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Variable scale Kernel density estimation for simple linear degradation model

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In this study, we proposed the variable scale kernel estimator for analyzing the degradation data. The properties of the proposed method are investigated and compared with the classical method such as; maximum likelihood and ordinary least square methods via simulation technique. The criteria bias and MSE are used for comparison. Simulation results showed that the performance of the variable scale kernel estimator is acceptable as a general estimator. It is nearly the best estimator when the assumption of the distribution is invalid. Application to real data set is also given.

keywords: Bandwidth selection; Classical kernel; Degradation; Failure time; Maximum likelihood; Ordinary least square; Variable scale kernel estimation.

1 Introduction

The reliability of the product will depend on the reliability of its units. The reliability of a unit is defined as the probability that a unit will perform its intended function until a specified point of time under encountered used conditions (Meeker and Escobar, 2014). Lu and Meeker (1993) proposed the two-stage method to estimate the nonlinear mixed effect degradation model parameters, hence to estimate the time-to-failure distribution. Meeker et al. (1998) used the maximum likelihood approach to estimate the nonlinear mixed effect degradation. Robinson and Crowder (2000) described a Bayesian approach to estimate the unknown parameters of time-to-failure distribution of nonlinear degradation model. Alodat and Al-Haj Ebrahem (2009) used the maximum likelihood method to estimate the parameters of time-to-failure distribution of a linear degradation model

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based on the ranked set sampling technique. Al-Haj Ebrahem et al. (2009a) proposed the Bayesian approach based on the idea of grouped and non-grouped data to estimate the parameters of time-to-failure distribution and its percentile for a simple linear degradation model. Al-Haj Ebrahem et al. (2009b) introduced the nonparametric classical kernel density method to estimate the time-to-failure distribution and its percentiles for simple linear degradation model. Naji Ba Dakhn et al. (2017) introduced semi-parametric method to estimate the time-to-failure distribution and its percentiles for simple linear degradation model. Eidous et al. (2017) estimated the time-to-failure distribution and its percentile using double kernel method.
In this paper, estimating the random effect distribution hence estimating the time-to-failure distribution and its percentiles by using an adaptation nonparametric method of the classical kernel estimator is considered, namely, the variable scale kernel estimator. The new estimator beside the classical kernel estimator are introduced and described in this paper. The performance of the proposed method will be compared with the existing parametric methods (OLS, MLE).

2 Time-To-Failure Distribution and its Percentiles

Degradation analysis is a useful tool when it is not possible to observe a significant number of failures (Lu and Meeker, 1993). To perform the analysis, we assume that the failure time is occurred at time t, if the actual degradation path, D(t) crosses the critical degradation level D_f. In general, the time-to-failure distribution of t,$F_T(t)$ in degradation models cannot posses in a closed form and therefore numerical methods are used to obtained it. In this paper, we consider the simple linear degradation model:

$$y_{ij} = \beta_i t_j + \epsilon_{ij}, \quad i = 1, \ldots, n; \quad j = 1, \ldots, m \tag{1}$$

where,y_{ij} is the observed degradation measurement of unit i at time t_j, D_{ij} is the actual degradation path of the unit i at time t_j, β_i is the i-th a random effect parameter assumed to be distributed as $g(\beta)$, where $g(\beta)$ may be $lognormal(\mu, \sigma^2)$, $halfnormal(\sigma^2)$; n is the number of the tested units, m is the total number of inspections on the i-th unit, ϵ_{ij} is the measurement error of unit i at time t_j, which is assumed to have zero mean and constant variance, ϵ_{ij} and β_i are assumed to be independent and t_j is the time of j^{th} measurement.
The failure time T is obtained by solving $D_f = D(t)$, then

$$D_f = \beta T.$$

The distribution function of the time-to-failure is

$$F_T(t) = p(T \leq t) = p\left(\frac{D_f}{\beta} \leq t\right) = p\left(\beta \geq \frac{D_f}{t}\right) = 1 - G_\beta\left(\frac{D_f}{t}\right), \tag{2}$$

where$G_\beta(.)$ is the distribution function of the random effect parameter.

To derive t_p, we need to solve $p = F_T(t_p)$ with respect to t_p. Therefore,

$$p = 1 - G_\beta\left(\frac{D_f}{t_p}\right).$$

3 Estimating the Time-To-Failure Distribution using kernel methods

Let $\beta_1, \beta_2, \ldots, \beta_n$ be a random sample from unknown probability density function $g_\beta(b)$, we will describe in this section how we can use the two kernel methods to estimate the time-to-failure distribution and its percentiles, then compare them by the most important parametric methods; Ordinary Least Squares (OLS) method and the Maximum Likelihood (ML) method by using simulation technique.

3.1 Classical kernel Method

The classical kernel density estimator of $g_\beta(b)$ is,

$$\hat{g}_{\beta\_classical}(b,h) = \frac{1}{nh}\sum_{i=1}^{n} K\left(\frac{b-\beta_i}{h}\right), \tag{3}$$

where $K(u)$ is the kernel function and h is the smoothing parameter. In this paper, we assume that the kernel function is Gaussian kernel, which is given by

$$K(u) = \frac{1}{\sqrt{2\pi}} e^{-u^2/2}$$

then,

$$\hat{g}_{\beta\_classical}(b,h) = \frac{1}{nh}\sum_{i=1}^{n} \frac{1}{\sqrt{2\pi}} exp\left[-\frac{1}{2h^2}(b-\beta_i)^2\right] \tag{4}$$

Silverman (1986) gave the following optimal rule for h, as follows,

$$h_{int} = 1.587\ \widehat{\sigma} n^{-1/3} \tag{5}$$

where $\widehat{\sigma} = \sqrt{\frac{\sum_{i=1}^{n} x_i^2}{n}}$ is the ML estimator for σ when the distribution of the random effect data is assumed to be $N(0, \sigma^2)$.

Thus, the time-to-failure distribution,$F_T(t)$ can be estimated based on the estimator (4). By using formula (2) we have,

$$\hat{F}_{T\_classical}(t) = 1 - \int_{-\infty}^{\frac{D_f}{t}} \hat{g}_{\beta\_classical}(b,h)db$$

$$= 1 - \int_{-\infty}^{\frac{D_f}{t}} \frac{1}{nh} \sum_{i=1}^{n} \frac{1}{\sqrt{2\pi}} exp\left[-\frac{1}{2h^2}(b-\beta_i)^2\right] db = 1 - \frac{1}{n}\sum_{i=1}^{n} p\left(B \leq \frac{D_f}{t}\right) .$$

Where B is a random variable distributed as $N(\beta_i, h^2)$. Let $U_i = \frac{B-\beta_i}{h}$ then B_i has standard normal distribution, i.e $U_i \sim N(0,1)$. Therefore,

$$\hat{F}_{T\_classical}\left(t\right) = 1 - \frac{1}{n}\sum_{i=1}^{n} \Phi\left(\frac{\frac{D_f}{t} - \beta_i}{h}\right)$$

Where $\Phi(u)$ is the standard normal cumulative distribution function. To estimate the $100p^{th}$ percentile t_p we need to solve $\hat{F}_{T\_classical}\left(\hat{t}_{p\_classical}\right) = p$ with respect to $\hat{t}_{p\_classical}$. This can be achieved by solving the following equation numerically with respect to $\hat{t}_{p\_classical}$

$$p = 1 - \frac{1}{n}\sum_{i=1}^{n} \Phi\left(\frac{\frac{D_f}{\hat{t}_{p_{classical}}} - \beta_i}{h}\right). \tag{6}$$

3.2 Variable Scale Kernel Method

While the smoothing parameter h in the classical kernel estimator (formula 3) is taken to be constant for a random sample $\beta_1, \beta_2, \ldots, \beta_n$, the variable scale estimator for $g_\beta(b)$ allows h to be vary for each value of X_i (*i=1,2,...,n*). For a random sample $X_1, X_2, \ldots, X_n$, the variable scale kernel estimator for $g_\beta(b)$is: (Abramson, 1982)

$$\hat{g}_{\beta-VS}\left(b,h\right) = \frac{1}{n}\sum_{i=1}^{n} \frac{1}{h\lambda_i} K\left(\frac{b-\beta_i}{h\lambda_i}\right)$$

By using the Gaussian kernel function, we obtain

$$\hat{g}_{\beta-VS}\left(b,h\right) = \frac{1}{n}\sum_{i=1}^{n} \frac{1}{h\lambda_i}\frac{1}{\sqrt{2\ \pi}} exp\left(-\frac{\left(b-\beta_i\right)^2}{2\left(h\lambda_i\right)^2}\right).$$

Therefore, from (2) we have

$$\hat{F}_{T-VS}\left(t\right) = 1 - \int_{-\infty}^{\frac{D_f}{t}} \hat{g}_{\beta-VS}\left(b,h\right) db$$

$$= 1 - \int_{-\infty}^{\frac{D_f}{t}} \frac{1}{n}\sum_{i=1}^{n} \frac{1}{h\lambda_i}\frac{1}{\sqrt{2\ \pi}} exp\left(-\frac{\left(b-\beta_i\right)^2}{2\left(h\lambda_i\right)^2}\right) db$$

$$= 1 - \frac{1}{n}\sum_{i=1}^{n} p\left(B \leq \frac{D_f}{t}\right),$$

where B is a random variable with distribution $N(\beta_i, (h\lambda_i)^2)$. Now, let $U_i = \frac{B-\beta_i}{h\lambda_i}$, then U_i has standard normal distribution, i.e $U_i \sim N(0,1)$. Thus,

$$\hat{F}_{T-VS}\left(t\right) = 1 - \frac{1}{n}\sum_{i=1}^{n}\Phi\left(\frac{\frac{D_f}{\hat{t}_{p-VS}} - \beta_i}{h\lambda_i}\right)$$

To estimate the $100p^{th}$percentile t_p, we need to solve $\hat{F}_{T-VS}\left(\hat{t}_{p-VS}\right) = p$ with respect to $\hat{t}_{p-VS}$ numerically to obtain the estimator value $\hat{t}_{p-VS}$. That is, solve the following equation numerically with respect to $\hat{t}_{p-VS}$

$$p = 1 - \frac{1}{n}\sum_{i=1}^{n}\Phi\left(\frac{\frac{D_f}{\hat{t}_{p-VS}} - \beta_i}{h\lambda_i}\right) \tag{7}$$

where λ_i is computed by using the formula: $\lambda_i = \left(\frac{\tilde{f}(x_i)}{\exp\left(\frac{1}{n}\sum_{i=1}^{n}\ln\left(\tilde{f}(x_i)\right)\ \right)}\right)^{-\frac{1}{2}}$, where $\tilde{f}(x_i)$ is the classical kernel estimator given by $\tilde{f}\left(x_i\right) = \frac{1}{nh}\sum_{i=1}^{n} K\left(\frac{x_i - x_j}{h}\right)$. By using the Gaussian Kernel function, we obtain $\tilde{f}\left(x_i\right) = \frac{1}{nh}\sum_{j=1}^{n}\frac{1}{\sqrt{\pi}}\exp\left(-\frac{(x_i-x_j)^2}{2h^2}\right)$.

3.3 Estimating the Time-To-Failure Distribution and its Percentiles by Ordinary Least Square (OLS) Method

Suppose that $\beta_1, \beta_2, \ldots, \beta_n$ is a random sample from a distribution with pdf $g_\beta(b, \mu)$ and distribution function $G_\beta(b, \mu)$. The OLS estimator $\widehat{\mu}_{OLS}$ of μ is obtained by minimizing

$$Q\left(\mu\right) = \sum_{i=1}^{n}\sum_{j=1}^{m}\left(y_{ij} - E(y_{ij})\right)^2$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{m}\left(y_{ij} - t_j E\beta_i\right)^2$$

with respect to μ. Therefore, the OLS estimator for time-to-failure distribution is

$$F_{OLS}\left(t\right) = 1 - G_\beta\left(\frac{D_f}{\hat{t}_{p\_OLS}}, \widehat{\mu}_{OLS}\right),$$

and the OLS estimator $\hat{t}_{p\_OLS}$ of t_p is obtained by solving

$$p = 1 - G_\beta\left(\frac{D_f}{\hat{t}_{p\_OLS}}, \widehat{\mu}_{OLS}\right),$$

with respect to $\hat{t}_{p\_OLS}$. In the rest of this subsection, we derive the OLS estimator for t_p when the distribution of the random effect β is assumed to be $halfnormal(\sigma^2)$ or $loglosistic(\alpha, 2)$.

Case I : if $\beta \sim\ halfnormal(\sigma^2)$ The OLS estimator of σ^2 is the value of $\widehat{\sigma}^2$ that minimizes $Q\left(\sigma^2\right)$, where $Q\left(\sigma^2\right) = \sum_{i=1}^{n}\sum_{j=1}^{m}\left(y_{ij} - E(y_{ij})\right)^2$. Since $E\left(y_{ij}\right) = \sqrt{\frac{2}{\pi}\sigma^2} t_j$, hence

$$\widehat{\sigma}^2_{OLS} = \frac{\pi}{2}\left(\frac{\sum_{i=1}^{n}\sum_{j=1}^{m} y_{ij}t_j}{n\ \sum_{j=1}^{m} {t_j}^2}\right)^2.$$

To obtain $\hat{t}_{p-OLS}$we need to solve $p = 1 - \hat{G}_{\beta-OLS}\left(\frac{D_f}{\hat{t}_{p-OLS}}, \widehat{\sigma}^2_{OLS}\right)$ with respect to $\hat{t}_{p-OLS}$ numerically. That is, we want to solve

$$p = 1 - \int_0^{D_f/\hat{t}_{p-OLS}} \frac{1}{\widehat{\sigma}_{OLS}}\sqrt{\frac{2}{\pi}} exp\left[-\frac{b^2}{2\widehat{\sigma}^2_{OLS}}\right] db \tag{8}$$

with respect to $\hat{t}_{p-OLS}$.

Case II : if $\beta \sim\ loglogistic(\alpha, 2)$ By considering the same procedures in case I, we can derive the OLS estimator of α by minimizing $Q\left(\alpha\right) = \sum_{i=1}^{n}\sum_{j=1}^{m}\left(y_{ij} - E(y_{ij})\right)^2, where E\left(y_{ij}\right) = \frac{\alpha\pi}{2\sin(\pi/2)} t_j$, Therefore,

$$\widehat{\alpha}_{OLS} = \frac{2\sin\left(\pi/2\right)}{n\pi}\left(\frac{\sum_{i=1}^{n}\sum_{j=1}^{m} y_{ij}t_j}{\sum_{j=1}^{m} {t_j}^2}\right)$$

Then to obtain $\hat{t}_{p-OLS}$ we need to solve $p = 1 - G_\beta\left(\frac{D_f}{\hat{t}_{p-OLS}}, \widehat{\alpha}_{OLS}\right)$ with respect to $\hat{t}_{p-OLS}$, which gives,

$$p = 1 - \int_{-\infty}^{D_f/\hat{t}_{p-OLS}} \frac{2b}{\widehat{\alpha}^2_{OLS}\left[1 + (b/\widehat{\alpha}_{OLS})^2\right]^2} db = 1 - \frac{1}{1 + \left(D_f/\widehat{\alpha}_{OLS}\hat{t}_{pOLS}\right)^{-2}}$$

Therefore, the OLS estimator of t_p is,

$$\hat{t}_{p-OLS} = \frac{D_f}{\widehat{\alpha}_{OLS}}\left(\frac{p}{1-p}\right)^{\frac{1}{2}} \tag{9}$$

3.4 Estimating the Time-To-Failure Distribution and its Percentiles by Maximum Likelihood (ML) Method

Consider the simple linear degradation model (1) and recall that our main aim is to estimate the time-to-failure distribution and then to estimate t_p. we will find the MLE of t_p for the following cases:

Case I: if $\beta \sim\ halfnormal(\sigma^2)$, then the time-to-failure distribution by using formula (2) is given by:

$$F_T(t) = 1 - \int_0^{D_f/t} \frac{1}{\sigma}\sqrt{\frac{2}{\pi}} exp\left[-\frac{b^2}{2\sigma^2}\right] db \tag{10}$$

So, by using Leibnit's Rule and by differentiate both sides of (10) with respect to t, then we obtain the probability density function of T which is,

$$f_T\left(t, \sigma^2\right) = \frac{D_f}{\sigma t^2}\sqrt{\frac{2}{\pi}} exp\left[-\frac{D_f^2}{2t^2\sigma^2}\right].$$

Now, let $t_1, t_2, \ldots, t_n$ be a random sample from $f_T\left(t, \sigma^2\right)$ then the likelihood function of σ^2 is ,

$$L\left(\sigma^2\right) = \left(\frac{2}{\pi\sigma^2}\right)^{\frac{n}{2}} D_f^n \left(\prod_{i=1}^n \left(\frac{1}{t_i^2}\right)\right) exp\left(-\frac{D_f^2}{2\sigma^2}\sum_{i=1}^n \frac{1}{t_i^2}\right).$$

Therefore,

$$\ln L(\sigma^2) = -\frac{n}{2}\ln\left(\frac{\pi}{2}\sigma^2\right) + n\ln D_f - \sum_{i=1}^n \ln t_i^2 - \frac{D_f^2}{2\sigma^2}\sum_{i=1}^n \frac{1}{t_i^2}.$$

The ML estimator of σ^2 is obtained by solving $\frac{d\ln L(\sigma^2)}{d\sigma^2} = 0$

$$\frac{d\ln L\left(\sigma^2\right)}{d\sigma^2} = -\frac{n}{2}\frac{1}{\sigma^2} + \frac{1}{\left(\sigma^2\right)^2}\frac{D_f^2}{2}\sum_{i=1}^n \frac{1}{t_i^2} = 0.$$

The ML estimator $\widehat{\sigma}^2_{MLE}$of $\sigma^2$is,

$$\widehat{\sigma}^2_{MLE} = \frac{D_f^2}{n}\sum_{i=1}^n \frac{1}{t_i^2}$$

Therefore, the ML estimator $\hat{t}_{p-MLE}$ of t_p is obtained by solving

$$p = 1 - \int_0^{D_f/\hat{t}_{p-MLE}} \frac{1}{\widehat{\sigma}_{MLE}}\sqrt{\frac{2}{\pi}} exp\left[-\frac{b^2}{2\widehat{\sigma}^2_{MLE}}\right] db \tag{11}$$

with respect to $\hat{t}_{p-MLE}$.

Case II: if $\beta \sim\ loglogistic(\alpha, 2)$, then the time-to-failure distribution by using formula

(2) is given by:

$$F_T(t) = 1 - \int_{-\infty}^{D_f/t} \frac{2b}{\alpha^2\left[1+(b/a)^2\right]^2} db = 1 - \frac{1}{1+(D_f/\alpha t)^{-2}} \tag{12}$$

So, the probability density function of T is

$$f_T(t,\alpha) = \frac{2}{t}\frac{\left(\frac{D_f}{\alpha t}\right)^2}{\left(1+\left(\frac{D_f}{\alpha t}\right)^2\right)^2}$$

Let $t_1, t_2, \ldots, t_n$ be a random sample from $f_T(t,\alpha)$ then the likelihood function of $\alpha$is ,

$$L(\alpha) = \frac{2^n}{\prod_{i=1}^n t_i^3}\left(\frac{D_f}{\alpha}\right)^{2n}\frac{1}{\prod_{i=1}^n\left(1+\left(\frac{D_f}{\alpha t_i}\right)^2\right)^2}$$

$$\ln L(\alpha) = -2n\ln(\alpha) - \sum_{i=1}^n \ln\left(1+\left(\frac{D_f}{\alpha t_i}\right)^2\right)^2 + constant$$

The ML estimator $\widehat{\alpha}_{MLE}$ of α is now obtained by solving the following equation numerically with respect to $\widehat{\alpha}_{MLE}$,

$$\frac{d\ln L(\alpha)}{d\alpha} = -\frac{2\ n}{\widehat{\alpha}_{MLE}} + 4\sum_{i=1}^n \frac{\widehat{\alpha}_{MLE} D_f^2 t_i^2}{\left(D_f^2 + \widehat{\alpha}_{MLE} t_i^2\right)^2} = 0 \tag{13}$$

Now, substitute the value $\widehat{\alpha}_{MLE}$ that obtained from (13) back into (12) to obtain the ML estimator for time-to-failure distribution. To find the ML estimator$\hat{t}_{p-MLE}$of t_p, we need to solve the following equation with respect to $\hat{t}_{p-MLE}$,

$$p = 1 - \frac{1}{1+\left(D_f/\widehat{\alpha}_{MLE}\hat{t}_{p-MLE}\right)^{-2}},$$

which gives the ML estimator of t_p,

$$\hat{t}_{p-MLE} = \frac{D_f}{\widehat{\alpha}_{MLE}}\left(\frac{p}{1-p}\right)^{\frac{1}{2}} \tag{14}$$

4 Simulation Study and Results

In this section a simulation study is conducted to compare the performances of the classical kernel, variable scale kernel, OLS and ML estimators of the $100p^{th}$ percentile of time-to-failure distribution, t_p. The performances of the different estimators are studied by

computing the bias and the MSE of classical estimator$\hat{t}_{p_classical}$, variable scale estimator $\hat{t}_{p\_VS}$, OLS estimator $\hat{t}_{p\_OLS}$ and ML estimator $\hat{t}_{p\_MLE}$.
The smoothing parameter h is then computed by using formula (5), the kernel estimators of t_p are then computed by solving equations (6) and (7) numerically, this given $\hat{t}_{p-classical}$ and $\hat{t}_{p-VS}$ for classical kernel, variable scale kernel estimators respectively. Using equations (8) and (11) to find the OLS estimator $\hat{t}_{p\_OLS}$ and ML estimator $\hat{t}_{p\_MLE}$ when $\beta \sim half\ normal\,(5)$ and equations (9) and (14) when $\beta \sim loglogistic(2, 5)$. Note that $D_f = 20$ and $p = \{0.5,\ 0.3\ and\ 0.5\}$

From tables 1-6 we conclude that:

1. The method of ML is the most efficient method to estimate t_p when the distribution of random effect parameter β is assumed to be known. The MLE has the smallest MSE with most accurate.

2. The biases (B) of the classical kernel and variable scale kernel methods are negative in most cases which indicates that the different kernels method are under estimate the true value of t_p.

3. The exact t_p increases as p increases and also the MSE of $\hat{t}_p$ (for all estimators) increases as p increases for the same sample size.

4. For the same value of p, the MSE of $\hat{t}_p$ decreases as n (sample size) increases for all estimators.

Table (1) B and MSE of the different estimators when the sample is taken from $half normal(5)$, n=20

p	Exact t_p	Classical kernel estimator		Variable scale kernel estimator		OLS		MLE	
		Bclass	MSEclass	Bvs	MSEvs	B_{OLS}	MSE_{OLS}	Bmle	MSE_{MLE}
0.1	2.4318	-0.265	0.1991	-0.216	0.18522	0.051	0.1935	0.096	0.18636
0.3	3.8593	-0.395	0.5445	-0.224	0.5166	0.122	0.5353	0.151	0.4411
0.5	5.9304	-0.391	1.25	-0.2025	1.3432	0.181	1.1239	0.225	1.1511

Table (2) B and MSE of the different estimators when the sample is taken from $half normal(5)$, n=40

p	Exact t_p	Classical kernel estimator		Variable scale kernel estimator		OLS		MLE	
		Bclass	MSEclas	Bvs	MSEvs	B_{OLS}	MSE_{OLS}	Bmle	MSE_{MLE}
0.1	2.4318	-0.1834	0.1043	-0.127	0.0947	0.0387	0.0928	0.0416	0.08137
0.3	3.8593	-0.2987	0.2927	-0.102	0.2576	0.0606	0.2443	0.0651	0.20265
0.5	5.9304	-0.3738	0.7544	-0.1793	0.7746	0.0703	0.5365	0.0889	0.47814

Table (3) B and MSE of the different estimators when the sample is taken from $halfnormal(5)$, n=60

P	Exact t_p	Classical kernel estimator		Variable scale kernel estimator		OLS		MLE	
		Bclaa	MSEclass	Bvs	MSEvs	B_{OLS}	MSE_{OLS}	Bmle	MSE_{MLE}
0.1	2.4318	-0.145	0.070036	-0.091	0.062117	0.034	0.060242	0.032	0.05224
0.3	3.8594	-0.237	0.19398	-0.043	0.170390	0.045	0.151175	0.059	0.13389
0.5	5.9304	-0.328	0.5430	-0.137	0.544929	0.049	0.360412	0.091	0.33079

Table (4) B and MSE of the different estimators when the sample is taken from $loglogistic(2, 5)$, n=20

P	Exact t_p	Classical kernel estimator		Variable scale kernel estimator		OLS		MLE	
		Bclass	MSEclass	Bvs	MSEvs	B_{OLS}	MSE_{OLS}	Bmle	MSE_{MLE}
0.1	1.3333	-0.200	0.18606	-0.139	0.167	0.134	0.17345	0.023	0.085604
0.3	2.6186	-0.647	0.71297	-0.505	0.57	0.289	0.72891	0.061	0.33949
0.5	4.00	-0.665	0.97786	-0.512	0.802	0.426	1.63329	0.067	0.78178

Table (5) B and MSE of the different estimators when the sample is taken from $loglogistic(2, 5)$, n=40

P	Exact t_p	Classical kernel estimator		Variable scale kernel estimator		OLS		MLE	
		Bclass	MSEclass	Bvs	MSEvs	B_{OLS}	MSE_{OLS}	Bmle	MSE_{MLE}
0.1	1.3333	-0.200	0.12624	-0.134	0.10827	0.064	0.09389	0.011	0.03353
0.3	2.6186	-0.590	0.52429	-0.425	0.37481	0.162	0.37384	0.024	0.16164
0.5	4.00	-0.630	0.66096	-0.44	0.47972	0.244	0.843	0.054	0.3123

Table (6) B and MSE of the different estimators when the sample is taken from $loglogistic(2, 5)$, n=60

P	Exact t_p	Classical kernel estimator		Variable scale kernel estimator		OLS		MLE	
		Bclass	MSEclass	Bvs	MSEvs	B_{OLS}	MSE_{OLS}	Bmle	MSE_{MLE}
0.1	1.3333	-0.167	0.09662	-0.099	0.0808	0.055	0.07058	0.013	0.02291
0.3	2.6186	-0.538	0.4111	-0.359	0.2671	0.123	0.24414	0.024	0.08855
0.5	4.00	-0.597	0.53747	-0.401	0.3535	0.169	0.59379	0.031	0.2025

5 Real Data Application

The analysis of the real data for estimating the time-to-failure distribution and its percentile by parametric and nonparametric methods is demonstrated in this section. We will use the Laser data from Meeker and Escobar (2014), Table C.17 page 742. The analysis of the real data is divided into two subsections. In the first section we will use the real data to estimate the time-to-failure distribution; $\hat{F}_{classical}(t)$, $\hat{F}_{VS}(t)$, $\hat{F}_{OLS}(t)$ and $\hat{F}_{MLE}(t)$. In the second subsection we will use the real data to estimate t_p using the different estimation methods then make a comparison between these methods.

5.1 Data Description

Meeker and Escobar (2014) presented the percent increase in laser operating current for GaAs laser test at 80° C, as a real data of degradation. In our analysis we will assume that the degradation level is $D_f = 5$. The data contains fifteen units and sixteen times (fifteen units, each unit of size sixteen), the times ranges from 250-4000 hours with step equal to 250 hours. Under the case when there is no assumption on the distribution of the random effect parameter β, the classical kernel method and variable scale kernel method are used to obtain $\hat{F}_{classical}(t)$ and $\hat{F}_{VS}(t)$. Then under the case of the random effect parameter β has a known distribution function (e.g $halfnormal\left(\sigma^2\right)$ or $loglogistic(\alpha, 2)$, the OLS and MLE methods are used to estimate the time-to-failure distribution; $\hat{F}_{OLS}(t)$ and $\hat{F}_{MLE}(t)$. Table (7) presents the estimate of the time-to-failure distribution. For the propose of comparison, we calculate $\hat{F}_{T\_emp}(t)$ the empirical distribution of the failure times.

5.2 Estimating the 50^{th} percentile of time-to-failure distribution

The 50^{th} percentile of time-to-failure distribution is also estimated by two cases. First case we estimate it with no assumption on the random effect parameter by kernel estimator's; $\hat{t}_{p\_classical}$ or $\hat{t}_{p\_scale}$, then the B and MSE are computed.
Table (8) represents the B and MSE of the estimate of 50^{th} percentiles of time-to-failure distribution by classical and variable scale kernel methods.
Second case, we assume that the random effect parameter β has known distribution, then we estimate the 50^{th} percentile of time-to-failure distribution by OLS and ML estimator. Finally, we evaluate the **B** and **MSE**. Table (9) represents the B and MSE of the estimate of 50^{th} percentile of the time-to-failure by OLS and MLE methods.

5.3 Conclusions for Real Data

1. Table (7) shows that the time-to-failure distribution which was estimated by the classical kernel method and the variable scale kernel method are approximately correspondence (in real data) and they are closed to the empirical distribution function rather than the OLS and MLE.

Table (7) The estimate of time-to-failure distribution

t_i^*	$\hat{F}_{emp}(t_i)$	$\hat{F}_{classical}(t_i)$	$\hat{F}_{VS}(t_i)$	$halfnormal(\sigma^2)$		$loglogistic(\alpha, 2)$	
				$\hat{F}_{OLS}(t_i^*)$	$\hat{F}_{MLE}(t_i^*)$	$\hat{F}_{OLS}(t_i^*)$	$\hat{F}_{MLE}(t_i^*)$
6.6667	0.06667	0.250306	0.247999	0.0682	0.00716	0.0682	0.00716
7.1479	0.13333	0.297225	0.294535	0.0889	0.01207	0.0889	0.01207
7.5732	0.20000	0. 336397	0.333469	0.1084	0.01783	0.1084	0.01783
8.0192	0.26667	0.374722	0.371693	0.1295	0.02526	0.1295	0.02526
8.9730	0.33333	0.446989	0.444029	0.1754	0.04554	0.1754	0.04554
9.4884	0.40000	0.480798	0.477973	0.2000	0.05862	0.2000	0.05862
10.2093	0.46667	0.52257	0.519991	0.2337	0.07884	0.2337	0.07884
10.5000	0.53333	0.537767	0.535297	0.2469	0.08748	0.2469	0.08748
10.6066	0.60000	0.54312	0.540691	0.2516	0.09071	0.2516	0.09071
11.2931	0.66667	0.574995	0.572832	0.2816	0.11210	0.2816	0.11210
11.6707	0.73333	0.590771	0.588753	0.2975	0.1242	0.2975	0.1242
12.0667	0.80000	0.606121	0.604251	0.3136	0.13703	0.3136	0.13703
12.4933	0.86667	0.62141	0.61965	0.3304	0.15096	0.3304	0.15096
12.5926	0.93333	0.624796	0.623117	0.3343	0.15420	0.3343	0.15420
12.600	1.0000	0.625046	0.623369	0.3345	0.15445	0.3345	0.15445

2. The distribution function of time-to-failure estimated by the variable scale kernel method is the closet distribution to the empirical comparing with the distribution function estimated by ML and OLS methods.

3. Tables (8) and (9) show that the bias and MSE for the estimate of 50^{th} percentile of time-to-failure by parametric and nonparametric. In the most cases, the variable scale kernel estimator records the smallest MSE and closet bias to zero while the ML estimator has the largest MSE and the farthermost bias to zero.

Table (8) B and MSE of the $\hat{t}_{p\_classical}$, $\hat{t}_{p\_location}$ and $\hat{t}_{p\_scale}$

Estimators	B	MSE
$\hat{t}_{p\_classical}$	-0.646416	0.738476
$\hat{t}_{p\_scale}$	-0.606964	0.699004

Table (9) The B and MSE of the estimate of $50p^{th}$ percentile of the time-to-failure by OLS and MLE methods

Estimators	$halfnormal(\sigma^2)$		$loglogistic(\alpha, 2)$	
	B	**MSE**	**B**	**MSE**
$\hat{t}_{p\_OLS}$	1.10716	1.66081	-0.568	0.621998
$\hat{t}_{p\_MLE}$	3.60531	13.6942	4.874	24.5268

6 Conclusions

When the assumption on the distribution of random effect parameter β is invalid then the performances of the MLE and OLS estimators are very poor and the performances of the variable scale kernel estimator is nearly the best one.

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