ of GIED(α, λ) are generated as follows: First, the random variables $u_1, u_2, ..., u_n$, $n \geq m$ are generated from $U(0, 1)$ and t_i, $i = 1, 2, ..., n$ from GIED(α, λ) are calculated by inverting $t_i = -\lambda/\ln(1 - u^{1/\alpha})$.
Then X_i within $(t_{i-1}, t_i]$ and R_i, $i = 1, 2, ..., m$ at the pre-specified inspection times $t_1 < t_2 < ... < t_m$ and the pre-specified percentage $p = (p_{(1)}, p_{(2)}, .., p_{(m-1)}, 1)$ can be generated as follows: set $X_0 = 0$ and $R_0 = 0$ for $i = 1, 2, ..., m$.

$$\begin{aligned} X_i| \quad X_{i-1} \quad & , ..., X_0, R_{i-1}, ..., R_0 \\ \sim \quad & rBinom\left[n - \sum_{j=1}^{i-1}(X_j + R_j), \frac{F(t_i) - F(t_{i-1})}{1 - F(t_{i-1})]}\right] \\ = \quad & rBinom\left[n - \sum_{j=1}^{i-1}(X_j + R_j), \frac{(1 - e^{-\lambda/t_{i-1}})^\alpha - (1 - e^{-\lambda/t_i})^\alpha}{(1 - e^{-\lambda/t_{i-1}})^\alpha}\right] \\ R_i| \quad X_i \quad & , ..., X_0, R_{i-1}, ..., R_0 = floor\left[p_{(i)} \times \left(n - \sum_{j=1}^{i-1}(X_j + R_j) - X_i\right)\right], \end{aligned} \tag{33}$$

where $rBinom(n, p)$ generates a random variable from the Binomial distribution with parameters n and p. Also $floor()$ returns the largest integer not greater than the argument; see Aggarwala* (2001) and Peng and Yan (2013). In our simulation study, progressive type-I interval censored samples are generated for various true values of parameters (α, λ) as $(1.5, 1)$, $(2, 1)$, $(3, 1)$, $(1.5, 2)$, and $(3, 2)$ under different sample sizes as $n = 20, 50, 200$ and $m = 8$ with inspection times $t_1 < t_2 < ... < t_m$. We consider the following three progressive interval censoring schemes which are also used in Peng and Yan (2013):

$$\begin{aligned} p_{(1)} &= (0.2, 0, 0, 0, 0, 0, 0, 1) \\ p_{(2)} &= (0.2, 0, 0, 0.2, 0, 0, 0, 1) \\ p_{(3)} &= (0, 0, 0, 0, 0, 0, 0, 1), \end{aligned}$$

where censoring scheme $p_{(3)}$ corresponds to the traditional type-I interval censoring. In the Bayesian case, we choose $(a_1, b_1, a_2, b_2) = (1.6, 3.2, 2, 2)$ as also used in Dube et al. (2016). Estimators of reliability and hazard functions are computed for t_3. All simulations are performed using programs written in the open source statistical package R based on 10.000 Monte-Carlo runs, see TeamR (2017). Also we use $N = 1000$ sampling to obtain the Bayesian estimates and the corresponding credible intervals based on Gibbs sampling procedure. We discarded the first 200 iterations as burn-in.

Performances of the ML and the Bayesian estimators of for parameters parameters, reliability and hazard functions of GIED are evaluated in terms of their bias and MSE by using

$$\text{Bias}(\theta) = E(\hat{\theta}) - \theta \tag{34}$$

and

$$\text{MSE}(\hat{\theta}) = \hat{E}\{(\hat{\theta} - \theta)^2\} \tag{35}$$

for $\hat{\theta} = \hat{\alpha}, \hat{\lambda}, \hat{S}(t)$ and $\hat{h}(t)$. $\hat{E}(.)$ denotes the means of the observed values.

We also compare the performance of 95% asymptotic confidence intervals based on the ML estimators obtained using EM algorithm and 95% credible intervals based on Gibbs sampling procedure with respect to their nominal coverage probablities (CPs) and average lengths (ALs) via simulation study.

In Table 1, we report the biases and MSEs of the ML estimates obtained using EM algorithm and the Bayesian estimates of parameters, reliability and hazard functions of GIED based on TK approximation and Gibbs sampling methods under different censoring schemes. The simulation results show that all estimators have negligble bias. As sample size increases, the MSEs decrease for all estimators under each censoring schemes. The Bayes estimates outperform the ML estimates for α, λ, $S(t)$ and $h(t)$ according to MSE criteria in all cases considered. It is also observed from Table 1, the Bayes estimates obtained using Gibbs sampling method have smaller MSE values than the corresponding estimates obtained using TK approximation. Overall, Gibbs sampling method is more preferable than the other methods to obtain estimators for parameters, reliability and hazard functions of GIED. Following Gibbs sampling method, MSEs of the estimators obtained by TK approximation are smaller than those of the ML estimators in the most of the cases.

(Table 1)

In Table 2, we have the observed CP and AL of the 95% asymptotic CIs based on the ML and the Bayesian credible intervals of parameters and lifetime parameters of GIED. It is observed that the CPs of the all type of CIs are not significantly different from the nominal level. It is seen that the ALs of the intervals decrease as the sample sizes increase as expected. Among two types of CIs, the Bayesian credible intervals show the best performance with the largest observed CP for all cases. Furthermore, credible intervals have better performance in terms of AL.

(Table 2)

We can also conclude according to censoring schemes that proposed estimators are more efficient for $p_{(1)}$ and $p_{(2)}$ than those for $p_{(3)}$ for most cases. However, for $(\alpha = 2, \lambda = 1)$, most estimators of $S(t)$ have the best efficiency for $p(3)$ censoring scheme with less MSE values. Also, AL of proposed CIs are shorter for $p_{(1)}$ and $p_{(3)}$ censoring schemes almost for all cases.

As shown in simulation study, we can conclude that the Bayes estimates of parameters, reliability and hazard functions of GIED and their 95% credible intervals are superior than the corresponding the ML estimates.

5 Real data analysis

In this section, we consider a real data set for illustration purposes. The data set in Table 3 involves survival times for 68 patients from the Standford Heart Transplantation program, Elandt-Johnson and Johnson (1980). This data set has been discussed earlier in Allison (2010). Also Arabi Belaghi et al. (2017) used this data set for modelling Burr XII distribution based on progressively type-I interval censoring. Further, Arabi Belaghi et al. (2017) compare the goodness of fit of the data set to Burr XII distribution with Weibull and generalized exponential (GE) distributions using negative log-likelihood (NL) and Kolmogorov Smirnov (KS) test. Therefore, we use NL and KS test to compare the goodness of fit of the data set to GIED with Weibull, GE and Burr XII distributions. For the KS test, the maximum distance between the emprical distribution $\hat{F}(t;\alpha,\lambda)$ of the progressive type-I interval censored data and the population distribution $F(\hat{\alpha},\hat{\lambda})$ is defined as

$$D_n(F) = \sup_{0\leq t\leq\infty}\left|\hat{F}(t;\alpha,\lambda) - \hat{F}(t;\alpha,\lambda)\right|. \tag{36}$$

Here $\hat{F}(t_i;\alpha,\lambda)$ can be estimated as

$$\hat{F}(t;\alpha,\lambda) = 1 - \Pi_{j=1}^{i}(1-\hat{p}_j), i = 1,2,...,m, \tag{37}$$

where $\frac{\hat{X_1}}{n}$ and $\hat{p}_j = \frac{X_j}{n-\sum_{k=1}^{j-1}}$, $j = 2,3,...,m$. We report the ML estimators of the parameters of the considered distributions, NL and KS values in Table 4. It is seen from Table 4, GIED fits the data well with smaller NL and KS values. We presented the ML and the Bayes estimates of the parameters and lifetime parameters in Table 5. In order to compute reliability and hazard functions, we use t_3. For Gibbs sampling procedure, we take $M = 1000$. We also obtain 95% ACIs and CRIs for the parameters. Width of ACI and CRD of parameters, reliability of hazard functions are also listed in Table 5.
According to Table 5, the ML and the Bayesian estimates of α, λ, $S(t)$ and $h(t)$ take similar values. However, CRIs perform better than ACIs in respect of width.

6 Conclusion

In this paper, we consider the estimation of parameters and lifetime parameters of GIED based on progressive type-I interval censoring by using the ML and the Bayesian methodologies. We use EM algorithm to obtain the ML estimators. We also discuss on the Bayesian estimation of parameters and lifetime parameters of GIED by using TK and Gibbs sampling methods.
Our simulation study shows the estimates proposed using Gibbs sampling method are less bias and more efficient than the ML estimates and the Bayesian estimates based on

TK approximation in most cases.
Furthermore, the asymptotic confidence intervals and the Bayesian credible intervals based on Gibbs sampling for parameters, reliability and hazard functions are constructed. We compare the performance of asymptotic confidence intervals and credible intervals via Monte Carlo simulation study. It is concluded that credible intervals perform better than asymptotic confidence intervals in terms of average level and coverage probability criteria.
As a future direction, it may be considered to estimate procedures of stress-stregth reliability for GIED based on progressive type-I interval censoring by using different estimation methods.

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Table 1: Simulated Bias and MSE values for the estimators of α, θ, $S(t)$ and $h(t)$.

				$\hat{\alpha}$			$\hat{\lambda}$			$\hat{S}(t)$			$\hat{h}(t)$		
	(α,λ)	Scheme		EM	TK	GIBBS	EM	TK	GIBBS	EM	TK	GIBBS	EM	TK	GIBBS
$n=20$	(1.5,1)	$p_{(1)}$	Bias	-0.009	0.011	0.008	-0.020	-0.022	-0.017	0.043	0.038	0.042	0.034	0.041	0.041
			MSE	0.025	0.023	0.021	0.040	0.036	0.028	0.019	0.018	0.017	0.013	0.012	0.012
		$p_{(2)}$	Bias	-0.009	0.012	0.008	-0.020	-0.022	-0.022	0.043	0.044	0.042	0.036	0.041	0.041
			MSE	0.029	0.028	0.024	0.019	0.018	0.015	0.018	0.018	0.017	0.017	0.017	0.017
		$p_{(3)}$	Bias	-0.014	0.013	0.013	-0.029	-0.031	-0.031	0.034	0.035	0.034	0.039	0.042	0.041
			MSE	0.064	0.064	0.062	0.094	0.094	0.093	0.098	0.091	0.090	0.017	0.018	0.017
	(2,1)	$p_{(1)}$	Bias	-0.072	0.080	0.046	0.030	0.043	-0.064	0.023	0.021	0.023	0.080	0.086	0.081
			MSE	0.055	0.051	0.034	0.025	0.024	0.024	0.054	0.044	0.042	0.036	0.036	0.032
		$p_{(2)}$	Bias	-0.033	0.028	0.025	0.060	0.055	0.057	0.023	0.021	0.023	0.079	0.086	0.081
			MSE	0.044	0.042	0.041	0.028	0.028	0.028	0.055	0.054	0.054	0.063	0.063	0.061
		$p_{(3)}$	Bias	0.020	0.020	0.020	0.021	0.021	0.021	0.019	0.017	0.020	0.083	0.083	0.081
			MSE	0.073	0.070	0.065	0.053	0.053	0.051	0.040	0.037	0.034	0.070	0.069	0.066
	(3,1)	$p_{(1)}$	Bias	0.040	0.044	0.045	-0.025	0.022	-0.231	0.091	0.092	0.091	0.012	0.013	0.012
			MSE	0.031	0.030	0.030	0.031	0.031	0.028	0.023	0.021	0.019	0.035	0.028	0.025
		$p_{(2)}$	Bias	0.045	0.044	0.045	-0.016	0.014	0.015	0.011	0.009	0.011	0.122	0.123	0.122
			MSE	0.032	0.030	0.030	0.010	0.009	0.009	0.013	0.008	0.008	0.494	0.463	0.449
		$p_{(3)}$	Bias	0.015	0.016	0.016	0.063	0.069	0.068	0.074	0.077	0.075	0.023	0.023	0.022
			MSE	0.052	0.053	0.054	0.039	0.040	0.038	0.014	0.013	0.013	0.015	0.017	0.015
	(1.5,2)	$p_{(1)}$	Bias	0.043	0.043	0.041	0.046	0.049	0.047	0.056	0.052	0.051	0.040	0.043	0.039
			MSE	0.041	0.040	0.036	0.058	0.058	0.054	0.053	0.053	0.053	0.052	0.052	0.051
		$p_{(2)}$	Bias	0.029	0.027	0.025	0.046	0.043	0.042	0.056	0.052	0.052	0.040	0.043	0.039
			MSE	0.075	0.073	0.071	0.074	0.073	0.075	0.040	0.037	0.035	0.045	0.043	0.041
		$p_{(3)}$	Bias	0.046	0.045	0.042	0.054	0.054	0.053	0.045	0.043	0.046	0.029	0.029	0.026
			MSE	0.052	0.049	0.045	0.062	0.060	0.058	0.031	0.031	0.031	0.030	0.026	0.024
	(3,2)	$p_{(1)}$	Bias	-0.057	0.058	0.071	0.029	0.022	0.027	0.039	0.037	0.040	0.087	0.088	0.088
			MSE	0.065	0.065	0.064	0.049	0.043	0.041	0.026	0.024	0.024	0.075	0.068	0.060
		$p_{(2)}$	Bias	-0.057	0.058	0.057	0.286	0.256	0.268	0.039	0.037	0.036	0.074	0.073	0.072
			MSE	0.063	0.054	0.052	0.058	0.054	0.052	0.055	0.052	0.049	0.057	0.056	0.052
		$p_{(3)}$	Bias	0.084	0.084	0.081	0.013	0.013	-0.014	0.035	0.033	0.032	0.082	0.081	0.079
			MSE	0.074	0.063	0.061	0.058	0.053	0.050	0.056	0.053	0.048	0.065	0.059	0.055
$n=50$	(1.5,1)	$p_{(1)}$	Bias	-0.021	0.028	0.023	-0.012	0.012	-0.017	0.032	0.031	0.032	0.060	0.063	0.061
			MSE	0.017	0.016	0.015	0.035	0.031	0.026	0.011	0.015	0.010	0.010	0.009	0.007
		$p_{(2)}$	Bias	-0.032	0.030	0.032	-0.036	0.032	-0.034	0.321	0.305	0.032	0.069	0.063	0.061
			MSE	0.027	0.022	0.017	0.025	0.015	0.012	0.044	0.039	0.035	0.044	0.041	0.041
		$p_{(3)}$	Bias	0.045	0.028	0.026	0.023	0.022	0.022	0.029	0.028	0.023	0.064	0.063	0.061
			MSE	0.059	0.053	0.050	0.089	0.086	0.076	0.084	0.077	0.073	0.041	0.042	0.037
	(2,1)	$p_{(1)}$	Bias	-0.056	0.061	0.062	0.029	0.024	0.023	0.023	0.022	0.023	0.081	0.085	0.081
			MSE	0.039	0.035	0.033	0.017	0.017	0.016	0.054	0.039	0.035	0.029	0.027	0.024
		$p_{(2)}$	Bias	-0.063	0.062	0.062	0.016	0.015	-0.016	0.023	0.022	0.023	0.081	0.085	0.081
			MSE	0.042	0.041	0.041	0.074	0.072	0.072	0.053	0.048	0.047	0.068	0.066	0.066
		$p_{(3)}$	Bias	0.054	0.054	0.052	0.198	0.161	0.157	0.210	0.198	0.183	0.085	0.084	0.081
			MSE	0.067	0.059	0.058	0.048	0.046	0.044	0.034	0.029	0.025	0.067	0.063	0.061
	(3,1)	$p_{(1)}$	Bias	0.077	0.077	0.072	0.020	0.017	0.017	0.115	0.107	0.105	0.224	0.211	0.220
			MSE	0.029	0.025	0.018	0.027	0.024	0.022	0.019	0.015	0.014	0.030	0.027	0.022
		$p_{(2)}$	Bias	0.067	0.065	0.022	0.022	0.023	0.022	0.115	0.107	0.105	0.122	0.131	0.122
			MSE	0.091	0.080	0.077	0.087	0.081	0.075	0.013	0.011	0.010	0.015	0.015	0.014
		$p_{(3)}$	Bias	0.408	0.359	0.322	0.272	0.253	-0.239	0.108	0.100	0.115	0.245	0.232	0.220
			MSE	0.023	0.026	0.019	0.023	0.027	0.023	0.012	0.010	0.013	0.016	0.017	0.015
	(1.5,2)	$p_{(1)}$	Bias	-0.283	0.247	0.274	-0.272	0.254	0.234	0.593	0.565	0.535	0.373	0.413	0.359
			MSE	0.036	0.033	0.032	0.054	0.053	0.052	0.035	0.032	0.029	0.042	0.042	0.041
		$p_{(2)}$	Bias	-0.038	0.035	0.030	-0.049	0.047	0.049	0.587	0.556	0.539	0.370	0.342	0.314
			MSE	0.041	0.041	0.040	0.058	0.058	0.054	0.035	0.031	0.031	0.038	0.037	0.035
		$p_{(3)}$	Bias	0.058	0.057	0.058	0.058	0.057	-0.058	0.054	0.052	0.050	0.042	0.043	0.039
			MSE	0.048	0.046	0.044	0.058	0.056	0.054	0.030	0.029	0.023	0.027	0.023	0.015
	(3,2)	$p_{(1)}$	Bias	-0.016	0.014	0.008	-0.034	0.023	-0.019	0.040	0.039	0.038	0.170	0.185	0.177
			MSE	0.053	0.050	0.043	0.038	0.034	0.031	0.016	0.015	0.015	0.060	0.057	0.053
		$p_{(2)}$	Bias	-0.159	0.159	0.153	0.028	0.023	-0.016	0.040	0.038	0.035	0.062	0.067	0.066
			MSE	0.050	0.049	0.048	0.032	0.032	0.028	0.049	0.048	0.045	0.054	0.047	0.043
		$p_{(3)}$	Bias	0.065	0.067	0.064	0.030	0.031	0.029	0.037	0.036	0.035	0.086	0.089	0.077
			MSE	0.069	0.058	0.054	0.046	0.041	0.037	0.050	0.047	0.042	0.057	0.054	0.049
$n=200$	(1.5,1)	$p_{(1)}$	Bias	-0.056	0.051	0.053	0.013	0.019	-0.016	0.032	0.032	0.032	0.061	0.062	0.061
			MSE	0.013	0.007	0.006	0.010	0.007	0.004	0.014	0.011	0.010	0.009	0.007	0.006
		$p_{(2)}$	Bias	-0.168	0.163	0.161	-0.132	0.131	-0.125	0.323	0.317	0.304	0.610	0.621	0.610
			MSE	0.011	0.009	0.010	0.008	0.006	0.005	0.103	0.093	0.091	0.044	0.039	0.037
		$p_{(3)}$	Bias	0.175	0.149	0.122	0.030	0.039	-0.036	0.031	0.032	0.032	0.062	0.064	0.061
			MSE	0.050	0.044	0.043	0.068	0.067	0.065	0.016	0.011	0.011	0.039	0.041	0.037
	(2,1)	$p_{(1)}$	Bias	-0.056	0.047	0.041	0.031	0.032	0.031	0.023	0.023	0.023	0.081	0.084	0.081
			MSE	0.022	0.022	0.019	0.015	0.012	0.009	0.052	0.029	0.025	0.023	0.017	0.017
		$p_{(2)}$	Bias	-0.016	0.015	0.014	0.062	0.055	0.048	0.023	0.023	0.020	0.081	0.084	0.081
			MSE	0.014	0.013	0.014	0.019	0.012	0.010	0.035	0.032	0.028	0.037	0.033	0.030
		$p_{(3)}$	Bias	0.027	0.023	0.021	0.079	0.075	0.074	0.023	0.022	0.023	0.083	0.086	0.081
			MSE	0.061	0.057	0.053	0.034	0.026	0.025	0.032	0.028	0.024	0.064	0.057	0.051
	(3,1)	$p_{(1)}$	Bias	-0.010	0.015	0.014	0.038	0.037	0.032	0.115	0.113	0.115	0.022	0.021	0.022
			MSE	0.012	0.011	0.008	0.014	0.012	0.010	0.017	0.013	0.013	0.020	0.017	0.015
		$p_{(2)}$	Bias	-0.052	0.050	0.043	0.027	0.025	0.019	0.115	0.113	0.115	0.122	0.129	0.121
			MSE	0.015	0.013	0.012	0.024	0.023	0.021	0.013	0.013	0.013	0.049	0.047	0.045
		$p_{(3)}$	Bias	0.064	0.058	0.062	0.086	0.080	-0.077	0.112	0.107	0.115	0.026	0.022	0.022
			MSE	0.069	0.059	0.048	0.033	0.034	0.028	0.013	0.011	0.013	0.059	0.057	0.055
	(1.5,2)	$p_{(1)}$	Bias	0.028	0.029	0.021	-0.069	0.072	0.072	0.059	0.057	0.059	0.039	0.040	0.039
			MSE	0.030	0.031	0.028	0.051	0.047	0.043	0.039	0.033	0.032	0.025	0.016	0.015
		$p_{(2)}$	Bias	-0.038	0.027	0.022	-0.017	0.048	0.063	0.059	0.058	0.059	0.054	0.040	0.039
			MSE	0.013	0.011	0.007	0.031	0.031	0.027	0.031	0.027	0.026	0.018	0.016	0.014
		$p_{(3)}$	Bias	0.045	0.046	0.041	0.060	0.062	-0.062	0.055	0.054	0.059	0.041	0.042	0.039
			MSE	0.043	0.043	0.039	0.046	0.044	0.043	0.030	0.028	0.017	0.022	0.018	0.015
	(3,2)	$p_{(1)}$	Bias	-0.016	0.014	0.008	-0.034	0.032	0.029	0.040	0.039	0.040	0.077	0.069	0.077
			MSE	0.043	0.039	0.036	0.033	0.024	0.018	0.015	0.013	0.011	0.058	0.054	0.048
		$p_{(2)}$	Bias	0.041	0.042	0.039	-0.013	0.012	0.017	0.040	0.039	0.033	0.078	0.076	0.074
			MSE	0.035	0.034	0.026	0.027	0.026	0.025	0.046	0.044	0.042	0.045	0.032	0.030
		$p_{(3)}$	Bias	0.095	0.085	0.083	0.049	0.049	-0.043	0.039	0.038	0.038	0.083	0.088	0.088
			MSE	0.046	0.045	0.040	0.034	0.032	0.027	0.045	0.041	0.038	0.039	0.033	0.031

Table 2: Observed CP and AL of 95% asymptotic confidence intervals (ACI) and credible intervals (CRI) of $\hat{\alpha}$, $\hat{\lambda}$, $\hat{S}(t)$ and $\hat{h}(t)$.

				$\hat{\alpha}$		$\hat{\lambda}$		$\hat{S}(t)$		$\hat{h}(t)$	
	(α,λ)	Scheme		ACI	CRI	ACI	CRI	ACI	CRI	ACI	CRI
$n=20$	(1.5,1)	$p_{(1)}$	CP	0.967	0.978	0.929	0.934	0.960	0.965	0.969	0.972
			AL	0.266	0.241	0.421	0.397	0.375	0.311	0.371	0.349
		$p_{(2)}$	CP	0.968	0.978	0.949	0.956	0.926	0.970	0.962	0.973
			AL	0.324	0.313	0.241	0.230	0.491	0.471	0.321	0.316
		$p_{(3)}$	CP	0.956	0.943	0.988	0.977	0.979	0.987	0.953	0.967
			AL	0.274	0.246	0.268	0.266	0.284	0.247	0.230	0.215
	(2,1)	$p_{(1)}$	CP	0.954	0.957	0.968	0.972	0.927	0.933	0.914	0.921
			AL	0.250	0.238	0.325	0.312	0.279	0.273	0.324	0.315
		$p_{(2)}$	CP	0.955	0.954	0.958	0.968	0.955	0.963	0.934	0.956
			AL	0.325	0.312	0.436	0.434	0.369	0.327	0.325	0.301
		$p_{(3)}$	CP	0.965	0.971	0.974	0.986	0.984	0.987	0.930	0.962
			AL	0.232	0.231	0.238	0.211	0.281	0.264	0.249	0.244
	(3,1)	$p_{(1)}$	CP	0.989	0.973	0.947	0.934	0.946	0.961	0.952	0.965
			AL	0.350	0.336	0.435	0.426	0.329	0.316	0.249	0.215
		$p_{(2)}$	CP	0.922	0.924	0.920	0.948	0.918	0.936	0.902	0.923
			AL	0.526	0.454	0.352	0.345	0.349	0.336	0.250	0.201
		$p_{(3)}$	CP	0.973	0.981	0.951	0.961	0.947	0.956	0.964	0.989
			AL	0.415	0.401	0.385	0.361	0.384	0.322	0.280	0.255
	(1.5,2)	$p_{(1)}$	CP	0.986	0.992	0.959	0.967	0.948	0.950	0.946	0.965
			AL	0.321	0.292	0.388	0.367	0.275	0.268	0.305	0.301
		$p_{(2)}$	CP	0.928	0.947	0.956	0.969	0.966	0.971	0.959	0.968
			AL	0.323	0.299	0.250	0.241	0.273	0.268	0.214	0.212
		$p_{(3)}$	CP	0.984	0.988	0.981	0.996	0.983	0.988	0.974	0.976
			AL	0.384	0.382	0.343	0.337	0.241	0.238	0.251	0.246
	(3,2)	$p_{(1)}$	CP	0.957	0.958	0.929	0.922	0.939	0.937	0.987	0.988
			AL	0.065	0.065	0.027	0.024	0.016	0.014	0.075	0.077
		$p_{(2)}$	CP	0.967	0.986	0.971	0.984	0.964	0.972	0.960	0.965
			AL	0.437	0.419	0.415	0.401	0.262	0.250	0.264	0.254
		$p_{(3)}$	CP	0.953	0.952	0.974	0.972	0.965	0.968	0.988	0.993
			AL	0.257	0.221	0.293	0.286	0.265	0.257	0.235	0.211
$n=50$	(1.5,1)	$p_{(1)}$	CP	0.968	0.970	0.933	0.936	0.901	0.903	0.941	0.958
			AL	0.174	0.155	0.348	0.329	0.305	0.295	0.361	0.339
		$p_{(2)}$	CP	0.947	0.967	0.947	0.982	0.944	0.970	0.945	0.966
			AL	0.218	0.196	0.212	0.209	0.364	0.359	0.214	0.210
		$p_{(3)}$	CP	0.918	0.942	0.927	0.949	0.951	0.958	0.957	0.973
			AL	0.183	0.144	0.211	0.208	0.269	0.238	0.214	0.209
	(2,1)	$p_{(1)}$	CP	0.955	0.972	0.950	0.952	0.951	0.970	0.941	0.959
			AL	0.232	0.214	0.270	0.259	0.254	0.249	0.265	0.247
		$p_{(2)}$	CP	0.960	0.966	0.975	0.971	0.951	0.956	0.979	0.984
			AL	0.242	0.241	0.381	0.372	0.353	0.305	0.268	0.266
		$p_{(3)}$	CP	0.954	0.968	0.957	0.967	0.974	0.981	0.973	0.977
			AL	0.150	0.146	0.211	0.210	0.260	0.261	0.247	0.225
	(3,1)	$p_{(1)}$	CP	0.977	0.975	0.973	0.989	0.921	0.937	0.953	0.961
			AL	0.277	0.252	0.347	0.338	0.234	0.224	0.216	0.210
		$p_{(2)}$	CP	0.966	0.973	0.950	0.962	0.979	0.985	0.969	0.975
			AL	0.337	0.322	0.262	0.235	0.257	0.252	0.216	0.209
		$p_{(3)}$	CP	0.934	0.958	0.944	0.967	0.904	0.904	0.913	0.930
			AL	0.343	0.334	0.275	0.261	0.221	0.216	0.258	0.215
	(1.5,2)	$p_{(1)}$	CP	0.964	0.973	0.960	0.964	0.984	0.987	0.988	0.989
			AL	0.296	0.273	0.322	0.320	0.241	0.239	0.250	0.246
		$p_{(2)}$	CP	0.965	0.969	0.974	0.984	0.981	0.988	0.962	0.963
			AL	0.304	0.296	0.214	0.209	0.237	0.229	0.205	0.205
		$p_{(3)}$	CP	0.975	0.986	0.969	0.979	0.952	0.985	0.965	0.962
			AL	0.373	0.369	0.341	0.335	0.218	0.215	0.215	0.211
	(3,2)	$p_{(1)}$	CP	0.965	0.968	0.965	0.974	0.973	0.976	0.983	0.986
			AL	0.097	0.095	0.003	0.002	0.033	0.036	0.062	0.058
		$p_{(2)}$	CP	0.957	0.958	0.929	0.938	0.938	0.947	0.974	0.977
			AL	0.354	0.349	0.370	0.359	0.252	0.242	0.257	0.236
		$p_{(3)}$	CP	0.946	0.964	0.926	0.930	0.943	0.973	0.908	0.911
			AL	0.243	0.218	0.234	0.223	0.249	0.238	0.216	0.212

(Table 2 continued)

				$\hat{\alpha}$		$\hat{\lambda}$		$\hat{S}(t)$		$\hat{h}(t)$	
	(α,λ)	Scheme		ACI	CRI	ACI	CRI	ACI	CRI	ACI	CRI
$n=200$	(1.5,1)	$p_{(1)}$	CP	-0.056	0.051	0.013	0.019	0.032	0.032	0.061	0.062
			AL	0.135	0.123	0.213	0.171	0.145	0.121	0.210	0.151
		$p_{(2)}$	CP	0.975	0.979	0.964	0.968	0.984	0.986	0.959	0.980
			AL	0.181	0.178	0.144	0.126	0.234	0.225	0.179	0.155
		$p_{(3)}$	CP	0.978	0.984	0.964	0.975	0.975	0.976	0.966	0.976
			AL	0.108	0.104	0.174	0.163	0.175	0.153	0.166	0.160
	(2,1)	$p_{(1)}$	CP	0.942	0.963	0.957	0.965	0.950	0.976	0.948	0.955
			AL	0.179	0.177	0.217	0.202	0.227	0.218	0.195	0.165
		$p_{(2)}$	CP	0.979	0.978	0.965	0.962	0.962	0.971	0.954	0.959
			AL	0.191	0.187	0.316	0.302	0.263	0.246	0.244	0.216
		$p_{(3)}$	CP	0.979	0.981	0.960	0.967	0.970	0.977	0.969	0.975
			AL	0.108	0.074	0.192	0.197	0.230	0.228	0.207	0.205
	(3,1)	$p_{(1)}$	CP	0.978	0.977	0.975	0.980	0.965	0.968	0.966	0.972
			AL	0.208	0.194	0.278	0.220	0.155	0.153	0.181	0.176
		$p_{(2)}$	CP	0.972	0.981	0.966	0.982	0.978	0.977	0.966	0.984
			AL	0.231	0.215	0.214	0.198	0.178	0.174	0.181	0.173
		$p_{(3)}$	CP	0.971	0.975	0.967	0.974	0.970	0.979	0.975	0.989
			AL	0.279	0.275	0.210	0.201	0.227	0.214	0.247	0.206
	(1.5,2)	$p_{(1)}$	CP	0.960	0.962	0.960	0.972	0.956	0.952	0.940	0.943
			AL	0.241	0.224	0.258	0.237	0.195	0.153	0.205	0.202
		$p_{(2)}$	CP	0.929	0.927	0.946	0.943	0.956	0.960	0.940	0.943
			AL	0.241	0.234	0.182	0.179	0.231	0.217	0.180	0.151
		$p_{(3)}$	CP	0.939	0.968	0.933	0.956	0.957	0.959	0.956	0.957
			AL	0.292	0.280	0.240	0.233	0.205	0.192	0.206	0.202
	(3,2)	$p_{(1)}$	CP	0.953	0.950	0.965	0.978	0.967	0.958	0.983	0.986
			AL	0.097	0.095	0.073	0.072	0.033	0.036	0.062	0.058
		$p_{(2)}$	CP	0.977	0.946	0.981	0.985	0.953	0.966	0.967	0.975
			AL	0.298	0.275	0.273	0.250	0.233	0.204	0.221	0.206
		$p_{(3)}$	CP	0.968	0.978	0.967	0.969	0.954	0.957	0.983	0.990
			AL	0.229	0.214	0.213	0.210	0.234	0.231	0.196	0.192

Table 3: Data set for 68 patients from the Stanford Hearth Transplantation Program.

Number of days	Number of deaths	Number of censored
$[0, 50)$	16	3
$[100, 200)$	4	2
$[200, 400)$	5	4
$[400, 700)$	2	6
$[700, 1000)$	4	3
$[1000, 1300)$	1	2
$[1300, 1600)$	1	3
$[1600, 1900)$	0	1

Table 4: Summary fit for the real data set in Table 3.

	MLEs			
Distribution	$\hat{\alpha}$	$\hat{\lambda}$	NL	KS
Weibull	0.2366	0.4620	116.08	0.2589
Burr XII	1.1546	0.2889	115.36	0.2575
GED	0.3501	0.1303	116.62	0.2607
GIED	0.9871	5.6129	115.12	0.2495

Table 5: Bayesian estimates of lifetime parameters and width of the CIs (ACI,CRI) of them for real data set in Table 3.

				Widths	
Parameter	ML	TK	GIBBS	ACI	CRI
α	0.9871	0.9860	0.9743	0.7128	0.6910
λ	5.6129	5.5975	5.4253	4.5721	4.2362
$S(t)$	0.4293	0.4255	0.4182	0.2117	0.1935
$H(t)$	0.2877	0.2785	0.2617	0.3358	0.3127