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By Hashmi, Haq, Usman

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A generalized exponential distribution with increasing, decreasing and constant shape hazard curves

Sharqa Hashmi[a,c], Muhammad Ahsan ul Haq*[b,c], and Rana Muhammad Usman[c]

[a]*Lahore College of Women University (LCWU), Lahore, Pakistan,*
[b]*Quality Enhancement Cell, National College of Arts, Lahore, Pakistan,*
[c]*College of Statistical and Actuarial Sciences, University of the Punjab, Lahore, Pakistan,*

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This paper introduces a generalization of moment exponential distribution so called Kumaraswamy Moment Exponential distribution. The limit behaviour of its density and hazard functions are described. Some properties of the proposed distribution are discussed including moments, skewness, kurtosis, quantile function, and mode. Characterizations based on truncated moments and hazard function are presented. Ri and q-entropies, mean residual life (MRL) and mean inactivity time (MIT) of X, and order statistics are determined. The maximum likelihood estimation (MLE) is used to estimate the model parameters. Two real data sets are used to compare the KwME distribution with other competitive models and concluded that it could serve as a better alternative lifetime distribution than existing well known models.

keywords: Hazard function, Moment Exponential distribution, moments, maximum likelihood estimation.

1 Introduction

The modeling of lifetime data has a vital role in statistical analysis in the various fields of science and human life. Therefore many generalizations have been developed by introducing shape parameter(s) to make baseline model more flexible. The interest in such

*Corresponding author: ahsanshani36@gmail.com

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generalized models remains increased these days. Since last few decades, generalized models are more useful in the field of medical and health, economic and growth, finance, reliability analysis and engineering. These generalizations includes beta-G family of distributions proposed by Eugene et al. (2002), Hashmi and Memon (2016) studied beta exponentiated Weibull distribution, Weibull-G family of distributions defined by Bourguignon et al. (2014), the generalized transmuted-G family of distributions introduced by Nofal et al. (2017), Haq et al. (2016) derived and studied transmuted Power function distribution, among others. One can get knowledge of recently developed generalizations from Lee et al. (2013) who discussed and compared the characteristics of these families.

The exponential distribution is a positively skewed distribution used worldwide for reliability analysis and to deal with lifetime data sets by Epstein (1958). Its generalizations include double exponential distribution by Norton (1984), exponentiated exponential distribution by Gupta and Kundu (2001), transmuted exponential distribution by Merovci (2013), moment exponential distribution by Dara and Ahmad (2012), exponentiated moment exponential distribution by Hasnain et al. (2015), the odd Fréchet-G family of probability distributions by Haq and Elgarhy (2018), the odd moment exponential family of distributions: its properties and applications by Haq et al. (2018), the Marshall-Olkin length-biased exponential distribution and its applications by Haq et al. (2017) and generalized exponentiated moment exponential distribution by Iqbal et al. (2014) among others.

Dara and Ahmad (2012) developed moment exponential distribution by assigning linear weights to exponential model. They developed some basic properties such as moment generating function, moments, skewness, kurtosis, explained the behaviour of distribution, its hazard curves with an application. Later on Hasnain et al. (2015) and Iqbal et al. (2014) generalized this moment exponential distribution for more litheness.

The probability density function (pdf) and cumulative distribution function (cdf) for one parametric moment exponential distribution is given as:

$$g\left(x;\beta\right)=\frac{x}{\beta^{2}}\mathrm{e}^{-\frac{x}{\beta}}\ ,\qquad x>0,\ \beta>0 \tag{1}$$

$$G\left(x;\beta\right)=1-\left(1+\frac{x}{\beta}\right)\mathrm{e}^{-\frac{x}{\beta}}\ ,\quad x>0,\ \beta>0 \tag{2}$$

where $\beta>0$ is its only scale parameter.

If G denoted the distribution function for parent distribution with a random variable X, then the cdf of $Kw-G$ can be defined as:

$$F\left(x\right)=1-\left[1-G^{a}\left(x\right)\right]^{b},\quad a>0,\ b>0 \tag{3}$$

Correspondingly, its pdf is given as:

$$f\left(x\right)=abg\left(x\right)G^{a-1}\left(x\right)\left[1-G^{a}\left(x\right)\right]^{b-1} \tag{4}$$

where $a>0,b>0$ are two new shape parameters which are responsible for skewness, tail weights and kurtosis. Its density function is a simple function and does not involve any particular function such as beta-G distribution and gamma-G distribution.

The arrangement of this paper is presented in subsequent sections. In section 2, we define Kw-ME distribution and present expansions for density, cumulative distribution and hazard functions, we also discuss limiting behaviour of its probability density and hazard functions. In section 3, different mathematical properties including moments, moment generating function, quantile function, mode and incomplete moments are derived. We also obtain Rényi entropy and q-entropy, mean residual life (MRL) and mean inactivity time (MIT) of X. The densities of smallest and largest order statistics are determined in section 4. In section 5, we obtain the maximum likelihood estimates (MLEs) of the model parameters. In section 6, some simulation results investigate the performance of these estimates. In Section7, we show the potentiality of the proposed distribution using two real data analysis. Finally, in Section 8, we state some remarks as a conclusion.

2 KwME Distribution

If $G(x;\beta)$ is the moment exponential distribution with parameter β then equation (2) yields a new model KwME cumulative distribution $(for\ x > 0)$, say $F(x) = F(x; a, b, \beta)$, reduces to

$$F(x) = 1 - \left[1 - \left\{1 - \left(1 + \frac{x}{\beta}\right)e^{-\frac{x}{\beta}}\right\}^{a}\right]^{b} \tag{5}$$

where $\beta > 0$ is a scale parameter while a and b are two positive real value shape parameters. The corresponding KwME pdf. is obtained by inserting (1) and (2) in equation (4).

$$f(x) = ab\frac{x}{\beta^2}e^{-\frac{x}{\beta}}\left[1 - \left(1 + \frac{x}{\beta}\right)e^{-\frac{x}{\beta}}\right]^{a-1}\left[1 - \left\{1 - \left(1 + \frac{x}{\beta}\right)e^{-\frac{x}{\beta}}\right\}^{a}\right]^{b-1}, \quad x \geq 0 \tag{6}$$

In Figures (1) and (2), we present the plots of the p.d.f. and failure (hazard) rate functions of the KwME distribution for specified values of parameters.

2.1 Expansions used for density function

Here, we explain some binomial function used for the expansion of probability density function for $(0 < a < 1)$

$$(1+a)^v = \sum_{k=0}^{\infty}\binom{k}{v}a^k$$

where

$$\binom{k}{v} = \frac{n(n-1)(n-2)\ldots\ldots(v-k+1)}{k!}$$

Also

$$(1-z)^{b-1} = \sum_{j=0}^{\infty}(-1)^j\binom{b}{j}z^j, \qquad |Z| > 0$$

We can write Eq. (6) as

$$f(x) = ab\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{i+j}\binom{b-1}{i}\binom{a(i+1)-1}{j}\binom{j}{k}\frac{1}{\beta^{k+2}}x^{k+1}e^{-\frac{x}{\beta}(j+1)}. \quad (7)$$

The survival function $S(x)$ and hazard or failure function $h(x)$ of X are given as

$$S(x) = \left[1-\left\{1-\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}\right\}^{a}\right]^{b}. \quad (8)$$

$$h(x) = \frac{ab\frac{x}{\beta^2}e^{-\frac{x}{\beta}}\left[1-\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}\right]^{a-1}}{1-\left\{1-\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}\right\}^{a}}. \quad (9)$$

A flexible model approaches to other distributions when parameters assume different values. If Y is a random variable of KwME distribution with pdf defined in Eq. (6), then we have following special cases for proposed distribution:

1. If $a = b = 1$, KwME converts into moment exponential distribution $ME(\beta)$.
2. When $a = 1$, we have the generalized moment exponential distribution $GME(\beta, a)$.
3. When $b = 1$, we get the exponentiated moment exponential distribution $EME(\beta, b)$.

2.2 Shape of density function

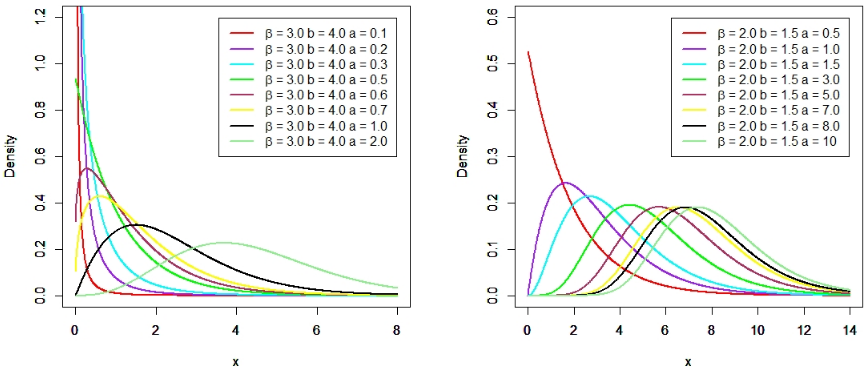
Figures 1a, 1b, 2a, 2b display some graphs of the KwME probability density, and hazard rate curves for some specific parametric values β, a and b. Further, the hrf (failure or hazard rate function) of the KwME model is very flexible in accommodating different form and thus it becomes an important model to fit real lifetime data.

Figure 1a & 1b shows the various shapes of KwME density curve with different set of parameter a and fixed values of $b = 4$ and $\beta = 3$. Figure 2a & 2b represent the hazard (failure) curves for KwME models with an increasing, decreasing and semi bathtub shapes.

2.3 Limiting behavior of KwME density and hazard functions

Theorem 1: The limit of KwME probability density function as $x \to \infty$ is zero and the limit at origin are

$$\lim_{x\to 0} f(x) = \begin{cases} \infty & for\ 0 < a < \frac{1}{2} \\ \frac{b}{\beta\sqrt{2}} & for\ a = \frac{1}{2} \\ 0 & for\ a > \frac{1}{2} \end{cases} \quad (10)$$



(a) (b)

Figure 1: Plots of probability density function for some specific parametric values

Proof:

$$\lim_{x\to 0} f(x;a,b,\beta) = \lim_{x\to 0}\left[ab\frac{x}{\beta^2}e^{-\frac{x}{\beta}}\left[1-\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}\right]^{a-1}\left[1-\left\{1-\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}\right\}^{a}\right]^{b-1}\right]$$

$$= \frac{ab}{\beta^2}\lim_{x\to 0}\left[xe^{-\frac{x}{\beta}}\left[1-\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}\right]^{a-1}\right]$$

$$= \frac{ab}{\beta^2}\lim_{x\to 0}\left\{xe^{-\frac{x}{\beta}}\left(\left(1+\frac{\mathrm{x}}{\beta}\right)e^{-\frac{x}{\beta}}\right)^{a-1} - (a-1)\left(\left(1+\frac{\mathrm{x}}{\beta}\right)e^{-\frac{x}{\beta}}\right)^{a-2} + \frac{(a-1)(a-2)}{2}\left(\left(1+\frac{\mathrm{x}}{\beta}\right)e^{-\frac{x}{\beta}}\right)^{a-3} - \ldots\right\}$$

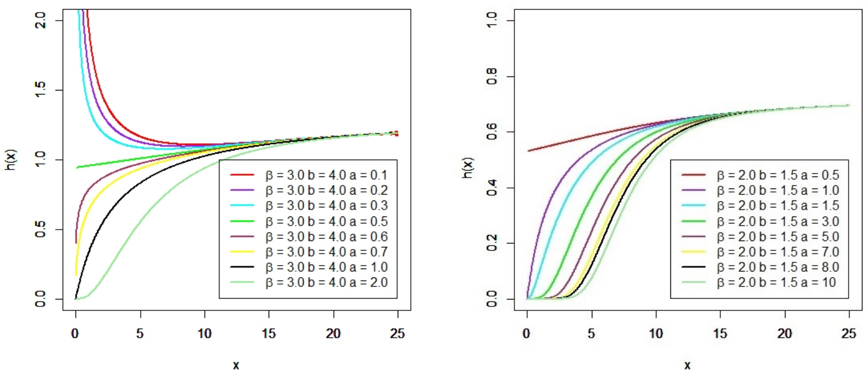
Expand exponent series and the result can be demonstrated easily.

Theorem 2: The limit of KwME hazard function as $x \to \infty$ is $\frac{b}{\beta}$ and the limit at origin are

$$\lim_{x\to 0} h(x) = \begin{cases} \infty & for\ 0 < a < \frac{1}{2} \\ \frac{b}{\beta\sqrt{2}} & for\ a = \frac{1}{2} \\ 0 & for\ a > \frac{1}{2} \end{cases} \tag{11}$$

Proof: The limit of KwME hazard function at origin is obtained by

$$\lim_{x\to 0} h(x;a,b,\beta) = \lim_{x\to 0}\left[\frac{ab\frac{x}{\beta^2}e^{-\frac{x}{\beta}}\left[1-\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}\right]^{a-1}}{1-\left\{1-\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}\right\}^{a}}\right]$$



(a) (b)

Figure 2: Plots of the failure rate function for some specific parametric values

It is straightforward to prove the result from this equation. Now for the limit of KwME hazard function at infinity, we use following lemma.

Lemma 1: If X be a random variable, then for $\beta > 0$

$$\lim_{x\to\infty}\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}} = 0$$

Proof: Applying L' Hospital rule and result follow. Using Lemma, we get

$$\lim_{x\to\infty} h\left(x;a,b,\beta\right) = \lim_{x\to\infty}\frac{ab\frac{x}{\beta^2}e^{-\frac{x}{\beta}}\left[1-\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}\right]^{a-1}}{1-\left\{1-\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}\right\}^{a}} = \frac{b}{\beta}$$

Remark 1: The shapes of density function of KwME distribution have the following properties:

1. The density curve is modal for $a > \frac{1}{2}$.

2. The pdf curve begins from a specific point $\frac{b}{\beta\sqrt{2}}$ at origin and has increasing trend and reaches to zero as x approaches to infinity for $a = 0.5$.

3. The curve has decreasing trend starts from infinity and touches zero as x approaches to infinity.

Remark 2: The hazard rate function of KwME distribution has the following properties:

1. The hazard rate curve begins from infinity at the origin and goes to the point $\frac{b}{\beta}$ as x tends to infinity for $a < 0.5$. The curve may be decreasing or semi bathtub shape for this.

2. The hazard rate curve begins at particular point $\frac{b}{\beta\sqrt{2}}$ at origin and goes to the point $\frac{b}{\beta}$ x approaches to infinity for $a = 0.5$.
3. The hazard rate curve has an increasing trend and begins from zero at the origin and reaches to the point $\frac{b}{\beta}$ as x goes to infinity for $a > 0.5$.

3 Mathematical properties

It is efficient to illustrate the structural quantities of KwME model with algebraic expressions as compare to direct numerical integration of density function. Therefore, we derived expressions for some important statistical measures.

3.1 Moments

Theorem 3: Let X be a r.v. from KwME distribution then its r^{th} moment is

$$\mu'_r = ab\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{i+j}\binom{b-1}{i}\binom{a(i+1)-1}{j}\binom{j}{k}\frac{\beta^r}{(j+1)^{k+r+2}}\Gamma(k+r+2) \quad (12)$$

Proof: The r^{th} moment of KwME random variable X can be obtained from

$$\mu'_r = \int_0^{\infty} x^r f(x; a,\ b,\ \beta)\,dx$$

Using binomial expansions,we have

$$= ab\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{i+j}\binom{b-1}{i}\binom{a(i+1)-1}{j}\binom{j}{k}\frac{1}{\beta^{k+2}}\int_0^{\infty} x^{k+r+1}e^{-\frac{x}{\beta}(j+1)}dx$$

$$Let \qquad z = \frac{x}{\beta}(j+1) \Rightarrow\ x = \frac{z\beta}{(j+1)} \ \Rightarrow dx = \frac{\beta}{(j+1)}dz$$

$$= ab\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{i+j}\binom{b-1}{i}\binom{a(i+1)-1}{j}\binom{j}{k}\frac{1}{\beta^{k+2}}\int_0^{\infty}\frac{z^{k+r+1}\beta^{k+r+1}}{(j+1)^{k+r+1}}e^{-z}\frac{\beta}{(j+1)}dz$$

$$= ab\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{i+j}\binom{b-1}{i}\binom{a(i+1)-1}{j}\binom{j}{k}\frac{\beta^r}{(j+1)^{k+r+2}}\int_0^{\infty} z^{k+r+1}e^{-z}dz$$

and the result follows.

3.2 Incomplete Moments

Theorem 4: Let X has KwME distribution then its r^{th} incomplete moment is

$$\varphi(x) = ab\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{i+j}\binom{b-1}{i}\binom{a(i+1)-1}{j}\binom{j}{k}\frac{\beta^r\gamma(k+r+2,x)}{(j+1)^{k+r+2}} \tag{13}$$

Proof: The r^{th} incomplete moment of X can be obtained from

$$\varphi(x) = \int_0^x v^r\ f(v)\,\mathrm{d}v$$

$$= ab\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{i+j}\binom{b-1}{i}\binom{a(i+1)-1}{j}\binom{j}{k}\frac{1}{\beta^{k+2}}\int_0^x v^{k+r+1}e^{-\frac{v}{\beta}(j+1)}dv$$

$$Let \qquad z = \frac{v}{\beta}(j+1) \Rightarrow\ x = \frac{z\beta}{(j+1)}\ \Rightarrow dx = \frac{\beta}{(j+1)}dz$$

$$= ab\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{i+j}\binom{b-1}{i}\binom{a(i+1)-1}{j}\binom{j}{k}\frac{1}{\beta^{k+2}}\int_0^x \frac{z^{k+r+1}\beta^{k+r+1}}{(j+1)^{k+r+1}}e^{-z}\frac{\beta}{(j+1)}dz$$

$$= ab\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{i+j}\binom{b-1}{i}\binom{a(i+1)-1}{j}\binom{j}{k}\frac{\beta^r}{(j+1)^{k+r+2}}\int_0^x z^{k+r+1}e^{-z}dz$$

Incomplete gamma function completes the proof.

3.3 Moment generating function

$$M_x(t) = E\left(e^{tx}\right) = \int_0^\infty e^{tx}f(x;a,b,\beta)\,dx$$

$$= ab\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{i+j}\binom{b-1}{i}\binom{a(i+1)-1}{j}\binom{j}{k}\frac{1}{\beta^{k+2}}\int_0^\infty x^{k+1}e^{-\frac{x}{\beta}(j+1)}e^{tx}dx$$

After integration and simplification, we get the moment generating function as

$$= ab\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{i+j}\binom{b-1}{i}\binom{a(i+1)-1}{j}\binom{j}{k}\frac{1}{(j+1+\beta t)^{k+2}}\Gamma(k+2) \tag{14}$$

3.4 Quantile and random number generation

If U is a uniform random variate with unit interval (0, 1), then the random variable $X = Q(U)$ has density Eq.(6). The quantile function of X corresponding to (Eq. 5) is

$$x = Q\left(u\right) = \beta\left[\left\{1-\left(1-u^{\frac{1}{b}}\right)^{\frac{1}{a}}\right\}e^{\frac{x}{\beta}}-1\right] \tag{15}$$

Since it is a complex equation so by iteration method, the equation provides the quartiles and random numbers of the KwME distribution.

3.5 Mode

The mode of $f(x)$ is an important measure of average. $f'(x) = \frac{d\,f(x)}{dx} = 0$, for the mode, simplifies to

$$1-\frac{1}{\beta}+\frac{x}{\beta}\mathrm{e}^{-\frac{x}{\beta}}\left(-\frac{(\mathrm{a}-1)}{1-\mathrm{e}^{-\frac{x}{\beta}}\left(1+\frac{x}{\beta}\right)}+\frac{\mathrm{a}\left(b-1\right)}{1-\left(1-\mathrm{e}^{-\frac{x}{\beta}}\left(1+\frac{x}{\beta}\right)\right)^{a}}\right)=0 \tag{16}$$

The second derivative may be used if required.

3.6 Skewness and kurtosis

Skewness is the measure of the asymmetry of the probability density function and kurtosis is the measure of peakedness of the probability density function. Both measures are the descriptive measures of the shape of the probability distribution. Skewness and kurtosis can be easily determined by the following expressions based on first four mean moments calculated by Eq. (11) or Eq. (12).

$$\gamma_1\left(sk\right) = \frac{\mu_3}{\mu_2^{\frac{3}{2}}}, \qquad \beta_2 = \frac{\mu_4}{\mu_2^2} \tag{17}$$

4 Characterizations

Characterization of a distribution is theoretically important as it is the unique way of identifying the distribution. Characterizing a distribution is an important problem which helps researcher to see if proposed model is the correct one.

4.1 Characterization based on two truncated moments

For characterization of KwME distribution we use the proposition based on the ratio of two truncated moments Glänzel (1987).

Theorem 5: Let X: $\Omega \to (0,\infty)$ be distributed as Eq.(6) and

$$q_1\left(x\right) = \left\{1-\left\{1-\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}\right\}^{a}\right\}^{1-b} \tag{18}$$

$$q_2(x) = q_1(x)\left\{1-\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}\right\}^a for\ x > 0. \tag{19}$$

The random variable X follows Kw ME distribution if and only if the function η is of the form

$$\eta(x) = \frac{1}{2}\left\{1+\left\{1-\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}\right\}^a\right\}. \tag{20}$$

Proof: It can be seen that

$$(1-F(x))\,E[q_1(X)\,|X \geq x] = b\left[1-\left\{1-\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}\right\}^a\right], \qquad x > 0$$

$$(1-F(x))\,E[q_2(X)\,|X \geq x] = \frac{b}{2}\left[1-\left\{1-\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}\right\}^{2a}\right], \qquad x > 0$$

and so

$$\eta(x) = \frac{1}{2}\left\{1+\left\{1-\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}\right\}^a\right\}.$$

As

$$\eta(x)\,q_1(x) - q_2(x) = \frac{q_1(x)}{2}\left\{1-\left\{1-\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}\right\}^a\right\} \neq 0 \ \ for\ all\ x.$$

the proof follows.

Conversely, given $q_1(x), q_2(x)$ and $\eta(x)$we show that the random variable X has Kw ME distribution. Here,

$$\acute{s}(x) = \frac{\acute{\eta}(x)\,q_1(x)}{\eta(x)\,q_1(x) - q_2(x)} = \frac{\acute{\eta}(x)}{\eta(x) - \left\{1-\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}\right\}^a}$$

$$\acute{s}(x) = \frac{ax\left(1-\frac{\mathrm{e}^{-\frac{x}{\beta}}(x+\beta)}{\beta}\right)^a}{\beta\left(x+\beta-\mathrm{e}^{x/\beta}\beta\right)\left\{\left(1-\frac{\mathrm{e}^{-\frac{x}{\beta}}(x+\beta)}{\beta}\right)^a - 1\right\}}, \qquad x > 0$$

and so

$$s(x) = -\ln\left[\left(1-\frac{\mathrm{e}^{-\frac{x}{\beta}}(x+\beta)}{\beta}\right)^a - 1\right], \qquad x > 0 \tag{21}$$

Now

$$= c\int_0^x \frac{\acute{\eta}(u)}{\eta(u)\, q_1(u) - q_2(u)} exp\left[-s(u)\right] du$$

$$= c\int_0^x \frac{au\left(1 - \frac{e^{-\frac{u}{\beta}}(u+\beta)}{\beta}\right)^a \left\{1 - \left\{1 - \left(1 + \frac{u}{\beta}\right) e^{-\frac{u}{\beta}}\right\}^a\right\}^{b-1}}{\beta\left(u + \beta - e^{u/\beta}\beta\right)} du$$

which can be simplified to

$$\int_0^x f_{KwMED}(u)\, du = F_{KwMED}(x)$$

4.2 Characterization based hazard function

Hamedani and Najibi (2016) presented the characterization based on hazard rate function. Using this idea, the characterization of KwME distribution is presented here. **Theorem 6:** The pdf of KwME distribution is (6) if and only if its hazard function $h(x)$ satisfies the differential equation

$$\acute{h}(x) - x^{-1} h(x) = \frac{abx\left(1 - \frac{e^{-\frac{x}{\beta}}(x+\beta)}{\beta}\right)^a \left[ax + \left(e^{x/\beta} - 1\right)\beta\left\{\left(1 - \frac{e^{-\frac{x}{\beta}}(x+\beta)}{\beta}\right)^a - 1\right\}\right]}{\beta^2\left(x + \beta - e^{x/\beta}\beta\right)^2\left[\left(1 - \frac{e^{-\frac{x}{\beta}}(x+\beta)}{\beta}\right)^a - 1\right]^2} \tag{22}$$

Proof: If X has pdf (6), then

$$\acute{h}(x) - x^{-1} h(x) = \frac{\left[ab\left(1 - \frac{e^{-\frac{x}{\beta}}(x+\beta)}{\beta}\right)^a \left\{ax^2 + \beta\left(e^{x/\beta}(x-\beta) + \beta\right)\left(\left(1 - \frac{e^{-\frac{x}{\beta}}(x+\beta)}{\beta}\right)^a - 1\right)\right\}\right]}{\left[\beta^2\left(x + \beta - e^{x/\beta}\beta\right)^2\left\{\left(1 - \frac{e^{-\frac{x}{\beta}}(x+\beta)}{\beta}\right)^a - 1\right\}^2\right]} -$$

$$- x^{-1}\left(\frac{abe^{-\frac{x}{\beta}} x\left(1 - e^{-\frac{x}{\beta}}\left(1 + \frac{x}{\beta}\right)\right)^{a-1}}{\left[1 - \left(1 - e^{-\frac{x}{\beta}}\left(1 + \frac{x}{\beta}\right)\right)^a\right]\beta^2}\right) \tag{23}$$

simplification follows (21). Now if (21) holds then

$$\frac{d}{dx}\left[x^{-1} h(x)\right] = ab\frac{d}{dx}\left(\frac{e^{-\frac{x}{\beta}}\left(1 - e^{-\frac{x}{\beta}}\left(1 + \frac{x}{\beta}\right)\right)^{a-1}}{\left[1 - \left(1 - e^{-\frac{x}{\beta}}\left(1 + \frac{x}{\beta}\right)\right)^a\right]\beta^2}\right)$$

$$\mathrm{h}\left(\mathrm{x}\right)=ab\frac{x\ \mathrm{e}^{-\ \frac{x}{\beta}}\left(1-\mathrm{e}^{-\frac{x}{\beta}}\left(1+\frac{x}{\beta}\right)\right)^{a-1}}{\left[1-\left(1-\mathrm{e}^{-\ \frac{x}{\beta}}\left(1+\frac{x}{\beta}\right)\right)^{a}\right]\beta^{2}} \tag{24}$$

5 Entropies

Entropy of a r. v. X is used to measure the variation of the uncertainty. Mostly, Ri entropy is used as a common measure of entropy.

5.1 Rényi Entropy

Theorem 7: If the random variable X is defined as Eq. 5, then the Ri entropy is given by

$$I_R\left(\delta\right)=\frac{\delta\log a}{1-\delta}+\frac{\delta\log b}{1-\delta}+\log\beta+\frac{1}{1-\delta}$$
$$\log\left[\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{i+j}\binom{\delta\left(b-1\right)}{i}\binom{ai+\delta\left(a-1\right)}{j}\binom{j}{k}\frac{\Gamma\left(\delta+k+1\right)}{\left(\delta+j\right)^{\delta+k+1}}\right] \tag{25}$$

Proof: If X has the KwME distribution then Ri entropy is defined as

$$I_R\left(\delta\right)=\frac{1}{1-\delta}\log\left[I\left(\delta\right)\right] \tag{26}$$

where $\delta>0$ and $\delta=1$ and $I\left(\delta\right)=\int\limits_{0}^{\infty}f^{\delta}\left(x\right)dx$

$$I\left(\delta\right)=\int\limits_{0}^{\infty}\left(ab\right)^{\delta}\frac{x^{\delta}}{\beta^{2\delta}}e^{-\frac{x\delta}{\beta}}\left[1-\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}\right]^{\delta(a-1)}\left[1-\left\{1-\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}\right\}^{a}\right]^{\delta(b-1)}dx$$

After simplification final expression is

$$=\frac{\left(ab\right)^{\delta}}{\beta^{\delta-1}}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{i+j}\binom{\delta\left(b-1\right)}{i}\binom{ai+\delta\left(a-1\right)}{j}\binom{j}{k}\frac{1}{\left(\delta+j\right)^{\delta+k+1}}\Gamma\left(\delta+k+1\right) \tag{27}$$

Substituting Eq. (26) in Eq. (25), the result follows.

5.2 q-Entropy

The q entropy (Hq) is defined by

$$\mathrm{H_q}=\frac{1}{\mathrm{q}-1}\log\left(1-\left(1-\mathrm{q}\right)\mathrm{I_R}\left(\delta\right)\right) \tag{28}$$

Substitution of Eq. (24) completes the proof.

6 Mean Residual Life (MRL) and Mean Inactivity Time (MIT)

If T is a continuous r. v. representing the life of a component having distribution function F(t) defined in Eq. (5), the mean residual life is defined by

$$\mu(t) = E\left(T - t\left|T\right\rangle t\right)\big) = \frac{1}{\bar{F}}\int_{t}^{\infty} S(v)\ \ \mathrm{d}v, \qquad t \geq 0 \tag{29}$$

where $\bar{F} = 1 - F = S(t)$ is the survival function.

Theorem 8: Let X be a r. v. having KwME distribution, the mean residual life is given by

$$\mu(t) = \frac{\left[1 - ab\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{i+j}\binom{b-1}{i}\binom{a(i+1)-1}{j}\binom{j}{k}\frac{\beta^{r}\,\gamma(k+r+2,x)}{(j+1)^{k+r+2}}\right]}{1-\left[1-\left\{1-\left(1+\frac{t}{\beta}\right)e^{-\frac{t}{\beta}}\right\}^{a}\right]^{b}} - t. \tag{30}$$

Proof: from

$$\mu(t) = \frac{1}{S(t)}\int_{t}^{\infty} S(v)\ \mathrm{d}v$$

$$= \frac{1}{1-\left[1-\left\{1-\left(1+\frac{t}{\beta}\right)e^{-\frac{t}{\beta}}\right\}^{a}\right]^{b}}\int_{t}^{\infty} 1-\left[1-\left\{1-\left(1+\frac{v}{\beta}\right)e^{-\frac{v}{\beta}}\right\}^{a}\right]^{b} dv$$

Also, the mean residual life can be obtained as

$$\mu(t) = \frac{\left[1-\varphi_1(t)\right]}{S(t)} - t = \frac{\int_{t}^{\infty} v\ f(v)\,\mathrm{d}v}{S(t)} - t, \qquad t \geq 0, \tag{31}$$

where $\varphi_1(t) = \int_{0}^{t} v\ f(v)\,\mathrm{d}v$ is first incomplete moment of V. Substituting of Eq. (12) in Eq. (30), completes the proof.

Theorem 9: Let X be a r.v. with KwME distribution, the mean inactivity life is given by

$$M(t) = t - \frac{ab\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{i+j}\binom{b-1}{i}\binom{a(i+1)-1}{j}\binom{j}{k}\frac{\beta^{r}\,\gamma(k+r+2,x)}{(j+1)^{k+r+2}}}{\left[1-\left\{1-\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}\right\}^{a}\right]^{b}}. \tag{32}$$

Proof: The mean Inactivity time (MIT) is defined by

$$M(t) = E(t-T|T\le t) = t - \frac{[\varphi_1(t)]}{F(t)}, \qquad t>0,$$

By inserting Eq. (12) and Eq. (5), the result follows.

7 Order Statistics

Let $X_{(i)}$ be random variables and its ordered values is denoted as $X_{(1)},\ X_{(2)}, X_{(3)},\ \ldots\ldots, X_{(n)}$. The probability density function (p.d.f.) of order statistics is obtained by the following function

$$f_{s:n}(x) = \frac{n!}{(s-1)!\,(n-s)!}[F(x)]^{s-1}[1-F(x)]^{n-s}f(x)$$

The density of the s^{th} ordered statistics of the KwME distribution is derived as follows

$$f_{s:n}(x) = \frac{n!}{(s-1)!\,(n-s)!}\left[1-\left\{1-\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}\right\}^{a}\right]^{bs-1}\left[1-\left[1-\left\{1-\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}\right\}^{a}\right]^{b}\right]^{n-s}$$
$$ab\frac{x}{\beta^2}e^{-\frac{x}{\beta}}\left[1-\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}\right]^{a-1}$$

The density of the smallest order statistics obtained as

$$f_{1:n}(x) = n\left[1-\left\{1-\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}\right\}^{a}\right]^{b-1}\left[1-\left[1-\left\{1-\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}\right\}^{a}\right]^{b}\right]^{n-1}ab\frac{x}{\beta^2}$$
$$e^{-\frac{x}{\beta}}\left[1-\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}\right]^{a-1} \tag{33}$$

The density of the largest order statistics obtained as

$$f_{n:n}(x) = n\left[1-\left\{1-\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}\right\}^{a}\right]^{bn-1}ab\frac{x}{\beta^2}e^{-\frac{x}{\beta}}\left[1-\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}\right]^{a-1}. \tag{34}$$

8 Estimation

Numerous estimation methods are recommended in statistical theory but the maximum likelihood estimation method is the supreme used. The MLEs provide the maximum information about the properties of distribution and useful during the construction of confidence intervals and also use in examination of test statistics. In large sample theory, the normal approximation for ML estimators can easily be managed either numerically or critically. Therefore, we preferably used ML estimation for the estimation of KwME's parameters. Here, we explore the MLEs of unknown parameters of the $KwME(a,b,\beta)$ model.

Let X be random variable following KwME distribution of size n with vector of parameters $(a,b,\beta)^T$. Then sample likelihood and Log-Likelihood functions of KwME are obtained as

$$\prod_{i=1}^{n} f(x) = \frac{a^n b^n}{\beta^{2n}} \prod_{i=1}^{n} x e^{-\frac{x}{\beta}} \left[1-\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}\right]^{a-1} \left\{1-\left[1-\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}\right]^{a}\right\}^{b-1}$$

Log-likelihood function is:

$$L = n\log a + n\log b + n\sum \log x - 2n\log\beta - \sum\frac{x}{\beta} + (a-1)\sum\log\left[1-\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}\right] +$$
$$+ (b-1)\sum\log\left[1-\left\{1-\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}\right\}^{a}\right] \quad (35)$$

Therefore, The MLE's of parameters (a,b and β) can be found by maximizing the above log-likelihood function Eq. (34). Take the first derivative of the above log-likelihood equation with respect to parameters and equate to zero respectively.

$$\frac{\partial L}{\partial a} = \frac{n}{a} + \sum\log\left[1-\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}\right] - (b-1)\sum\frac{\left\{1-\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}\right\}^{a}\log a}{1-\left\{1-\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}\right\}^{a}} = 0 \quad (36)$$

$$\frac{\partial L}{\partial b} = \frac{n}{b} + \sum\log\left[1-\left\{1-\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}\right\}^{a}\right] = 0 \quad (37)$$

$$\frac{\partial L}{\partial \beta} = \frac{-2n}{\beta} + \sum\frac{x}{\beta^2} - \frac{(a-1)}{\beta^2}\sum\frac{x e^{-\frac{x}{\beta}}}{1-\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}} + \frac{a(b-1)}{\beta^2}\sum\frac{x\left\{1-\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}\right\}^{a} e^{-\frac{x}{\beta}}}{1-\left\{1-\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}\right\}^{a}} = 0 \quad (38)$$

The exact solution of above derived ML estimator for unknown parameters is not possible. So it is well-situated to use Newton-Raphson algorithm to maximize the above likelihood function numerically. We can use R (optim function or maxBFGS function), or MATHEMATICA (NMaximize function).

9 Simulation study

Monte Carlo simulation study under 10,000 repetitions is used to inspect the performance for MLEs of KwME parameters for various sample sizes (n). The simulations are achieved as:

- Data sets are produced from the relation $F(x) = R$, where $R\tilde{}U(0,\ 1)$.
- The values of true parameters (a, b, β) are taken as $(2.5, 1.5, 1.5), (1.25, 4, 2), (1.5, 3, 4)$ respectively for example.
- n=50, 150, 300 and 500 are the sample sizes used for simulation.
- The experiment is replicated 100 00 times for each sample size.

We calculate the average estimates (AEs), mean square errors (MSEs) and biases. The outcomes of the Monte Carlo simulation study are presented in Table 1. The findings of simulated results indicate that as 'n' increases the MSE decrease and approaching towards zero, as usually expected under the first-order asymptotic theory. The average parameter estimates tend to be closer to the true parameters as the sample size 'n' increases. An obvious fact can be seen during estimation of parameters is that the asymptotic normal distribution provides a satisfactory approximation to the finite sample distribution of the estimates. This normal approximation can be upgraded by the adjustment of bias to the estimates. First-order bias correction plays an excellent role in bias reduction but MSE might increase. Correction of bias is beneficial in practice depends mainly on the shape of the bias function and the variance of the MLE.

The figures in Table 1 indicate that the MSE of ML estimators of a, b, and β decreases and their biases decay towards 0 as sample size increases.

10 Data Analysis

This section explains and proves the litheness of the new distribution KwME by its application to the two real data sets empirically. We compare it with the fits of the Exponential (E) distribution Epstein (1958), beta-exponential (BE) distribution Nadarajah and Kotz (2006), Kumaraswamy exponential (KwE) distribution Cordeiro et al. (2010), moment exponential (ME) distribution Dara and Ahmad (2012), Exponentiated inverted Weibull (EIW) distribution Flaih et al. (2012), Exponentiated moment exponential (EME) distribution Hasnain et al. (2015) and generalized moment exponential (GEME) distribution Iqbal et al. (2014). Their probability density functions are given by:

$$KwE:\ f(x) = ab\ \lambda\ e^{-\lambda x}\left[1 - e^{-\lambda x}\right]^{b-1}\left(1 - \left(1 - e^{-\lambda x}\right)^{b}\right)^{a-1}$$

$$GEME:\ f(x) = a\lambda\frac{x^{2\lambda-1}}{\beta^2}e^{-\frac{x^\lambda}{\beta}}\left[1 - \left(1 + \frac{x^\lambda}{\beta}\right)e^{-\frac{x^\lambda}{\beta}}\right]^{a-1}$$

$$EME:\ f(x) = a\frac{x}{\beta^2}e^{-\frac{x}{\beta}}\left[1 - \left(1 + \frac{x}{\beta}\right)e^{-\frac{x}{\beta}}\right]^{a-1}$$

$$BE: f(x) = \frac{\lambda}{B(a, b)}e^{-b\lambda x}\left[1 - e^{-\lambda x}\right]^{a-1}$$

Table 1: Mean estimates, bias and MSE of Estimated parameters

a	b	β	Sample size	Parameter	Mean	Bias	MSE
2.5	3.5	1.5	50	a	4.335	1.835	43.60
				b	10.69	7.193	226.9
				β	2.067	0.567	2.683
			150	a	2.759	0.259	1.111
				b	7.997	4.497	136.8
				β	1.911	0.411	1.582
			300	a	2.614	0.114	0.390
				b	5.570	2.070	46.46
				β	1.718	0.218	0.706
			500	a	2.259	0.061	0.195
				b	4.551	1.051	16.19
				β	1.624	0.124	0.343
2.5	2	2	50	a	2.259	1.009	12.44
				b	12.12	8.123	246.9
				β	3.149	1.149	8.839
			150	a	1.393	0.143	0.249
				b	11.43	7.432	255.3
				β	3.175	1.175	8.501
			300	a	1.303	0.053	0.059
				b	9.891	5.891	196.7
				β	2.989	0.989	6.481
			500	a	1.278	0.028	0.031
				b	8.316	4.316	133.3
				β	2.760	0.760	4.586
3.5	2.5	2	50	a	2.917	1.417	23.89
				b	7.589	5.089	108.5
				β	4.619	1.619	18.54
			150	a	1.707	0.209	0.489
				b	6.399	3.899	93.07
				β	4.373	1.373	1.373
			300	a	1.599	0.099	0.131
				b	4.769	2.269	55.86
				β	2.041	0.041	0.091
			500	a	1.564	0.064	0.066
				b	3.691	1.191	21.57
				β	3.454	0.454	4.311

$$EIW: \; f(x) = \lambda\beta x^{-(\beta+1)}\left(e^{-x^{-\beta}}\right)^{\lambda}$$

$$ME: \; f(x) = \frac{x}{\beta^2}\mathrm{e}^{-\frac{x}{\beta}}$$

$$Exp: \; f(x) = \lambda e^{-\lambda x}$$

All parameters of the these mentioned densities are positive real numbers.

Data: In this section, we offer a data analysis for a simple uncensored data set to see how the new distribution works in practice. The first data set has been obtained from Nichols and Padgett (2006), the data concerning tensile strength of 100 observations of carbon fibers.

The second data set has been obtained from Al-Aqtash et al. (2014) and consists of 66 observations. The observations are breaking stress of carbon fibers of 50 mm length (GPa). This data was used by Cordeiro and Lemonte (2011) in the application of the four-parameter beta-Birnbaum-Saunders distribution (BBS) and compared it with two-parameter Birnbaum-Saunders distribution by Birnbaum and Saunders (1969).

Table 2 provides some descriptive statistics about both data sets. Table 3 and 4 provide MLEs of the model parameters and some comparative measures for the first data sets. MLEs of the model parameters and corresponding some comparative measures for second data set are given in Table 5 and 6 respectively. Comparative measure includes Akaike information criterion (AIC), Bayesian information criterion (BIC) and A2, W2 (Anderson Darling comparison measures). Since the values of these measures are smaller for KwME distribution compared with those values of other distributions, the new proposed model seems to be very competitive model for these data sets.

Table 2: Descriptive statistics

Data	Q1	Median	Q3	Mean	Max.	skewness	kurtosis
Data 1	1.840	2.675	3.198	2.611	5.560	0.392	3.18
Data 2	2.555	2.95	3.295	2.962	4.900	-0.130	0.34

Table 3: MLEs for first data set

Distribution	$\hat{a}$	$\hat{b}$	$\hat{\beta}$	$\hat{\lambda}$	
KwME	1.7317	27.210	-	4.2949	-
E	-	-	-	0.3829	
BE	7.0046	15.842	-	0.1270	
KwE	3.4966	21.921	-	0.1633	
ME	-	-	1.3057	-	
EIW	-	-	1.7737	3.0855	

Table 4: Some statistics for models fitted to first data set

Distribution	AIC	BIC	-2L	A	W
KwME	287.44	290.65	281.44	0.44260	0.07783
E	393.97	396.18	391.98	17.5557	3.48081
BE	291.15	294.35	285.14	4.95641	0.78112
KwE	287.45	296.66	281.46	1862.62	33.4998
ME	333.85	341.06	331.86	8.14797	1.45650
EIW	348.99	354.20	344.99	5.45403	0.90695

Table 5: MLEs for second data set

Distribution	$\hat{a}$	$\hat{b}$	$\hat{\beta}$	$\hat{\lambda}$
KwME	1.72223	$2.9496 * 10^9$	1214.64	-
E	-	-	-	0.362379
BE	7.48244	217.156	-	0.012304
ME	-	-	1.37977	-
EME	-	-	0.779259	-
GEME	0.4403	-	44.1001	3.4466

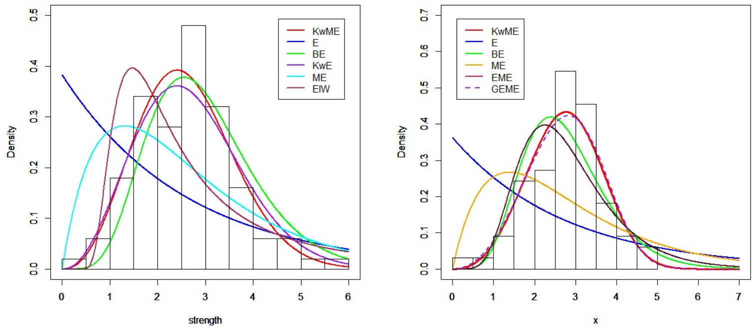
Figures 3 and 4 show that KwME distribution is better than the other models in the model fitting. Table 4 and Table 5 also shows that KWME distribution provide a good fit for both datasets.

11 Conclusions

We introduce a three parameter probability distribution, named the Kumaraswamy moment exponential (KwME) distribution, which extends the moment exponential distribution. We provide detailed study of its mathematical properties including moment generating function, moments, the densities of the order statistics, and entropies. We study the estimation procedure by maximum likelihood method. Since biases of MLEs decreases as 'n' increases. Thus, this simulation study supports the use of the Kumaraswamy moment exponential distribution for describing the data sets. Both applications to real data sets show that the fit of the new model is superior to the existing models in probability theory. We hope that KwME attract wider applications in probability theory.

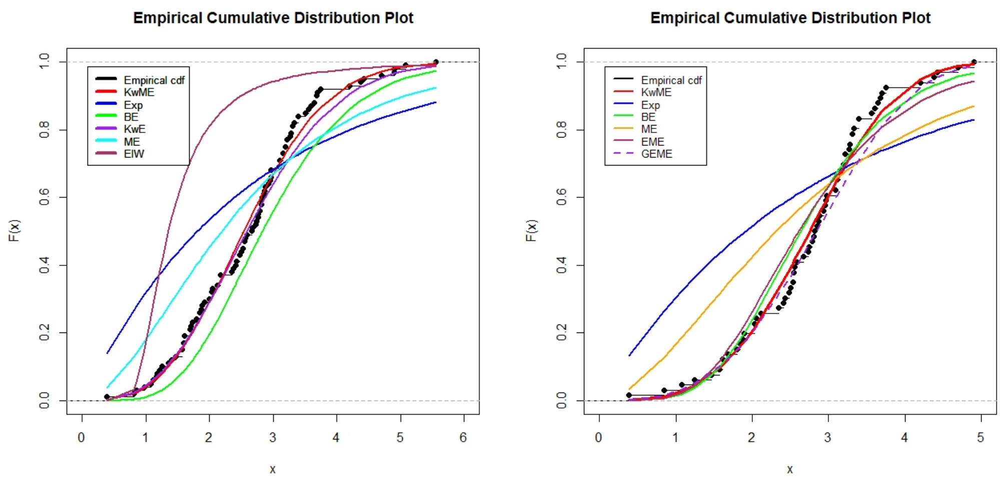
Table 6: Some statistics for models fitted to second data set

Distribution	AIC	BIC	-2L	A	W
KwME	177.889	184.458	172.138	0.49185	0.08432
E	267.989	270.178	265.989	17.6309	3.69494
BE	188.336	194.905	182.336	1.32817	0.24846
ME	226.008	228.197	224.008	8.09264	1.54250
EME	192.130	196.510	188.13	1.90614	0.34345
EIW	178.740	184.643	172.74	0.50671	0.08594



(a) (b)

Figure 3: The fitted KwME density curves.



(a) (b)

Figure 4: The fitted KwME cdf curves.

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