Electronic Journal of Applied Statistical Analysis
EJASA, Electron. J. App. Stat. Anal.
http://siba-ese.unisalento.it/index.php/ejasa/index
e-ISSN: 2070-5948
DOI: 10.1285/i20705948v11n2p140

A comparison of process capability indices for seasonal and non-seasonal autoregressive auto-correlated data

By Al-Zoubi, Smadi

Published: 26 April 2019

Electronic Journal of Applied Statistical Analysis
Vol. 12, Issue 01, April 2019, 140-152
DOI: 10.1285/i20705948v11n2p140

A comparison of process capability indices for seasonal and non-seasonal autoregressive auto-correlated data

Ayat I. Al-Zoubi[a] and Abdullah A. Smadi*[b]

[a]*Department of Applied Science, Al-Balqa' Applied University, Ajloun, Jordan*
[b]*Department of Statistics, Yarmouk University, Irbid, Jordan*

Published: 26 April 2019

Traditionally, the main assumptions often used in practice for calculating process capability indices are that process data being independent and normally distributed. In this research we study the effects of the autocorrelation structure among process data on the distributional properties for the sample version of the most three common process capability indices in terms of bias, MSE and empirical distribution. Previous studies investigated ordinary autocorrelation structures among data modeled by some ordinary *ARMA* models. Here, we investigate a different autocorrelation structure, namely the first order seasonal autoregressive ($SAR(1)$) model in which the seasonal auto-correlation structure is apparent. For the sake of completeness, we have also considered two other models for process data, namely the ordinary first order autoregressive ($AR(1)$) model and the multiplicative seasonal AR model of orders (1, 1). Assuming that the process data follow those models with normal error terms, Monte-Carlo simulations are carried out using R. The results showed that the characteristics of sample process capability indices are negatively affected by the autocorrelation among data, especially for the multiplicative seasonal AR model. Besides, we found that the empirical distributions of the three sample capability indices are positively skewed and leptokurtic, a fact which is merely true when the data are independent and normal.

keywords: Process Capability Indices, Autocorrelation, Statistical Process Control, Seasonal ARMA Models.

*Corresponding author: asmadi@yu.edu.jo

ISSN: 2070-5948
http://siba-ese.unisalento.it/index.php/ejasa/index

1 Introduction

Quality control is a process that ensures a certain level of quality is achieved in a product or service. The main statistical techniques for quality control include statistical process control (SPC), design of experiments and acceptance sampling. The SPC method includes control chart and process capability. For more details on these issues see Montgomery (2009).

The process capability is defined as the ability of the production process to produce conforming products. Indices of process capability give a clear indication of the capability of a manufacturing process. The main assumption in process capability is the process data are independent and normally distributed. However, in real industries data are often autocorrelated (see for example, Shore (1997) and Guevara and Vargas (2007).

A process capability index (PCI) is a numerical measure defined to evaluate process capability. The essence of such measure is to compare the behavior of a product or some of its characteristics with its engineering specifications (Wen-lea and Samuel, 2006). Usually, a large value of such indices indicates that the current process is capable (Steiner et al., 1997). In the literature of statistical process control (SPC) a large number of PCIs were defined (see Guevara and Vargas (2007)). The most popular indices used in the industry, which we also consider in this research, are defined as:

$$C_p = \frac{USL - LSL}{6\sigma},$$

$$C_{pk} = (\frac{USL - \mu}{3\sigma}, \frac{\mu - LSL}{3\sigma}),$$

and

$$C_{pm} = \frac{USL - LSL}{6\sqrt{\sigma^2 + (\mu - T)^2}}$$

with sample versions, given respectively as:

$$\widehat{C}_p = \frac{USL - LSL}{6s} \tag{1}$$

$$\widehat{C}_{pk} = (\frac{USL - \overline{X}}{3s}, \frac{\overline{X} - LSL}{3s}) \tag{2}$$

and

$$\widehat{C}_{pm} = \frac{USL - LSL}{6\sqrt{s^2 + (\overline{X} - T)^2}} \tag{3}$$

where LSL and USL are respectively the lower and the upper specification limits, σ is the (theoretical) process standard deviation, s is the sample standard deviation, μ is the process theoretical mean, $(\overline{X})$ is the sample mean, and T is the target specification value (Wen-lea and Samuel, 2006). Notice that $\widehat{C}_p$ is solely based on the variability of data while the other two measures depend on both center and variability of data.

Under the assumptions of normality and independence of process data, Chou and Owen (1989) derived the probability density function, the mean and the variance for

$\widehat{C}_p$ defined in (1) while Pearn et al. (1998) obtained the UMVUE of C_p. Kotz et al. (1993) derived the r^{th} moments of $\widehat{C}_{pk}$ and proved that this estimator is asymptotically unbiased. They also proved that $\widehat{C}_{pm}$ defined in (3) is the MLE of C_{pm}. A detailed account on the distributional properties of the three sample PCIs defined in (1) to (3) for independent and normal data is found in Wen-lea and Samuel (2006). They reported that all the three sample measures are biased but asymptotically unbiased with $\widehat{C}_p$ and $\widehat{C}_{pk}$ having positive bias while $\widehat{C}_{pm}$ have a negative bias especially for large sample size.

Besides, many researches questioned the validity of the independence assumption in real process data and examined the effect of such dependence on sample PCIs. For instance,Shore (1997) discussed the effects of autocorrelation data on capability indices in some depth. He concluded that bias and variability of the sample PCIs are considerably affected when autocorrelation among data is present. In addition, in a comparison between a process with independent observations and a process data following the ordinary first order autoregressive model, Guevara and Vargas (2007) concluded that the bias of sample PCIs is affected by the direction and strength of autocorrelation in data. Alzoubi (2013) investigated the effect of autocorrelation among data based on low-orders autoregressive moving average (ARMA) models. She concluded that bias, MSE and empirical distributions of sample PCIs are affected by the direction, the strength and the type of autocorrelation within data.

Having a deep look at the previous research regarding sample PCIs with autocorrelated data, we noticed that ordinary autocorrelation structures are considered via some selected ARMA models. An interesting type of autocorrelation structure, namely the seasonal autocorrelation structure is the main issue in this research. In general, seasonality in time series data is frequently encountered in practice. Therefore, we will consider process data that have such kind of autocorrelation structure by assuming that those data follow the first order seasonal autoregressive $SAR(1)$ model. In addition, and in order to link the results of this research with previous works in the area we considered also the ordinary first order autoregressive ($AR(1)$) model. Besides, we considered a third model with autocorrelation structure that mixes both the ordinary and seasonal types, namely, the multiplicative seasonal autoregressive model of orders (1, 1). More details about these models and their autocorrelation functions are given in the next section.

Thus, our main objective in this article is to study the distributional properties of the three sample PCIs defined in (1) to (3) including bias, MSE and some aspects of the shape of their empirical distributions, namely skewness and kurtosis. The study will be carried out using Monte-Carlo simulation assuming that process data are autocorrelated according to the three models mentioned above. The parameters of simulations and related settings are explained in section 3. In the next section we explain the time series models considered in this research along their autocorrelation functions and some other properties.

2 The Selected Time Series Models

Now, we will define three different time series models that belong to the autoregressive type. These models will be considered in this research for simulating autocorrelated process data. These three models are special cases of the general seasonal autoregressive moving average (SARMA) models.

The zero-mean SARMA model of orders (p, q) and (P, Q) is written as (Box et al., 2008):

$$\phi_p(B)\Phi_P(B^\omega)X_t = \theta_q(B)\Theta_Q(B^\omega)\alpha_t \tag{4}$$

where

$$\phi_p(B) = 1 - \phi_1 B - \phi_2 B^2 - \cdots - \phi_p B^p$$
$$\Phi_P(B^\omega) = 1 - \Phi_1 B^\omega - \Phi_2 B^{2\omega} - \cdots - \Phi_P B^{P\omega}$$
$$\theta_q(B) = 1 - \theta_1 B - \theta_2 B^2 - \cdots - \theta_q B^q$$

and

$$\Theta_Q(B^\omega) = 1 - \Theta_1 B^\omega - \Theta_2 B^{2\omega} - \cdots - \Theta_Q B^{Q\omega}$$

are, respectively, the ordinary AR, seasonal AR, ordinary MA and seasonal MA polynomials of orders p, Q, q, Q while B is the backshift operator and ω is the seasonal length. This model is denoted by the $SARMA(p, q) \times (P, Q)_\omega$ in which p and q represent the non-seasonal orders and P and Q represent the seasonal orders. If $P = Q = 0$, then model (4) reduces to the traditional $ARMA(p, q)$ model while if $p = q = 0$, then it reduces to the purely seasonal $ARMA(P, Q)$ model. In case at least one of the ordinary orders p and q is non-zero as well as at least one of the seasonal orders P and Q is non-zero, then the resulting model is called a multiplicative SARMA model.

In this research we will consider three special cases of model (4) that all belong to the autoregressive type. The first one is the zero-mean ordinary first order autoregressive $AR(1)$ model, which results from the general SARMA model above with $p = 1$, $P = Q = q = 0$, and ω =1. Thus, it is written a

$$X_t = \phi X_{t-1} + \alpha_t \tag{5}$$

where ϕ is the AR parameter satisfying $|\phi| < 1$ for stationarity and $\{\alpha_t\}$ is a purely random process which we assume iid $N(0, \sigma_\alpha^2)$ through out this article (for all selected models). This model is widely used for modeling autocorrelated data. The autocorrelation function (ACF) of this model is given by

$$\rho_k = \phi^k, k = 0, 1, \cdots$$

which has an exponential decay pattern. An example of this ACF is shown in Figure (1) for positive and negative values of ϕ. Notice that autocorrelations exist at all lags with an alternating behavior for negative values of ϕ.

The second model is the zero-mean first order seasonal autoregressive $SAR(1)_\omega$ model, where ω is the seasonal period which we set in this article to 4. The $SAR(1)_4$ model is a special case of the $SARMA(p,q) \times (P,Q)_\omega$ model (4) with $P = 1, p = q = Q = 0$, and $\omega = 4$. Thus the zero-mean $SAR(1)_4$ model is written as:

$$X_t = \Phi X_{t-4} + \alpha_t \tag{6}$$

where Φ is the seasonal AR parameter satisfying $|\Phi| < 1$ for stationarity. The ACF of this model is written as:

$$\rho_k = \begin{cases} \Phi^{(\frac{k}{4})} & k = 0, 4, 8, \cdots \\ 0 & otherwise \end{cases}$$

Again, an example of this ACF is shown in Figure (1) . It is seen that autocorrelations exist only at seasonal lags. However, exponential decay is also seen for autocorrelations at seasonal lags. Thus, the ACF here has gaps of non-autocorrelations. Those gaps increase as the seasonal length ω gets larger.

Moreover, the zero-mean multiplicative seasonal autoregressive $SAR(1) \times (1)_\omega$ model is a special case of the $SARMA(p,q) \times (P,Q)_\omega$ model (4) when $P = p = 1, Q = q = 0$ and $\omega = 4$. The $SAR(1) \times (1)_\omega$ model combines the idea of seasonal and nonseasonal autoregressive models. The zero-mean $SAR(1) \times (1)_4$ model is derived from (4) as:

$$(1 - \phi B)((1 - \Phi B^4)X_t = \alpha_t$$

which is then simplified as:

$$X_t = \phi X_{t-1} + \Phi X_{t-4} - \phi\Phi X_{t-5} + \alpha_t \tag{7}$$

where ϕ and Φ are respectively the ordinary and seasonal AR parameters. This model is stationary iff $|\phi| < 1$ and $|\Phi| < 1$(see, Cryer and Chan (2008)). Unfortunately, there is no closed formula for the ACF of the $SAR(1) \times (1)_\omega$ model, although Yule-Walker equations can be developed for recursive comparison of autocorrelation of this model (see Cryer and Chan (2008)). For computation purposes we developed an R-code that utilizes the R-command (ARMAacf). In Figure (2) we illustrate the ACF of the $SAR(1) \times (1)_4$ model in (7) for four pairs of ϕ and Φ. Note that the ACF of this model can take several patterns according to the choice of its parameters ϕ and Φ. Sometimes seasonality in the ACF of this model can be clearly seen as in Figure (2, A) and Figure (2, B). However, autocorrelations are present at seasonal as well as other lags.

A detailed account on the three models introduced earlier as well as other SARMA models, their ACF, other properties and applications is found, for example, in Cryer and Chan (2008).

In the next section we define the settings of our simulation study that focuses on some statistical properties of the sample PCIs defined in (1) - (3) for autocorrelated data governed by models (5), (6) and (7). Then the results of our study are summarized and then discussed.

3 A Simulation study

As we have mentioned above, the main objective of this article is to study the bias, MSE as well as some statistical properties of the empirical distribution for the sample capability indices $\widehat{C}_p$, $\widehat{C}_{pk}$,$\widehat{C}_{pm}$ defined in (1) to (3) when the process data are autocorrelated according the $AR(1), SAR(1)$ and $SAR(1) \times (1)_4$ models. Therefore, we developed an R-code to obtain the empirical relative bias (RB) as well as the relative root mean square error ($RRMSE$) defined respectively as:

$$RB(\widehat{\theta}) = \frac{1}{\theta}\widehat{B}ias(\widehat{\theta}) = \frac{1}{\theta}(\frac{\sum \widehat{\theta}_i}{r} - \theta), \qquad \theta \neq 0$$

and

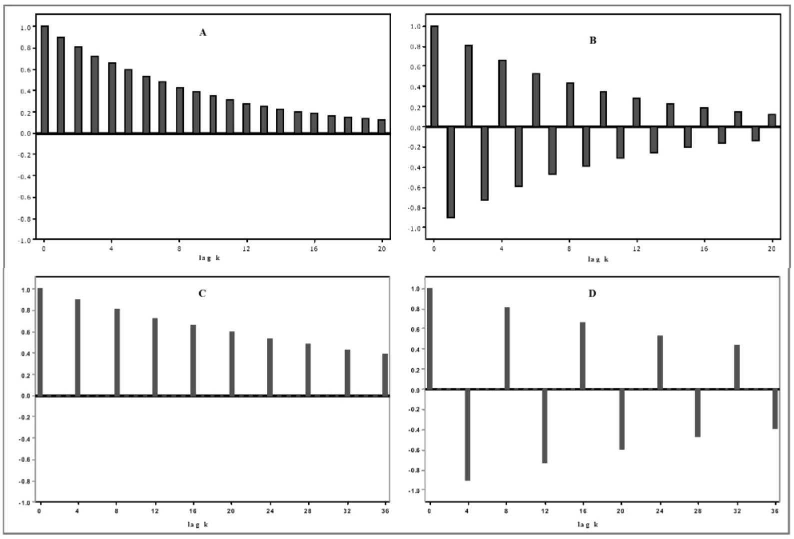

Figure 1: The theoretical ACF of $AR(1) : \phi = 0.9, AR(1) : \phi = -0.9$, (A and B),and the theoretical ACF of $SAR(1)_4 : \Phi = 0.9$, $SAR(1)_4 : \Phi = -0.9$, (C and D).

$$RRMSE(\widehat{\theta}) = \frac{1}{\theta}\sqrt{\frac{1}{r}\sum(\widehat{\theta}_i - \theta)^2}, \qquad \theta \neq 0$$

where r is the number of iterations, θ is any parameter and $\widehat{\theta}_i$ is the estimate of θ from the i-th iteration, $i = 1, \cdots, r$. The simulated data from various time series models were carried out using the R-command "arima.sim". All the computations were done based on R, Version 3.4.2 (Team, 2017).

Now, for the parameters of our simulations we have selected $r = 10,000, LSL = -3, USL = 3$and $\mu = T = 0$. Besides, the variance of error terms σ^2_α is selected so that $Var(X_t) = 1$ for all selected models. Accordingly, the values of the three theoretical measuresC_p, C_{pm} and C_{pk} are all equal 1.

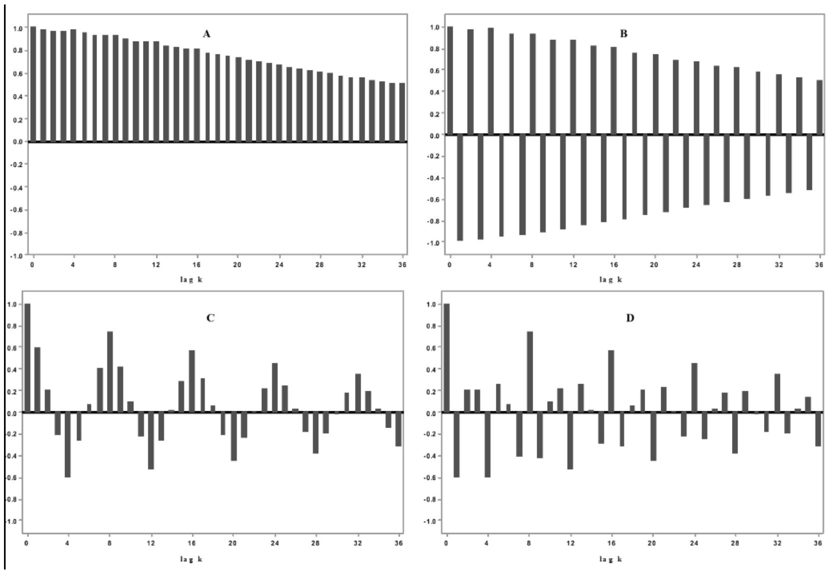

Figure 2: The theoretical ACF of $SAR(1) \times (1)_4$: $(\phi, \Phi) = (0.9, 0.9), (-0.9, 0.9), (0.9, -0.9)$ and $(-0.9, -0.9)$, in sub-graphs (A) to (D), respectively.

We recall that $Var(X_t)$ (usually denoted as γ_0) for models (5), (6) and (7) are given respectively by (see, Cryer and Chan (2008)):

$$\gamma_0 = \frac{\sigma_\alpha^2}{1-\phi^2},$$

$$\gamma_0 = \frac{\sigma_\alpha^2}{1-\Phi^2},$$

and

$$\gamma_0 = \frac{\sigma_\alpha^2}{1-\phi\rho_1-\Phi\rho_4-\phi\Phi\rho_5}$$

where ρ_1, ρ_4 and ρ_5 are the autocorrelation values at lags 1, 4 and 5, respectively. Thus, fixing $Var(X_t) = \gamma_0$ at one, we use the formulas above to find σ_α^2 at each selected value(s) of AR parameters. As for the AR parameters, we chose them in $\{-0.8, -0.4, 0, 0.4, 0.8\}$ for both ϕ and Φ for all models. Finally the sample size (or, realization length) is chosen as $\{75, 150, 300\}$.

Now, the simulation results regarding the RB and $RRMSE$ for the three sample capability indices defined in (1) to (3) for the $AR(1)$, $SAR(1)_4$ and $SAR(1) \times (1)_4$ models are summarized in Tables (1) to (3), respectively.

Another issue we are concerned with in this research is to study the behaviors of the empirical distributions for the sample capability indices defined in (1) to (3) in view of the autocorrelation structure of data and the sample size. To accomplish this objective, we may use the empirical density function (obtained using R-command "density"). Figure (3) is an illustration for this method which presents the empirical distribution for $\widehat{C}_p$ when the data follow the $SAR(1)_4$ model for $n = 75$ and $n = 300$. However, due to the large number of cases we have in this study as well the difficulty to accurately compare such curves; we decided to investigate the skewness and kurtosis of the sample distributions for simulated data. To do this we have considered the $SAR(1) \times (1)_4$ model with both ϕ and Φ are selected in $\{-0.8, -0.4, 0, 0.4, 0.8\}$ which covers the ordinary $AR(1)$ as well as the $SAR(1)_4$ models corresponding to $\Phi = 0$ and $\phi = 0$, respectively. The measures of skewness and kurtosis are computed based on the same setting as above using the R-commands "skewness" and "kurtosis" involved in the R-package "moments". The results regarding skewness and kurtosis are summarized in Table (4).

Table 1: The values of RB and RRMSE (in parentheses) of several capability indices for AR(1) data

		$\widehat{C}_p$			$\widehat{C}_{pk}$			$\widehat{C}_{pm}$		
	$n =$	75	150	300	75	150	300	75	150	300
		0.036	0.020	0.008	0.025	0.012	0.003	0.042	0.022	0.010
	-0.8	(0.179)	(0.124)	(0.089)	(0.175)	(0.123)	(0.088)	(0.181)	(0.125)	(0.089)
		0.009	0.005	0.002	-0.011	-0.010	-0.008	0.013	0.006	0.003
	-0.4	(0.098)	(0.068)	(0.049)	(0.097)	(0.068)	(0.049)	(0.098)	(0.068)	(0.049)
ϕ		0.010	0.005	0.003	-0.020	-0.017	-0.012	0.010	0.005	0.003
	0	(0.085)	(0.058)	(0.041)	(0.088)	(0.061)	(0.044)	(0.085)	(0.058)	(0.041)
		0.021	0.011	0.005	-0.026	-0.022	-0.018	0.012	0.007	0.003
	0.4	(0.101)	(0.071)	(0.048)	(0.104)	(0.076)	(0.054)	(0.098)	(0.070)	(0.048)
		0.098	0.050	0.024	0.000	-0.017	-0.023	0.041	0.023	0.010
	0.8	(0.210)	(0.136)	(0.091)	(0.184)	(0.130)	(0.094)	(0.181)	(0.125)	(0.087)

4 Discussion of Results

Starting with Figure (3), it is seen that for the $SAR(1)_4$ model, the bias, variability and shape of distribution of $\widehat{C}_p$ is affected by the sample size as well as the value of seasonal AR parameter (Φ). This result is assured for all sample capability indices in view of Tables (1) to (4).

Now, for the RB and $RRMSE$, Tables (1) to (3) show that both measures decrease when the size of data n increases. These measures are also affected by the autocorrelation structure, the strength and direction of autocorrelation. For $AR(1)$ and $SAR(1)_4$ models, as each of $|\phi|$ and $|\Phi|$ approaches one, then RB and $RRMSE$ increase for all the three

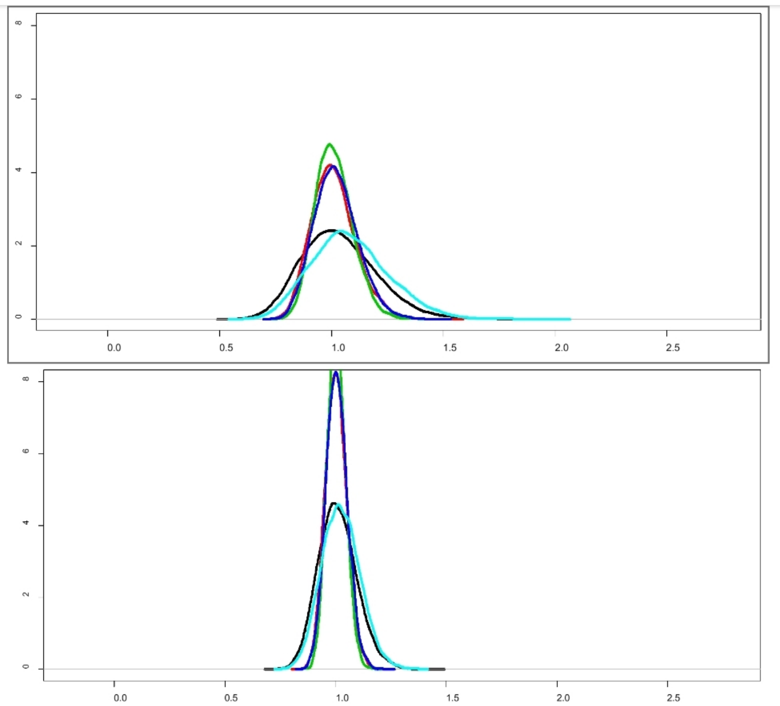

Figure 3: The empirical distribution of $\widehat{C}_p$ under $SAR(1)_4$ model with $\Phi \in \{-0.8, -0.4, 0, 0.4, 0.8\}$ for $n = 75$ (above) and $n = 300$ (down)

capability indices, respectively. A similar conclusion applies for the $SAR(1) \times (1)_4$ model when either $|\phi|$ and $|\Phi|$ approaches one.

Again, for $AR(1)$ and $SAR(1)_4$ models, at the same value of n and for the same values of ϕ and Φ, it seems that corresponding values of RB and $RRMSE$ are nearly the same, although slightly smaller for the $SAR(1)_4$ model. This point shows that the gaps within the ACF of the $SAR(1)_4$ as compared to that of the $AR(1)$ model (pictured in Figure (1)) did not strongly impact the values of RB and $RRMSE$. On the other hand, for the same models, it is seen from Tables (1) and (2) that for $\widehat{C}_p$ and $\widehat{C}_{pk}$ the values of RB and $RRMSE$ are often smaller for negative AR parameter as for corresponding positive values. For $\widehat{C}_{pm}$ it seems that the values of RB and $RRMSE$ are affected by the strength of autocorrelation but not its direction.

For the $SAR(1) \times (1)_4$ data, it is seen in Table (3) that RB and $RRMSE$ values for the three measures are affected by the values of both AR parameters (ϕ, Φ). It is however seen that the largest values of RB and $RRMSE$ for $\widehat{C}_p$ and $\widehat{C}_{pk}$ are seen when $(\phi, \Phi) = (0.8, 0.8)$ and for $\widehat{C}_{pm}$ when $(\phi, \Phi) = (0.8, 0.8)$ as well as $(\phi, \Phi) = (-0.8, 0.8)$.

As far as the bias of the three sample capability indices is concerned, it is seen in Tables(1) to (3) that for all the models we have in this research, $\widehat{C}_p$ and $\widehat{C}_{pm}$ overestimate C_p and C_{pm}, respectively, while $\widehat{C}_{pk}$ (often) underestimates C_{pk}. This point agrees with the result regarding the bias of the three measures for independent and normal data

Table 2: The values of RB and RRMSE (in parentheses) of several capability indices for $SAR(1)_4$ data

		$\widehat{C}_p$			$\widehat{C}_{pk}$			$\widehat{C}_{pm}$		
	$n =$	75	150	300	75	150	300	75	150	300
	-0.8	0.035	0.017	0.009	0.023	0.009	0.003	0.041	0.020	0.010
		(0.172)	(0.122)	(0.087)	(0.169)	(0.120)	(0.087)	(0.174)	(0.123)	(0.088)
	-0.4	0.010	0.004	0.002	-0.011	-0.010	-0.008	0.013	0.006	0.003
		(0.098)	(0.068)	(0.048)	(0.097)	(0.069)	(0.049)	(0.098)	(0.068)	(0.048)
Φ	0	0.010	0.005	0.003	-0.020	-0.017	-0.012	0.010	0.005	0.003
		(0.085)	(0.058)	(0.041)	(0.088)	(0.061)	(0.044)	(0.085)	(0.058)	(0.041)
	0.4	0.020	0.011	0.005	-0.027	-0.022	-0.018	0.011	0.007	0.003
		(0.100)	(0.070)	(0.049)	(0.103)	(0.075)	(0.054)	(0.097)	(0.069)	(0.048)
	0.8	0.084	0.047	0.023	-0.004	-0.017	-0.022	0.039	0.022	0.010
		(0.193)	(0.132)	(0.089)	(0.173)	(0.127)	(0.092)	(0.171)	(0.123)	(0.086)

reported by Wen-lea and Samuel (2006) mentioned previously.

Now, from Table (4) it is clear that the shapes of the empirical distributions for all sample capability indices are skewed to right. This skewness vanishes as the sample size n gets larger. Similarly, the measure of kurtosis for all cases was greater than three, which means that the empirical distributions of all sample capability indices and all autocorrelation structures are leptokurtic. Also, these measures become closer to three as n increases.

Finally, for the three time series models considered here the measures of skewness and kurtosis were affected by the values and signs of AR parameter(s). More specifically, for the $AR(1)$ and $SAR(1)_4$ models, those measures were nearly the largest when $n = 75$ and the AR parameter is closest to -1, whereas for the $SAR(1) \times (1)_4$ model, the measures of skewness and kurtosis were the largest when $n = 75$ for $\widehat{C}_p$ and $\widehat{C}_{pk}$ when $(\phi, \Phi) = (-0.8, 0.8)$ and for $\widehat{C}_{pm}$ when $(\phi, \Phi) = (0.8, 0.8)$ as well as $(\phi, \Phi) = (-0.8, 0.8)$.

5 Conclusions

In this article we investigated some distributional properties for three common sample capability indices when data are autocorrelated according to three time series models that belong to the AR type. The focus was on seasonal autocorrelation among process data. In view of the extensive simulation results, we conclude that RB and $RRMSE$ are affected by the sample size, autocorrelation structure and AR parameters, although minor differences are seen in the results between pure ordinary AR(1) and seasonal

Table 3: The values of RB and RRMSE(in parentheses) of several capability indices for SAR(1) $\times(1)_4$ data

		$\widehat{C}_p$			$\widehat{C}_{pk}$			$\widehat{C}_{pm}$		
ϕ	Φ	$n = 75$	150	300	75	150	300	75	150	300
-0.8	-0.8	0.035	0.020	0.009	0.029	0.016	0.006	0.042	0.023	0.010
		(0.175)	(0.127)	(0.091)	(0.174)	(0.126)	(0.091)	(0.178)	(0.128)	(0.092)
	-0.4	0.021	0.008	0.005	0.013	0.003	0.001	0.028	0.011	0.007
		(0.137)	(0.096)	(0.067)	(0.135)	(0.095)	(0.067)	(0.139)	(0.096)	(0.067)
	0.4	0.071	0.038	0.017	0.057	0.028	0.011	0.077	0.040	0.019
		(0.251)	(0.174)	(0.120)	(0.244)	(0.171)	(0.119)	(0.253)	(0.175)	(0.120)
	0.8	0.173	0.093	0.053	0.150	0.077	0.042	0.175	0.095	0.054
		(0.431)	(0.290)	(0.204)	(0.416)	(0.282)	(0.199)	(0.429)	(0.290)	(0.203)
-0.4	-0.8	0.040	0.023	0.012	0.032	0.017	0.008	0.047	0.026	0.013
		(0.184)	(0.133)	(0.096)	(0.181)	(0.132)	(0.095)	(0.186)	(0.134)	(0.096)
	-0.4	0.012	0.006	0.003	-0.001	-0.004	-0.004	0.018	0.008	0.004
		(0.109)	(0.076)	(0.053)	(0.108)	(0.076)	(0.053)	(0.110)	(0.077)	(0.053)
	0.4	0.021	0.010	0.005	-0.010	-0.011	-0.010	0.205	0.010	0.005
		(0.121)	(0.083)	(0.059)	(0.118)	(0.083)	(0.060)	(0.120)	(0.083)	(0.059)
	0.8	0.072	0.040	0.022	0.017	-0.001	-0.007	0.056	0.031	0.018
		(0.216)	(0.154)	(0.109)	(0.199)	(0.146)	(0.106)	(0.203)	(0.149)	(0.107)
0.4	-0.8	0.040	0.022	0.013	0.022	0.011	0.005	0.044	0.025	0.014
		(0.187)	(0.135)	(0.096)	(0.182)	(0.133)	(0.095)	(0.188)	(0.136)	(0.096)
	-0.4	0.018	0.008	0.004	-0.014	-0.014	-0.011	0.017	0.008	0.004
		(0.111)	(0.077)	(0.053)	(0.110)	(0.075)	(0.055)	(0.110)	(0.077)	(0.053)
	0.4	0.047	0.027	0.012	-0.025	-0.023	-0.023	0.019	0.013	0.005
		(0.128)	(0.087)	(0.061)	(0.126)	(0.091)	(0.068)	(0.119)	(0.084)	(0.060)
	0.8	0.161	0.091	0.048	0.023	-0.010	-0.022	0.056	0.031	0.017
		(0.249)	(0.170)	(0.115)	(0.201)	(0.151)	(0.113)	(0.202)	(0.147)	(0.106)
0.8	-0.8	0.049	0.026	0.013	0.004	-0.005	-0.009	0.041	0.022	0.011
		(0.184)	(0.130)	(0.091)	(0.173)	(0.126)	(0.090)	(0.177)	(0.128)	(0.090)
	-0.4	0.058	0.027	0.015	-0.015	-0.024	-0.021	0.028	0.012	0.008
		(0.152)	(0.100)	(0.070)	(0.143)	(0.103)	(0.074)	(0.140)	(0.096)	(0.068)
	0.4	0.176	0.089	0.044	0.042	-0.002	-0.018	0.072	0.038	0.020
		(0.309)	(0.196)	(0.130)	(0.254)	(0.175)	(0.126)	(0.249)	(0.172)	(0.121)
	0.8	0.478	0.249	0.121	0.228	0.087	0.015	0.174	0.097	0.050
		(0.650)	(0.395)	(0.239)	(0.479)	(0.308)	(0.205)	(0.430)	(0.294)	(0.201)

Table 4: The values of skewness and kurtosis (in parentheses) of several sample capability indices for SAR(1)$\times$(1)$_4$ data

		$\widehat{C}_p$		$\widehat{C}_{pk}$		$\widehat{C}_{pm}$	
ϕ	Φ	$n = 75$	300	75	300	75	300
-0.8	-0.8	0.408 (3.342)	0.195 (3.027)	0.409 (3.343)	0.195 (3.032)	0.407 (3.342)	0.195 (3.027)
	-0.4	0.464 (3.348)	0.273 7 (3.165)	0.463 (3.347)	0.274 (3.165)	0.462 (3.345)	0.273 (3.165)
	0	0.550 (3.514)	0.253 (2.933)	0.556 (3.532)	0.253 (2.934)	0.549 (3.513)	0.252 (3.933)
	0.4	0.672 (3.575)	0.372 (3.130)	0.672 (3.582)	0.375 (3.136)	0.666 (3.564)	0.372 (3.130)
	0.8	0.850 (3.845)	0.606 (3.588)	0.857 (3.871)	0.610 (3.606)	0.833 (3.805)	0.605 (3.587)
-0.4	-0.8	0.448 (3.379)	0.251 (3.242)	0.448 (3.378)	0.251 (3.242)	0.446 (3.375)	0.251 (3.242)
	-0.4	0.366 (3.293)	0.165 (3.045)	0.359 (3.290)	0.251 (3.058)	0.362 (3.288)	0.165 (3.046)
	0	0.403 (3.203)	0.228 (3.088)	0.399 (3.213)	0.251 (3.111)	0.399 (3.203)	0.230 (3.092)
	0.4	0.307 (3.173)	0.157 (2.961)	0.318 (3.215)	0.251 (2.973)	0.308 (3.194)	0.156 (2.962)
	0.8	0.356 (3.169)	0.171 (3.055)	0.372 (3.202)	0.251 (3.056)	0.359 (3.212)	0.166 (3.048)
0	-0.8	0.477 (3.327)	0.247 (3.057)	0.475 (3.325)	0.249 (3.065)	0.474 (3.322)	0.247 (3.058)
	-0.4	0.457 (3.550)	0.171 (3.003)	0.466 (3.557)	0.159 (3.008)	0.458 (3.563)	0.170 (3.003)
	0	0.462 (3.425)	0.171 (3.002)	0.442 (3.450)	0.148 (2.998)	0.468 (3.458)	0.170 (3.001)
	0.4	0.393 (3.219)	0.184 (3.021)	0.349 (3.202)	0.115 (3.003)	0.395 (3.222)	0.173 (3.000)
	0.8	0.458 (3.335)	0.250 (3.075)	0.419 (3.226)	0.212 (3.052)	0.464 (3.243)	0.257 (3.069)
0.4	-0.8	0.440 (3.298)	0.204 (3.087)	0.438 (3.273)	0.206 (3.094)	0.430 (3.265)	0.204 (3.087)
	-0.4	0.436 (3.438)	0.193 (3.091)	0.429 (3.465)	0.184 (3.121)	0.436 (3.465)	0.194 (3.103)
	0	0.398 (3.332)	0.176 (3.078)	0.372 (3.272)	0.128 (3.017)	0.406 (3.302)	0.177 (3.062)
	0.4	0.338 (3.223)	0.172 (3.075)	0.242 (3.173)	0.115 (3.058)	0.313 (3.201)	0.181 (3.090)
	0.8	0.394 (3.223)	0.145 (2.999)	0.306 (3.162)	0.100 (2.909)	0.363 (3.171)	0.149 (2.929)
0.8	-0.8	0.417 (3.265)	0.190 (3.058)	0.416 (3.294)	0.192 (3.033)	0.404 (3.276)	0.188 (3.055)
	-0.4	0.468 (3.330)	0.268 (3.134)	0.414 (3.338)	0.198 (3.082)	0.471 (3.358)	0.256 (3.088)
	0	0.436 (3.196)	0.248 (3.041)	0.402 (3.239)	0.201 (3.074)	0.464 (3.302)	0.251 (3.063)
	0.4	0.521 (3.278)	0.300 (3.103)	0.611 (3.320)	0.276 (3.102)	0.604 (3.465)	0.315 (3.114)
	0.8	0.671 (3.443)	0.547 (3.425)	0.674 (3.514)	0.504 (3.388)	0.853 (3.851)	0.570 (3.508)

AR(1) models. Besides, we found that the empirical distributions for the three sample capability indices, by means of skewness and kurtosis, are also affected by sample size, autocorrelation structure and AR parameters. It is noticed that those distributions are always skewed to right and leptokurtic. It is worth mentioning that more agreement in terms of RB, $RRMSE$ and shape is seen between $\widehat{C}_p$ and $\widehat{C}_{pk}$ as compared to $\widehat{C}_{pm}$.

Finally, comparing our results with corresponding theory for independent and normal data, it seems that the autocorrelation among data does not alter the main facts about bias and shape of distributions for the three sample capability indices, but affect their magnitude or degree.

References

Alzoubi, A. I. (2013). On the distributional properties of some process capability measures for autocorrelated data. Master's thesis.

Box, G. E., Jenkins, G. M., and Reinsel, G. C. (2008). *Time series analysis: forecasting and control*. John Wiley & Sons.

Chou, Y.-m. and Owen, D. (1989). On the distributions of the estimated process capability indices. *Communications in Statistics-Theory and Methods*, 18(12):4549–4560.

Cryer, J. D. and Chan, K.-S. (2008). *Time series regression models*. Springer.

Guevara, R. D. and Vargas, J. A. (2007). Comparison of process capability indices under autocorrelated data. *Revista Colombiana de Estadística*, 30(2):301–316.

Kotz, S., Pearn, W. L., and Johnson, N. (1993). Some process capability indices are more reliable than one might think. *Journal of the Royal Statistical Society: Series C (Applied Statistics)*, 42(1):55–62.

Montgomery, D. C. (2009). *Introduction to statistical quality control*. John Wiley & Sons.

Pearn, W., Lin, G., and Chen, K. (1998). Distributional and inferential properties of the process accuracy and process precision indices. *Communications in Statistics-Theory and methods*, 27(4):985–1000.

Shore, H. (1997). Process capability analysis when data are autocorrelated. *Quality Engineering*, 9(4):615–626.

Steiner, S., Abraham, B., and MacKay, J. (1997). Understanding process capability indices. *Waterloo, Ontario*.

Team, R. C. (2017). R: a language and environment for statistical computing. r foundation for statistical computing, vienna. http s. *www. R-proje ct. org*.

Wen-lea, P. and Samuel, K. (2006). *Encyclopedia and handbook of process capability indices: a comprehensive exposition of quality control measures*, volume 12. World Scientific.