Electronic Journal of Applied Statistical Analysis
EJASA, Electron. J. App. Stat. Anal.
`http://siba-ese.unisalento.it/index.php/ejasa/index`
e-ISSN: 2070-5948
DOI: 10.1285/i20705948v12n3p619

Inference on $P(X < Y)$ in *bivariate Lomax* Model

By Musleh, Helu, Samawi

Published: 20 November 2019

Electronic Journal of Applied Statistical Analysis
Vol. 13, Issue 03, November 2019, 619-636
DOI: 10.1285/i20705948v12n3p619

Inference on $P(X < Y)$ in *bivariate Lomax* Model

Rola M. Musleh*[a], Amal Helu[a], and Hani Samawi[b]

[a]*Department of mathematics, The University of Jordan , Amman, Jordan*
[b]*Jiann-Ping Hsu College of Public Health, Georgia Southern University, USA*

Published: 20 November 2019

In this article we consider the estimation of the stress-strength reliability parameter, $R = P(X < Y)$ when the stress (X) and the strength (Y) are dependent random variables distributed as *bivariate Lomax* model. The maximum likelihood, moment and Bayes estimators are derived. We obtained Bayes estimators using symmetric and asymmetric loss functions via squared error loss and *Linex* loss functions respectively. Since there are no closed forms for the Bayes estimators, we used an approximation based on Lindley's method to obtain Bayes estimators under these loss functions. An extensive computer simulation is used to compare the performance of the proposed estimators using three criteria, namely, relative bias, mean squared error and Pitman nearness (PN) probability. Real data application is provided to illustrate the performance of our proposed estimators using bootstrap analysis.

keywords: *Bivariate Lomax* distribution, Lindley's approximation, Pitman nearness probability.

1 Introduction

Due to the diversity of purposes and applications, the study of reliability models received the attention of researchers from many diverse disciplines. As a result, reliability models have been considered from different perspectives. The most widely used approach for reliability estimation is the well known stress-strength model, $R = P(X < Y)$ where X and Y are random variables. In this model, the reliability R, of the system is the

*muslehrolla@hotmail.com

ISSN: 2070-5948
http://siba-ese.unisalento.it/index.php/ejasa/index

probability that the system is strong enough to overcome the stress imposed on it. The reliability of aircrafts' windshields is an example of aerodynamics and mechanical engineering. The windshields consist of several layers of materials to withstand extreme temperatures and pressure. In order to maintain a regular performance of aircrafts, it is a vital information to know what the probability of windshield failure is at different stages of the windshield life (after 1000, 2000…etc., of flight hours). Given a good estimate of the windshield reliability by defining the stress to be the temperature and/or the pressure differential and the strength to be the thickness and/or the composition of the windshield layers. One can then make a rational decision about when windshields need to be repaired or replaced.

Another example is the customer satisfaction which has been a main interest for manufacturers to produce reliable products. For their products to remain desired and thus profitable, they are motivated to develop high quality and long life products. This requires having knowledge about products failure time distributions which is achieved by performing life testing experiments on products before being released into the markets. After knowing the failure time distribution, the manufacturer finds the reliability characteristics such as hazard rate and mean time to system failure.

It is worth mentioning that R is of greater interest than just a stress-strength model since it provides a general measure of the difference between two populations and has applications in many areas. For example, in clinical studies we may be interested in comparing the effectiveness of two drugs, so X may represent the life time lived by a patient when treated with a certain drug, and Y represents the life time lived by another patient when treated with another drug.

A vast number of researchers dedicated so much of their work to study the stress-strength model. Birnbaum et al. (1956) was the first connected the stress-strength model with the Man-Whitney statistic in order to estimate R in case where X and Y are independent. More works have followed to provide point and interval estimation of R using different approaches. For example Kotz and Pensky (2003) provided a comprehensive review of the development of the stress-strength reliability and its applications until the year 2003. Recently, Rezaei et al. (2010) studied the estimation of R when X and Y are two independent generalized Pareto distribution with different parameters. RRL et al. (2010) studied the reliability in multicomponent stress-strength model when X and Y follow log-logistic distribution. Barbiero (2013) studied the reliability of stress-strength model when X and Y are independent Poisson random variables. Whereas, Al-Mutairi et al. (2013) considered the problem of estimating R when X and Y are distributed as Lindley with different shape parameters. Ghitany et al. (2015) derived a point and interval estimation of R using maximum likelihood, parametric and non-parametric bootstrap methods when X and Y are independent power Lindley random variables. Makhdoom et al. (2016) extended the work of Ghitany et al. (2015) and developed a Bayesian inference on R. In addition, Wong (2012) derived an asymptotic confidence intervals for R when X and Y are two independent generalized Pareto random variables with same scale parameter. However, several researchers have focused their interest to a more realistic problem that is the estimation of R in the case where

X and Y are dependent. For example, but not limited to, Barbiero (2012) assumed that (X,Y) are jointly normally distributed; Rubio et al. (2013) assumed that X and Y are marginally distributed as a skewed scale mixture of normal distribution and constructed the corresponding joint distribution using a Gaussian copula; Domma and Giordano (2013) constructed the joint distribution of (X, Y) using a Farlie-Gumbel-Morgenstern copula with marginal distributions belonging to the Burr system; Domma and Giordano (2012) considered Dagum distributed marginals and constructed their joint distribution using a Frank copula; Samawi et al. (2016) considered the problem of estimating R when X and Y are dependent random variables with a bivariate underlying distribution using kernel estimation and bivariate ranked set sampling; Nasiri (2016) considered the estimation of R for *Lomax* distribution with presence of outliers, among others (Gupta et al. (2013); Nadarajah (2005)).

Our focus is on estimating $R = P(X < Y)$ when X and Y follow a *bivariate Lomax* distribution with different parameters. This model was proposed by Lindley and Singpurwalla (1986), considering a two components system where in a given environment, η, the component lifetimes X and Y are conditionally independently exponentially distributed with failure rates $\eta\lambda_1$ and $\eta\lambda_2$ respectively, where λ_1 and λ_2 are the failure rates under the test environment. Then

$$f(x, y|\eta) = \int \eta\lambda_1 e^{-\eta\lambda_1} \eta\lambda_2 e^{-\eta\lambda_2} dG(\eta), \tag{1}$$

where $G(\eta)$ is the distribution function of η. In order to find the unconditional distribution of (X, Y), we assign Gamma distribution $g_\eta(c, b)$ as a distribution of η, with a density

$$g_\eta(c, b) = \frac{b^c \eta^{c-1} e^{-\eta b}}{\Gamma(c)}, \quad \eta, c, b > 0. \tag{2}$$

Then, the joint distribution function of (X, Y) is given by

$$\begin{aligned} f(x, y) &= \int f(x, y|\eta) g_\eta(c, b) d\eta \\ &= \frac{\lambda_1 \lambda_2 b^c c(c+1)}{(b + \lambda_1 x + \lambda_2 y)^{c+2}}. \end{aligned} \tag{3}$$

Let $\alpha_1 = \frac{\lambda_1}{b}$ and $\alpha_2 = \frac{\lambda_2}{b}$, hence (3) is reduced to

$$f(x, y) = \frac{\alpha_1 \alpha_2 c(c+1)}{(1 + \alpha_1 x + \alpha_2 y)^{c+2}}, \quad x, y, \alpha_1, \alpha_2, c\ > 0\ , \tag{4}$$

which is known as *bivariate Lomax* distribution. It can be shown that the joint survival function of (X, Y), $\overline{F}(x, y)$, the marginal of X, $h_1(x)$ and the marginal of Y, $h_2(y)$ are as follows:

$$\overline{F}(x,y) = (1+\alpha_1 x+\alpha_2 y)^{-c}, \quad x,y,c,\alpha_1,\alpha_2 > 0. \quad (5)$$

$$h_1(x) = \frac{c\alpha_1}{(1+\alpha_1 x)^{(c+1)}}, \quad x,c,\alpha_1,\alpha_2 > 0. \quad (6)$$

$$h_2(y) = \frac{c\alpha_2}{(1+\alpha_2 y)^{(c+1)}}, \quad y,c,\alpha_1,\alpha_2 > 0. \quad (7)$$

The quantity of interest is the parameter $R = P(X < Y)$ which is derived as

$$\begin{aligned} R &= P(X<Y) = \int_0^\infty\int_x^\infty \frac{\alpha_1\alpha_2 c(c+1)}{(1+\alpha_1 x+\alpha_2 y)^{c+2}} dydx \\ &= \frac{\alpha_1}{\alpha_1+\alpha_2}. \end{aligned} \quad (8)$$

The aim of the paper is to consider classical and Bayesian estimation of R. As for the classical estimation we suggest using the maximum likelihood estimator (MLE) and the moment estimator (MOM). While for the Bayes estimates we will derive them based on symmetric and asymmetric loss functions. It is observed that the Bayes estimates cannot be obtained in explicit forms, so instead of using numerical techniques, approximation method such as Lindley's approximation is applied. To compare the performance of the proposed estimators using real data, we use bootstrap approach to calculate the bootstrap bias, standard error, lower and upper confidence interval limits (see Efron and Tibshirani (1994)).

This paper is arranged as follows: In Section 2 we derive the classical estimates. The Bayes estimation is provided in Section 3. In Section 4 we provide the simulation study. Results of the simulation study are in Section 5. In Section 6 we illustrate the proposed procedures of estimation using a real data example. Our conclusion and remarks are presented in Section 7.

2 Classical estimation procedures

2.1 Maximum likelihood estimator of R

Let $(x_1, y_1), (x_2, y_2), ..., (x_m, y_m)$ be a random sample of size m from a *bivariate Lomax* distribution with pdf defined in (4), then the likelihood function is given by

$$L(\alpha_1,\alpha_2;\underline{x},\underline{y}) = \alpha_1^m\alpha_2^m(c(c+1))^m \prod_{j=1}^m (1+\alpha_1 x_j+\alpha_2 y_j)^{-(c+2)}, \quad (9)$$

and the corresponding log likelihood function is

$$l(\alpha_1,\alpha_2;\underline{x},\underline{y}) = m\ln\alpha_1 + m\ln(\alpha_2) + m\ln c(c+1) - (c+2)\sum_{j=1}^m \ln(1+\alpha_1 x_j+\alpha_2 y_j). \quad (10)$$

The MLEs of the parameters α_1, α_2 and c, denoted by $\widehat{\alpha}_1$, $\widehat{\alpha}_2$ and $\widehat{c}$ respectively, can be obtained by taking the first derivative of (10) with respect to α_1, α_2 and c and then equating the normal equations to 0 as follows:

$$\frac{\partial l(\alpha_1, \alpha_2; \underline{x}, \underline{y})}{\partial \alpha_1} = \frac{m}{\alpha_1} - (c+2)\sum_{j=1}^{m} \frac{x_j}{(1+\alpha_1 x_j + \alpha_2 y_j)} = 0 \quad (11)$$

$$\frac{\partial l(\alpha_1, \alpha_2; \underline{x}, \underline{y})}{\partial \alpha_2} = \frac{m}{\alpha_2} - (c+2)\sum_{j=1}^{m} \frac{y_j}{(1+\alpha_1 x_j + \alpha_2 y_j)} = 0 \quad (12)$$

$$\frac{\partial l(\alpha_1, \alpha_2; \underline{x}, \underline{y})}{\partial c} = \frac{m(2c+1)}{c(c+1)} - \sum_{j=1}^{m} \ln(1+\alpha_1 x_j + \alpha_2 y_j) = 0. \quad (13)$$

Note that there is no explicit solution to Eqs.(11)-(13). Therefore, we implement Newton-Raphson method by using $SAS\backslash IML$ language to obtain MLEs of α_1, α_2 and c . Once $\widehat{\alpha}_1$, $\widehat{\alpha}_2$ are obtained, the MLE of R, denoted by $\widehat{R}_{MLE}$, is obtained.

$$\widehat{R}_{MLE} = \frac{\widehat{\alpha}_1}{\widehat{\alpha}_1 + \widehat{\alpha}_2}. \quad (14)$$

2.2 Moment estimator of R

The method of moments, estimating the parameters of the probability distribution by matching the sample moment

$$m_x = \frac{1}{m}\sum_{i=1}^{m} X_i = \overline{X}, \quad (15)$$

with the theoretical moment

$$\mu_x = \int x h_1(x) dx = \frac{1}{\alpha_1(c-1)}. \quad (16)$$

Equating equations (15) and (16) and solving for $\alpha_{1,mom}$, we get

$$\alpha_{1,mom} = \frac{1}{\overline{X}(c-1)}. \quad (17)$$

Similarly,

$$m_y = \frac{1}{m}\sum_{i=1}^{m} Y_i = \overline{Y}, \quad (18)$$

$$\begin{aligned}\mu_y &= \int y h_2(y) dy \\ &= \frac{1}{\alpha_2(c-1)}.\end{aligned} \tag{19}$$

Equating equations (18) and (19) and solving for $\alpha_{2,mom}$, we get

$$\alpha_{2,mom} = \frac{1}{\overline{Y}(c-1)}. \tag{20}$$

m_x & m_y are viewed as an estimator of μ_x and μ_y respectively. From the law of large numbers: $m_x \to \mu_x$ and $m_y \to \mu_y$ in probability as $m \to \infty$.
Substitute $\alpha_{1,mom}$ and $\alpha_{2,mom}$ in equation (8), we get the moment estimator of R, denoted by $\widehat{R}_{MOM}$

$$\begin{aligned}\widehat{R}_{MOM} &= \frac{\frac{1}{\overline{X}(c-1)}}{\frac{1}{\overline{X}(c-1)} + \frac{1}{\overline{Y}(c-1)}} \\ &= \frac{1}{1 + \frac{\overline{X}}{\overline{Y}}} \\ &= \frac{\overline{Y}}{\overline{Y} + \overline{X}}\end{aligned} \tag{21}$$

3 Bayes estimator of R

Bayesian estimation for the probability $R = P(X < Y)$ under *bivariate Lomax* distribution is obtained. Bayes method, which considers the parameters to be random variables with distributions commonly known as prior distributions. The Bayes method is effected by the choice of the loss function not just by the choice of the prior distribution. In the literature, the most popular loss function is the symmetric squared error loss function (SEL). The SEL is widely employed in the Bayesian inference due to its computational simplicity. It is a symmetric loss function that gives equal weight to overestimation as well as underestimation. However, this is not a good criteria from a practical point of view. For example, Feynman (1987) stated that in the disaster of the space shuttle, Challenger, the management may have overestimated the average life or reliability of solid fuel rocket booster. In estimating reliability and failure rate functions, an overestimation causes more damage than underestimation. To resolve such situation, asymmetrical loss functions are more appropriate.
Varian (1975) introduced the *Linex* loss function (Linear- Exponential) in response to the criticism of the SEL. *Linex* loss function has been widely used by several authors such as Karimnezhad (2013), Rasheed and Sultan (2015), Yu and Xie (2016), Metiri et al. (2016) and Rizki et al. (2017). *Linex* loss function rises approximately exponentially on one side of zero and approximately linearly on the other side. The *Linex* loss function

is defined as follows:

$$L(\widehat{\theta},\theta)=\exp(\lambda(\widehat{\theta}-\theta))-\lambda(\widehat{\theta}-\theta)-1,\quad \lambda\neq 0 \tag{22}$$

To obtain the Bayes estimator of $R=P(X<Y)$, we assume that, the parameters α_1 and α_2 have independent gamma priors:

$$\alpha_1\sim\pi_1(\alpha_1)=\frac{\alpha_1^{a_1-1}e^{-\alpha_1}}{\Gamma(a_1)}, \tag{23}$$

$$\alpha_2\sim\pi_2(\alpha_2)=\frac{\alpha_2^{a_2-1}e^{-\alpha_2}}{\Gamma(a_2)}, \tag{24}$$

where a_1 and a_2 are assumed to be known and non-negative. Without loss of generality, we assume the scale parameters to be 1. We will use (23) and (24) to construct prior distribution of $R=\frac{\alpha_1}{\alpha_1+\alpha_2}$. Let $S=\alpha_1+\alpha_2$ which follows gamma distribution with a_1+a_2 as its shape parameter and 1 as its scale parameter,

$$\pi_S=\frac{S^{a_1+a_2-1}e^{-S}}{\Gamma(a_1+a_2)}. \tag{25}$$

Using change of variables, the prior distribution of R is Dirichelet (a_1,a_2):

$$\begin{aligned}\pi_R &= \frac{\Gamma(a_1+a_2)}{\Gamma(a_1)\Gamma(a_2)}\left(\frac{\alpha_1}{\alpha_1+\alpha_2}\right)^{a_1-1}\left(\frac{\alpha_2}{\alpha_1+\alpha_2}\right)^{a_2-1}\\ &= \frac{\Gamma(a_1+a_2)}{\Gamma(a_1)\Gamma(a_2)}R^{a_1-1}(1-R)^{a_2-1}.\end{aligned} \tag{26}$$

Hence, the joint prior distribution of R and S:

$$\pi(R,S)\propto S^{a_1+a_2-1}e^{-S}R^{a_1-1}(1-R)^{a_2-1}, \tag{27}$$

Next, we redefine the likelihood function in Eq. (9) so it will be a function of R and S:

$$L(R,S;\underline{x},\underline{y})=(RS)^m(S(1-R))^m(c(c+1))^m\prod_{j=1}^{m}(1+RS(x_j-y_j)+Sy_j)^{-(c+2)}. \tag{28}$$

Thus, the joint posterior distribution of R & S after observing $(x_1,y_1),...,(x_m,y_m)$ is as follows:

$$\pi^*(R,S|\underline{x},\underline{y})\propto R^{m+a_1-1}(1-R)^{m+a_2-1}S^{2m+a_1+a_2-1}e^{-S}\prod_{j=1}^{m}(1+RS(x_j-y_j)+Sy_j)^{-(c+2)}. \tag{29}$$

Therefore, the Bayes estimator of any function of R and S say $u(R,S)$ is the posterior expected value. Let $u(R,S)$ be a function of R and S, then the expected value of $u(R,S)$ is given by:

$$\widehat{u} = E_{\pi^*}(u(R,S)|\underline{x},\underline{y}) = \frac{\int_0^\infty\int_0^1 u(R,S)\pi^* dRdS}{\int_0^\infty\int_0^1 \pi^* dRdS}$$

$$= \frac{\int_0^\infty\int_0^1 u(R,S)e^{l(R,S|\underline{x},\underline{y}))+\rho(R,S)}dRdS}{\int_0^\infty\int_0^1 e^{l(R,S|\underline{x},\underline{y})+\rho(R,S)}dRdS} \quad (30)$$

where $l(R,S|\underline{x},\underline{y}) = log\ L(R,S|\underline{x},\underline{y})$, and $\rho(R,S) = log\ \pi(R,S)$.

It can be noticed that $\widehat{u}$ is in the form of ratio of two integrals which cannot be simplified to closed form. Hence, Lindley's approximation method is applied to obtain the Bayes estimator of R, see Lindley (1980). Then Eq. (30) is reduced to the following numerical expression:

$$\begin{aligned}\widehat{u} = {} & u(\widehat{R},\widehat{S}) + 0.5[(\widehat{u}_{RR} + 2\widehat{u}_R\widehat{\rho}_R)\widehat{\sigma}_{RR} + (\widehat{u}_{SR} + 2\widehat{u}_S\widehat{\rho}_R)\widehat{\sigma}_{SR} + (\widehat{u}_{RS} + 2\widehat{u}_R\widehat{\rho}_S) \\ & + \widehat{\sigma}_{RS} + (\widehat{u}_{SS} + 2\widehat{u}_S\widehat{\rho}_S)\widehat{\sigma}_{SS}] + 0.5[(\widehat{u}_R\widehat{\sigma}_{RR} + \widehat{u}_S\widehat{\sigma}_{RS})(\widehat{l}_{RRR}\widehat{\sigma}_{RR} + \widehat{l}_{RSR}\widehat{\sigma}_{RS} \\ & + \widehat{l}_{SRR}\widehat{\sigma}_{SR} + \widehat{l}_{SSR}\widehat{\sigma}_{SS}) + (\widehat{u}_R\widehat{\sigma}_{SR} + \widehat{u}_S\widehat{\sigma}_{SS})(\widehat{l}_{SRR}\widehat{\sigma}_{RR} + \widehat{l}_{RSS}\widehat{\sigma}_{RS} + \widehat{l}_{SRS}\widehat{\sigma}_{SR} \\ & + \widehat{l}_{SSS}\widehat{\sigma}_{SS})]. \end{aligned} \quad (31)$$

where $\widehat{R}$ and $\widehat{S}$ are the MLEs of R and S respectively, $\hat{u}_R = \frac{\partial^2 u(R,S)}{\partial R}|_{(\widehat{R},\widehat{S})}$, $\hat{u}_{RR} = \frac{\partial^2 u(R,S)}{\partial R \partial S}|_{(\widehat{R},\widehat{S})}$, $\widehat{\rho}_R = \frac{a_1-1}{\widehat{R}} - \frac{a_2-1}{(1-\widehat{R})}, \widehat{\rho}_S = \frac{a_1+a_2-1}{\widehat{S}} - 1$. Other expressions can be defined similarly (see Appendix).

3.1 Bayes estimate of R under squared error loss function

If $u(R,S) = R, u_R = 1, u_S = u_{SS} = u_{RR} = u_{RS} = u_{SR} = 0$, then,

$$\begin{aligned}\widehat{R}_{SEL} = {} & \widehat{R} + \widehat{\sigma}_{RR}(\widehat{\rho}_R + \widehat{l}_{RRS}\widehat{\sigma}_{RS}) + \widehat{\sigma}_{RS}(\widehat{\rho}_S + \widehat{l}_{RSS}\widehat{\sigma}_{RS}) + \\ & 0.5[\widehat{\sigma^2}_{RR}\widehat{l}_{RRR} + \widehat{\sigma}_{RR}(\widehat{\sigma}_{RS}\widehat{l}_{RRS} + \widehat{\sigma}_{SS}\widehat{l}_{RSS}) + \widehat{\sigma}_{RS}\widehat{l}_{SSS}\widehat{\sigma}_{SS}]. \end{aligned} \quad (32)$$

3.2 Bayes estimate of R under *Linex* loss function

If $u(R,S) = e^{-\lambda R}$, $u_R = -\lambda e^{-\lambda R}$, $u_{RR} = \lambda e^{-\lambda R}, u_S = u_{SS} = u_{RS} = u_{SR} = 0$ then

$$\begin{aligned}E_{\pi^*}(e^{-\lambda R}|\underline{x},\underline{y}) = {} & e^{-\lambda\widehat{R}} + 0.5\lambda^2 e^{-\lambda\widehat{R}}\widehat{\sigma}_{RR} - \lambda e^{-\lambda\widehat{R}}[\widehat{\sigma}_{RR}(\widehat{\rho}_R + \widehat{l}_{RRS}\widehat{\sigma}_{RS}) + \widehat{\sigma}_{RS}(\widehat{\rho}_S + \widehat{l}_{RSS}\widehat{\sigma}_{RS})] \\ & -0.5\lambda e^{-\lambda\widehat{R}}[\widehat{\sigma^2}_{RR}\widehat{l}_{RRR} + \widehat{\sigma}_{RR}(\widehat{\sigma}_{RS}\widehat{l}_{RRS} + \widehat{\sigma}_{SS}\widehat{l}_{RSS}) + \widehat{\sigma}_{RS}\widehat{l}_{SSS}\widehat{\sigma}_{SS}]. \end{aligned} \quad (33)$$

Hence, the Bayes estimate of R is obtained by

$$\widehat{R}_{LIN} = -\frac{1}{\lambda}\log E_{\pi^*}(e^{-\lambda R}|\underline{x}, \underline{y}), \tag{34}$$

where,

$$E_{\pi^*}(e^{-\lambda R}|\underline{x}, \underline{y}) = \frac{\int_0^\infty\int_0^1 e^{-\lambda R} e^{l(R,S|\underline{x},\underline{y}))+\rho(R,S)} dRdS}{\int_0^\infty\int_0^1 e^{l(R,S|\underline{x},\underline{y})+\rho(R,S)} dRdS}. \tag{35}$$

4 Simulation Study

The purpose of the simulation study is to compare the performance of the classical estimators (MLE, MOM) with the Bayes estimators under symmetric and asymmetric loss functions using independent gamma priors for the parameters α_1 and α_2.

The "Method of Mixture" has been used to generate new random samples of the *bivariate Lomax* distribution, which depends on the fact $f(x, y) = f(y|x).f(x)$. A bivariate pair (x, y) is generated by sequentially simulating steps starting by generating observation x from its marginal distribution, then using the conditional distribution $f(y|x)$ to generate y given the generated value of x.

Values of α_1 and α_2 are generated from π_1 and π_2 given in (23) and (24) with specified parameters a_1 and a_2. The resulted values of α_1 and α_2 are considered to be the true values that will be used to generate the *bivariate Lomax* random samples.
Our simulation is based on 5000 simulated sets over the following values of a_1 (= 1.5,2,3,4,5), a_2 (=3,5,5.5,6), $c = 10$ and $\lambda = 1$ and in each set the classical and Bayes estimators are computed. For each set of data, the classical and the Bayesian estimators are computed. We obtain the MLEs of α_1, α_2 and c by solving the nonlinear equations (11)-(13) using Newton-Raphson algorithm. $\widehat{R}_{MLE}$ is obtained by substituting $\widehat{\alpha}_1, \widehat{\alpha}_2$ in Eq. (14).

The three criteria used for comparing all the above estimators are the relative bias ($RBias$), mean squared error (MSE) and Pitman nearness (PN) probability. Suppose $\widehat{R}_i$ is the estimate of R for the i^{th} simulated data set, then the $RBias$, MSE and PN are computed as follows:

1. $RBias = \frac{(\widehat{R}_i - R)}{R} \times 100$.

2. $MSE = \frac{1}{5000}\sum_{j=1}^{5000}(\widehat{R}_j - R)^2$.

3. Pitman Nearness: Suppose $\widehat{R}_i$ and $\widehat{R}_j$ $(i \neq j)$ are two estimators of parametric function R. Then $\widehat{R}_i$ is said to be Pitman nearness to R relative to $\widehat{R}_j$ if $PN = P((\widehat{R}_i - R)'(\widehat{R}_i - R) < (\widehat{R}_j - R)'(\widehat{R}_j - R)) \geq 0.5$.

5 Results of the simulation study

Results are summarized in Tables 1-3, provided at the end of this section, as follows:

- Table 1 presents the $RBias$ and MSE values for the estimators of R.
- Tables 2 and 3 display the PN probabilities of the estimators of R relative to each other.

A summary of the results is provided below:

- $\widehat{R}_{MOM}$ and $\widehat{R}_{MLE}$ are equivalent in terms of the $RBias$ and MSE values.
- Classical estimators outperform the Bayes estimators in terms of the $RBias$ values especially for small values of m ($m = 10\&20$). However, they are equivalent as m increases ($m > 20$). In addition, classical and Bayesian estimators are equivalent in terms of MSE values.
- Among Bayes estimators, $\widehat{R}_{SEL}$ outperforms $\widehat{R}_{LIN}$ in terms of $RBias$ values although they are equivalent in terms of the MSE values.
- In terms of PN probabilities, we notice that
  - $\widehat{R}_{MLE}$ outperforms $\widehat{R}_{MOM}$ for all values of m, α_1 and α_2.
  - For small values of $m(m = 10)$ and under all choices of α_1 & α_2, it is observed that the classical estimators outperform the Bayes estimators. However, as m values increase ($m \geq 50$) Bayes estimators prevail over the classical estimators.
  - Among Bayes estimators, $\widehat{R}_{SEL}$ outperforms $\widehat{R}_{LIN}$ for small values of m ($m = 10\&20$), while for large values of m ($m \geq 50$) and for most choices of α_1 and α_2 $\widehat{R}_{LIN}$ is superior to $\widehat{R}_{SEL}$.

Table 1: $RBias$ and MSE of R using different estimation methods

		Estimation	$m = 10$		$m = 20$		$m = 50$		$m = 100$	
(a_1, a_2)	R	Method	$RBias$	MSE	$RBias$	MSE	$RBias$	MSE	$RBias$	MSE
$(1.5, 3.0)$	0.3327	MLE	1.4480	0.0105	1.0213	0.0052	0.6968	0.0021	0.8250	0.0010
		MOM	1.4415	0.0105	1.0399	0.0053	0.7164	0.0021	0.8195	0.0010
		SEL	5.9594	0.0264	1.9993	0.0049	0.8543	0.0019	0.5579	0.0010
		LIN	7.5781	0.0399	2.5248	0.0049	1.1584	0.0019	0.7130	0.0010
$(1.5, 5.0)$	0.2286	MLE	6.5896	0.0073	0.7879	0.0034	1.9230	0.0013	2.6419	0.0007
		MOM	6.5746	0.0073	0.8172	0.0035	1.9539	0.0014	2.6529	0.0007
		SEL	14.909	0.0495	3.2858	0.0054	2.3451	0.0014	2.7346	0.0006
		LIN	22.230	0.0856	3.5098	0.0059	2.6943	0.0013	2.9886	0.0006
$(2.0, 5.0)$	0.2850	MLE	3.6893	0.0092	1.7074	0.0045	1.5545	0.0018	0.0492	0.0009
		MOM	3.6792	0.0092	1.7303	0.0046	1.5790	0.0018	0.0571	0.0009
		SEL	12.485	0.0568	4.2431	0.0060	1.9016	0.0017	0.2677	0.0008
		LIN	17.271	0.0912	4.2583	0.0070	2.1798	0.0017	0.3166	0.0008
$(2.0, 6.0)$	0.2508	MLE	4.4180	0.0080	1.6960	0.0038	0.6952	0.0015	0.4692	0.0007
		MOM	4.4049	0.0080	1.7231	0.0039	0.7233	0.0015	0.4593	0.0007
		SEL	15.493	0.0724	4.8683	0.0068	1.2252	0.0015	0.1084	0.0007
		LIN	22.660	0.1215	5.3807	0.0086	1.4687	0.0015	0.2506	0.0007
$(3.0, 5.0)$	0.3749	MLE	1.6692	0.0116	0.3711	0.0058	0.9760	0.0023	0.3936	0.0011
		MOM	1.6650	0.0116	0.3563	0.0058	0.9916	0.0024	0.3901	0.0012
		SEL	7.0138	0.0647	1.2849	0.0058	1.3095	0.0020	0.2268	0.0010
		LIN	10.191	0.0896	1.3039	0.0070	1.0206	0.0020	0.3793	0.0010
$(3.0, 5.5)$	0.3518	MLE	2.6391	0.0111	1.5588	0.0056	1.4143	0.0022	0.2587	0.0011
		MOM	2.6334	0.0111	1.5752	0.0056	1.4321	0.0023	0.2542	0.0011
		SEL	9.3399	0.0758	3.3862	0.0064	1.5079	0.0020	0.0451	0.0010
		LIN	12.967	0.1030	3.3955	0.0085	1.8230	0.0019	0.1982	0.0010
$(3.0, 6.0)$	0.3329	MLE	2.2531	0.0106	0.5331	0.0053	2.4407	0.0021	0.7800	0.0010
		MOM	2.2465	0.0106	0.5512	0.0053	2.4602	0.0022	0.7854	0.0011
		SEL	10.748	0.0881	2.8326	0.0071	2.9072	0.0019	0.8479	0.0009
		LIN	14.845	0.1166	3.1287	0.0111	2.6286	0.0019	1.0018	0.0009
$(4.0, 5.0)$	0.4426	MLE	1.6715	0.0127	0.2837	0.0064	0.3054	0.0026	0.2083	0.0013
		MOM	1.6708	0.0126	0.2746	0.0065	0.2952	0.0026	0.2075	0.0013
		SEL	3.6069	0.0719	0.1561	0.0051	0.1230	0.0021	0.1282	0.0011
		LIN	6.3757	0.0694	0.4798	0.0067	0.3978	0.0021	0.2724	0.0011
$(4.0, 5.5)$	0.4198	MLE	1.1166	0.0124	0.1676	0.0062	0.3019	0.0025	0.3661	0.0012
		MOM	1.1145	0.0123	0.1786	0.0063	0.3137	0.0026	0.3678	0.0013
		SEL	4.8947	0.0844	1.2778	0.0057	0.2686	0.0020	0.3614	0.0011
		LIN	8.0000	0.0875	1.3222	0.0071	0.5479	0.0020	0.4131	0.0011
$(4.0, 6.0)$	0.3977	MLE	2.0033	0.0121	0.0603	0.0060	0.0272	0.0024	0.0446	0.0012
		MOM	2.0001	0.0120	0.0728	0.0061	0.0406	0.0025	0.0420	0.0012
		SEL	6.9575	0.0970	1.6621	0.0063	0.2022	0.0020	0.0353	0.0011
		LIN	10.454	0.1072	1.6842	0.0085	0.3050	0.0020	0.1148	0.0011
$(5.0, 5.5)$	0.4759	MLE	0.2033	0.0129	0.4036	0.0065	0.3465	0.0026	0.1123	0.0013
		MOM	0.2034	0.0128	0.4104	0.0066	0.3543	0.0027	0.1125	0.0013
		SEL	1.3013	0.0963	0.3059	0.0051	0.1222	0.0020	0.0643	0.0011
		LIN	4.8473	0.0753	0.6703	0.0058	0.3849	0.0020	0.2021	0.0011
$(5.0, 6.0)$	0.4554	MLE	0.5012	0.0128	0.0292	0.0064	0.4223	0.0026	0.8189	0.0013
		MOM	0.5005	0.0127	0.0209	0.0065	0.4130	0.0027	0.8193	0.0013
		SEL	2.6334	0.1085	0.5160	0.0055	0.3004	0.0020	0.6969	0.0011
		LIN	6.2677	0.0893	0.6028	0.0074	0.5672	0.0020	0.8395	0.0011

Table 2: PN comparisons

(a_1,a_2)	R		$m = 10$	$m = 20$	$m = 50$	$m = 100$
(1.5,3.0)	0.3327	$\hat{R}_{MLE}$ vs $\hat{R}_{MOM}$	0.5233	0.5248	0.5302	0.5285
		$\hat{R}_{MLE}$ vs $\hat{R}_{SEL}$	0.5264	0.4704	0.4331	0.4194
		$\hat{R}_{MLE}$ vs $\hat{R}_{LIN}$	0.5293	0.4683	0.4246	0.4113
		$\hat{R}_{MOM}$ vs $\hat{R}_{SEL}$	0.5279	0.4725	0.4396	0.4467
		$\hat{R}_{MOM}$ vs $\hat{R}_{LIN}$	0.5318	0.4724	0.4381	0.4494
		$\hat{R}_{SEL}$ vs $\hat{R}_{LIN}$	0.5650	0.4919	0.4791	0.5340
(1.5,5.0)	0.2286	$\hat{R}_{MLE}$ vs $\hat{R}_{MOM}$	0.5238	0.5259	0.5301	0.5270
		$\hat{R}_{MLE}$ vs $\hat{R}_{SEL}$	0.6207	0.5812	0.5058	0.4755
		$\hat{R}_{MLE}$ vs $\hat{R}_{LIN}$	0.6224	0.5821	0.5021	0.4672
		$\hat{R}_{MOM}$ vs $\hat{R}_{SEL}$	0.6215	0.5748	0.4902	0.4544
		$\hat{R}_{MOM}$ vs $\hat{R}_{LIN}$	0.6235	0.5771	0.4880	0.4504
		$\hat{R}_{SEL}$ vs. $\hat{R}_{LIN}$	0.6840	0.6053	0.4676	0.4138
(2.0,5.0)	0.2850	$\hat{R}_{MLE}$ vs $\hat{R}_{MOM}$	0.5237	0.5255	0.5296	0.5283
		$\hat{R}_{MLE}$ vs $\hat{R}_{SEL}$	0.6014	0.5464	0.4805	0.4547
		$\hat{R}_{MLE}$ vs $\hat{R}_{LIN}$	0.6044	0.5463	0.4757	0.4495
		$\hat{R}_{MOM}$ vs $\hat{R}_{SEL}$	0.6033	0.5435	0.4703	0.4513
		$\hat{R}_{MOM}$ vs $\hat{R}_{LIN}$	0.6060	0.5447	0.4673	0.4507
		$\hat{R}_{SEL}$ vs $\hat{R}_{LIN}$	0.6643	0.5623	0.4619	0.4969
(2.0,6.0)	0.2508	$\hat{R}_{MLE}$ vs $\hat{R}_{MOM}$	0.5237	0.5258	0.5299	0.5293
		$\hat{R}_{MLE}$ vs $\hat{R}_{SEL}$	0.6473	0.5893	0.5112	0.4730
		$\hat{R}_{MLE}$ vs $\hat{R}_{LIN}$	0.6489	0.5903	0.5089	0.4703
		$\hat{R}_{MOM}$ vs $\hat{R}_{SEL}$	0.6487	0.5861	0.4995	0.4645
		$\hat{R}_{MOM}$ vs $\hat{R}_{LIN}$	0.6502	0.5872	0.4981	0.4640
		$\hat{R}_{SEL}$ vs $\hat{R}_{LIN}$	0.7287	0.6195	0.5000	0.5165
(3.0,5.0)	0.3749	$\hat{R}_{MLE}$ vs $\hat{R}_{MOM}$	0.5234	0.5251	0.5302	0.5288
		$\hat{R}_{MLE}$ vs $\hat{R}_{SEL}$	0.5401	0.4804	0.4082	0.3917
		$\hat{R}_{MLE}$ vs $\hat{R}_{LIN}$	0.5441	0.4832	0.4025	0.3877
		$\hat{R}_{MOM}$ vs $\hat{R}_{SEL}$	0.5432	0.4852	0.4203	0.4292
		$\hat{R}_{MOM}$ vs $\hat{R}_{LIN}$	0.5467	0.4870	0.4192	0.4304
		$\hat{R}_{SEL}$ vs $\hat{R}_{LIN}$	0.6001	0.5370	0.4619	0.5183

Table 3: PN comparisons

(a_1, a_2)	R		$m = 10$	$m = 20$	$m = 50$	$m = 100$
(3.0,5.5)	0.3518	$\hat{R}_{MLE}$ vs $\hat{R}_{MOM}$	0.5239	0.5244	0.5296	0.5288
		$\hat{R}_{MLE}$ vs $\hat{R}_{SEL}$	0.5643	0.4995	0.4345	0.4152
		$\hat{R}_{MLE}$ vs $\hat{R}_{LIN}$	0.5680	0.5012	0.4291	0.4120
		$\hat{R}_{MOM}$ vs $\hat{R}_{SEL}$	0.5663	0.5006	0.4363	0.4339
		$\hat{R}_{MOM}$ vs $\hat{R}_{LIN}$	0.5711	0.5031	0.4346	0.4342
		$\hat{R}_{SEL}$ vs $\hat{R}_{LIN}$	0.6287	0.5329	0.4525	0.5101
(3.0,6.0)	0.3329	$\hat{R}_{MLE}$ vs $\hat{R}_{MOM}$	0.5231	0.5265	0.5283	0.5278
		$\hat{R}_{MLE}$ vs $\hat{R}_{SEL}$	0.5979	0.5339	0.4546	0.4314
		$\hat{R}_{MLE}$ vs $\hat{R}_{LIN}$	0.6004	0.5373	0.4483	0.4264
		$\hat{R}_{MOM}$ vs $\hat{R}_{SEL}$	0.5996	0.5350	0.4490	0.4352
		$\hat{R}_{MOM}$ vs $\hat{R}_{LIN}$	0.6028	0.5377	0.4466	0.4341
		$\hat{R}_{SEL}$ vs $\hat{R}_{LIN}$	0.6708	0.5750	0.4306	0.4665
(4.0,5.0)	0.4426	$\hat{R}_{MLE}$ vs $\hat{R}_{MOM}$	0.5232	0.5253	0.5291	0.5285
		$\hat{R}_{MLE}$ vs $\hat{R}_{SEL}$	0.4697	0.3757	0.3057	0.2815
		$\hat{R}_{MLE}$ vs $\hat{R}_{LIN}$	0.4736	0.3836	0.3130	0.2883
		$\hat{R}_{MOM}$ vss $\hat{R}_{SEL}$	0.4683	0.3885	0.3737	0.4026
		$\hat{R}_{MOM}$ vs $\hat{R}_{LIN}$	0.4731	0.3936	0.3788	0.4061
		$\hat{R}_{SEL}$ vs $\hat{R}_{LIN}$	0.5261	0.5007	0.5053	0.5117
(4.0,5.5)	0.4198	$\hat{R}_{MLE}$ vs $\hat{R}_{MOM}$	0.5239	0.5257	0.5295	0.5277
		$\hat{R}_{MLE}$ vs $\hat{R}_{SEL}$	0.5133	0.4239	0.3528	0.3284
		$\hat{R}_{MLE}$ vss $\hat{R}_{LIN}$	0.5174	0.4289	0.3516	0.3251
		$\hat{R}_{MOM}$ vs $\hat{R}_{SEL}$	0.5125	0.4306	0.3888	0.4041
		$\hat{R}_{MOM}$ vs $\hat{R}_{LIN}$	0.5164	0.4350	0.3901	0.4047
		$\hat{R}_{SEL}$ vs $\hat{R}_{LIN}$	0.5684	0.5081	0.4794	0.4838
(4.0,6.0)	0.3977	$\hat{R}_{MLE}$ vs $\hat{R}_{MOM}$	0.5245	0.5254	0.5301	0.5282
		$\hat{R}_{MLE}$ vs $\hat{R}_{SEL}$	0.5372	0.4648	0.3868	0.3702
		$\hat{R}_{MLE}$ vs $\hat{R}_{LIN}$	0.5410	0.4692	0.3854	0.3649
		$\hat{R}_{MOM}$ vs $\hat{R}_{SEL}$	0.5371	0.4689	0.4083	0.4143
		$\hat{R}_{MOM}$ vs $\hat{R}_{LIN}$	0.5417	0.4724	0.4103	0.4161
		$\hat{R}_{SEL}$ vs $\hat{R}_{LIN}$	0.5951	0.5337	0.4885	0.5001
(5.0,5.5)	0.4759	$\hat{R}_{MLE}$ vs $\hat{R}_{MOM}$	0.5247	0.5243	0.5304	0.5279
		$\hat{R}_{MLE}$ vs $\hat{R}_{SEL}$	0.4368	0.3096	0.2223	0.1906
		$\hat{R}_{MLE}$ vs $\hat{R}_{LIN}$	0.4674	0.3298	0.2444	0.2205
		$\hat{R}_{MOM}$ vs $\hat{R}_{SEL}$	0.4340	0.3270	0.3283	0.3806
		$\hat{R}_{MOM}$ vs $\hat{R}_{LIN}$	0.4650	0.3398	0.3366	0.3859
		$\hat{R}_{SEL}$ vs $\hat{R}_{LIN}$	0.5311	0.4887	0.4848	0.5077
(5.0,6.0)	0.4554	$\hat{R}_{MLE}$ vs $\hat{R}_{MOM}$	0.5237	0.5254	0.5306	0.5274
		$\hat{R}_{MLE}$ vs $\hat{R}_{SEL}$	0.4901	0.3700	0.2865	0.2515
		$\hat{R}_{MLE}$ vs $\hat{R}_{LIN}$	0.4967	0.3794	0.2977	0.2546
		$\hat{R}_{MOM}$ vs $\hat{R}_{SEL}$	0.4869	0.3800	0.3549	0.3797
		$\hat{R}_{MOM}$ vs $\hat{R}_{LIN}$	0.4937	0.3869	0.3609	0.3814
		$\hat{R}_{SEL}$ vs $\hat{R}_{LIN}$	0.5461	0.5009	0.5089	0.4600

6 Real Life Data

In order to illustrate the proposed estimators of $R = P(X < Y)$ under a *bivariate Lomax* distribution, we use the American Football League data from the matches on three consecutive weekends in 1986, which were first published in 'Washington Post' and proposed by Csörgő and Welsh (1989) after converting the seconds in the data to decimal points. The validity of exponential model is checked using Kolmogrov-Smirnov (K-S), as well as Anderson-Darling (A-D) and Chi-square tests. In this bivariate data set (X, Y), the variable X represents the game time to the first points scored by kicking the ball between goal posts, while the variable Y represents the game time by moving the ball into the end zone. The times are given in minutes and seconds and reported in Table (4).

Table 4: American Football League Data

X	Y	X	Y	X	Y
2.05	3.98	5.78	25.98	10.40	14.25
9.05	9.05	13.80	49.75	2.98	2.98
0.85	0.85	7.25	7.25	3.88	6.43
3.43	3.43	4.25	4.25	0.75	7.75
7.78	7.78	1.65	1.65	11.63	17.37
10.57	14.28	6.42	15.08	1.38	1.38
7.05	7.05	4.22	9.48	10.35	10.35
2.58	2.58	15.53	15.53	12.13	12.13
7.23	9.68	2.90	2.90	14.58	14.58
6.85	34.58	7.02	7.02	11.82	11.82
32.45	42.35	6.42	6.42	5.52	11.27
8.53	14.57	8.98	8.98	19.65	10.70
31.13	49.88	10.15	10.15	17.83	17.83
14.58	20.57	8.87	8.87	10.85	30.07

Without loss of generality we assume $\eta = 1$. We fit the exponential distribution for X with failure rate $\lambda_1 = 0.1102$, we observed that K-S = 0.17379 with P_{value}= 0.14023, A-D = 1.7151 and chi-square = 2.9102 with a corresponding $P_{value} = 0.40569$. While for Y we fit the exponential distribution with failure rate $\lambda_2 = 0.07449$, and we observed that K-S = 0.14201 with P_{value}= 0.3332, A-D = 0.80191 and Chi-square = 3.0078 with a corresponding $P_{value} = 0.55652$. Using Eq. (3) we found that $R = 0.5966$. A bootstrap approach is used to compute the bootstrap bias ($BootBias$) and standard error ($StdErr$). The 95% bootstrap confidence interval is calculated and reported in terms of ($LowerCI$, $UpperCI$). The output of the bootstrap analysis is summarized in Table (5). Notice that the bootstrap bias for all proposed estimators are small with respect to the true value of R. In addition, our proposed estimators provided small bootstrap standard error and short confidence limits.

Table 5: The bootstrap estimates output over 1000 resamples

	$BootBias$	$SdtErr$	$LowerCI$	$UpperCI$
$\widehat{R}_{MLE}$	0.0030	0.0231	0.5447	0.6344
$\widehat{R}_{MOM}$	0.0004	0.0238	0.5494	0.6421
$\widehat{R}_{SEL}$	0.0013	0.0248	0.5324	0.6299
$\widehat{R}_{LIN}$	0.0013	0.0248	0.5310	0.6286

7 Final remarks and conclusions

The importance of drawing inferences about $R = P(X < Y)$ arises naturally in many disciplines. Therefore, it is of interest to find a reliable estimates of R. In this paper, we have considered two types of inference procedures; the classical (MLE and MOM) and the Bayesians (SEL and $Linex$) to estimate $R = P(X < Y)$ when X and Y are distributed as *bivariate Lomax* distribution.

It is observed that the Bayes estimators do not have explicit forms, therefore we use the Lindley's approximation method, under the assumption of gamma priors.

The performance of the classical and Bayesian estimates are studied and compared based on extensive simulations. It is observed that the classical and Bayesian estimators are equivalent in terms of the MSE values for all choices of m. Moreover, the classical estimators outperform the Baysians in terms of $RBias$ and PN values especially for small values of m. However, the Bayes estimators outperform the classical estimators in terms of PN probabilities when $m \geq 50$. In addition, $\widehat{R}_{LIN}$ outperforms $\widehat{R}_{SEL}$ in terms of PN probabilities for large values of m. Based on all, for estimating the reliability R based on *bivariate Lomax* distribution, we suggest to use the classical estimators when sample size m is small and $\widehat{R}_{LIN}$ for larger values of m (> 50).

1 Appendix

The entries for Lindley's approximation are given by the following equations

$$\widehat{\sigma} = \begin{bmatrix} \widehat{\sigma}_{RR} & \widehat{\sigma}_{RS} \\ \widehat{\sigma}_{SR} & \widehat{\sigma}_{SS} \end{bmatrix}^{-1} = \begin{bmatrix} -\frac{\partial^2 l}{\partial R^2} & -\frac{\partial^2 l}{\partial R \partial S} \\ -\frac{\partial^2 l}{\partial S \partial R} & -\frac{\partial^2 l}{\partial S^2} \end{bmatrix}^{-1}_{R=\widehat{R},S=\widehat{S}}$$

$$\widehat{l}_{RR} = \frac{\partial^2 l}{\partial R^2}|_{R=\widehat{R},S=\widehat{S}} = -\frac{m}{(1-R)^2} - \frac{m}{R^2} + (2+c)\sum_{j=1}^{m}\left(\frac{S(x_j-y_j)}{(1+RS(x_j-y_j)+Sy_j)}\right)^2$$

$$\widehat{l}_{RRR} = \frac{\partial^3 l}{\partial R^3}|_{R=\widehat{R},S=\widehat{S}} = -\frac{2m}{(1-R)^3} + \frac{2m}{R^3} - 2(2+c)\sum_{j=1}^{m}\left(\frac{S(x_j-y_j)}{(1+RS(x_j-y_j)+Sy_j)}\right)^3$$

$$\widehat{l}_{RS} \equiv \widehat{l}_{SR} = \frac{\partial^2 l}{\partial R \partial S}|_{R=\widehat{R},S=\widehat{S}} = (2+c)\sum_{j=1}^{m}\left[\frac{S(x_j-y_j)(R(x_j-y_j)+y_j)}{(1+RS(x_j-y_j)+Sy_j)^2} - \frac{x_j-y_j}{1+RS(x_j-y_j)+Sy_j}\right]$$

$$\widehat{l}_{SS} = \frac{\partial^2 l}{\partial S^2}|_{R=\widehat{R},S=\widehat{S}} = -\frac{2m}{S^2} + (2+c)\sum_{j=1}^{m}\left(\frac{R(x_j-y_j)+y_j}{1++RS(x_j-y_j)+Sy_j}\right)^2$$

$$\widehat{l}_{SSS} = \frac{\partial^3 l}{\partial S^3}|_{R=\widehat{R},S=\widehat{S}} = \frac{4m}{S^3} - 2(2+c)\sum_{j=1}^{m}\left(\frac{R(x_j-y_j)+y_j}{1+RS(x_j-y_j)+Sy_j}\right)^3$$

$$\widehat{l}_{RSS} = \frac{\partial^3 l}{\partial S \partial R^2}|_{R=\widehat{R},S=\widehat{S}} = -2(2+c)\sum_{j=1}^{m}\left[\frac{S(x_j-y_j)(R(x_j-y_j)+y_j)^2}{(1+RS(x_j-y_j)+Sy_j)^3} - \frac{(x_j-y_j)(R(x_j-y_j)+y_j)}{(1+RS(x_j-y_j)+Sy_j)^2}\right]$$

$$\widehat{l}_{RRS} = \frac{\partial^3 l}{\partial R^2 \partial S}|_{R=\widehat{R},S=\widehat{S}} \quad -2(2+c)\sum_{j=1}^{m}\left[\frac{S^2(x_j-y_j)^2(R(x_j-y_j)+y_j)}{(1+RS(x_j-y_j)+Sy_j)^3} - \frac{S(x_j-y_j)^2}{(1+RS(x_j-y_j)+Sy_j)^2}\right]$$

References

Al-Mutairi, D., Ghitany, M., and Kundu, D. (2013). Inferences on stress-strength reliability from lindley distributions. *Communications in Statistics-Theory and Methods*, 42(8):1443–1463.

Barbiero, A. (2012). Interval estimators for reliability: the bivariate normal case. *Journal of Applied Statistics*, 39(3):501–512.

Barbiero, A. (2013). Inference on reliability of stress-strength models for poisson data. *Journal of Quality and Reliability Engineering*, 2013.

Birnbaum, Z. et al. (1956). On a use of the mann-whitney statistic. In *Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, Volume 1: Contributions to the Theory of Statistics*. The Regents of the University of California.

Csörgő, S. and Welsh, A. (1989). Testing for exponential and marshall–olkin distributions. *Journal of Statistical Planning and Inference*, 23(3):287–300.

Domma, F. and Giordano, S. (2012). A stress–strength model with dependent variables to measure household financial fragility. *Statistical Methods & Applications*, 21(3):375–389.

Domma, F. and Giordano, S. (2013). A copula-based approach to account for dependence in stress-strength models. *Statistical Papers*, 54(3):807–826.

Efron, B. and Tibshirani, R. J. (1994). *An introduction to the bootstrap*. CRC press.

Feynman, R. (1987). Mr. feynman goes to washington. engineering and science.

Ghitany, M., Al-Mutairi, D. K., and Aboukhamseen, S. (2015). Estimation of the reliability of a stress-strength system from power lindley distributions. *Communications in Statistics-Simulation and Computation*, 44(1):118–136.

Gupta, R. C., Ghitany, M., and Al-Mutairi, D. (2013). Estimation of reliability from a bivariate log-normal data. *Journal of Statistical Computation and Simulation*, 83(6):1068–1081.

Karimnezhad, A. (2013). Estimating a bounded normal mean under the linex loss function. *Journal of Sciences, Islamic Republic of Iran*, 24(2):157–164.

Kotz, S. and Pensky, M. (2003). *The stress-strength model and its generalizations: theory and applications*. World Scientific.

Lindley, D. V. (1980). Approximate bayesian methods. *Trabajos de estadística y de investigación operativa*, 31(1):223–245.

Lindley, D. V. and Singpurwalla, N. D. (1986). Multivariate distributions for the life lengths of components of a system sharing a common environment. *Journal of Applied Probability*, 23(2):418–431.

Makhdoom, I., Nasiri, P., and Pak, A. (2016). Bayesian approach for the reliability parameter of power lindley distribution. *International Journal of System Assurance Engineering and Management*, 7(3):341–355.

Metiri, F., Zeghdoudi, H., and Remita, M. R. (2016). On bayes estimates of lindley distribution under linux loss function: informative and non informative priors. *Global journal of Putre and Applied Mathematics*, 12:391–400.

Nadarajah, S. (2005). Reliability for some bivariate gamma distributions. *Mathematical Problems in Engineering*, 2005(2):151–163.

Nasiri, P. (2016). Estimation parameter of $r = p(y < x)$ for lomax distribution with presence of outliers. In *International Mathematical Forum*, volume 11, pages 239–248.

Rasheed, H. A. and Sultan, A. J. (2015). Bayesian estimation of the scale parameter for inverse gamma distribution under linex loss function. *International Journal of Advanced Research*, 3(2):369–375.

Rezaei, S., Tahmasbi, R., and Mahmoodi, M. (2010). Estimation of $p[y < x]$ for generalized pareto distribution. *Journal of Statistical Planning and Inference*, 140(2):480–494.

Rizki, S., Mara, M., and Sulistianingsih, E. (2017). Survival bayesian estimation of exponential-gamma under linex loss function. In *Journal of Physics: Conference*

Series, volume 855, page 012036. IOP Publishing.

RRL, K. et al. (2010). Estimation of reliability in multicomponent stress-strength model: Log-logistic distribution. *Electronic Journal of Applied Statistical Analysis*, 3(2):75–84.

Rubio, F. J., Steel, M. F., et al. (2013). Bayesian inference for $p(x < y)$ using asymmetric dependent distributions. *Bayesian Analysis*, 8(1):43–62.

Samawi, H. M., Helu, A., Rochani, H., Yin, J., and Linder, D. (2016). Estimation of $p(x > y)$ when x and y are dependent random variables using different bivariate sampling schemes. *Communications for Statistical Applications and Methods*, 23(5):385.

Varian, H. R. (1975). A bayesian approach to real estate assessment. *Studies in Bayesian econometric and statistics in Honor of Leonard J. Savage*, pages 195–208.

Wong, A. (2012). Interval estimation of $p(y < x)$ for generalized pareto distribution. *Journal of Statistical Planning and Inference*, 142(2):601–607.

Yu, W. and Xie, J. (2016). Bayesian estimation of reliability of geometric distribution under different loss functions. *Advances in Modelling and Analysis A*, 53:172–185.