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Classical and Bayesian estimation of Kumaraswamy distribution based on type II hybrid censored data

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In the literature, different estimation procedures are used for inference about Kumaraswamy distribution based on complete data sets. But, in many life-testing and reliability studies, a censored sample of data may be available in which failure times of some units are not reported. Unlike the common practice in the literature, this paper considers non-Bayesian and Bayesian estimation of Kumaraswamy parameters when the data are type II hybrid censored. The maximum likelihood estimates (MLE) and its asymptotic variance-covariance matrix are obtained. The asymptotic variances and covariances of the MLEs are used to construct approximate confidence intervals. In addition, by using the parametric bootstrap method, the construction of confidence intervals for the unknown parameter is discussed. Further, the Bayesian estimation of the parameters under squared error loss function is discussed. Based on type II hybrid censored data, the Bayes estimate of the parameters cannot be obtained explicitly; therefore, an approximation method, namely Tierney and Kadane's approximation, is used to compute the Bayes estimates of the parameters. Monte Carlo simulations are performed to compare the performances of the different methods, and one real data set is analyzed for illustrative purposes.

keywords: Hybrid censoring, Maximum likelihood estimators, Asymptotic confidence interval, Bayes estimators.

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1 Introduction

A random variable X is said to have Kumaraswamy distribution with two positive shape parameters α and λ, if its probability density function (pdf) and cumulative distribution function are given, respectively, by

$$f(x;\alpha,\lambda)=\alpha\lambda x^{\alpha-1}(1-x^{\alpha})^{\lambda-1},\qquad 0<x<1,\quad \alpha,\lambda>0, \tag{1}$$

and

$$F(x;\alpha,\lambda)=1-(1-x^{\alpha})^{\lambda}. \tag{2}$$

From now on Kumaraswamy distribution with parameters α and λ will be denoted by $KW(\alpha,\lambda)$. Kumaraswamy distribution is applicable to many natural phenomena whose outcomes have lower and upper bounds, such as the heights of individuals, scores obtained on a test, atmospheric temperatures, hydrological data, economic data such as unemployment data, etc. (Nadar et al. , 2012). The basic properties of this distribution were given by Jones (2009), who also investigated the properties of the skewness and kurtosis, derived the maximum likelihood estimators (MLEs) for the parameters and summarized the similarities and differences between the beta and Kumaraswamy distributions. Recently, Mitnik (2013) derived an expression for the moments of Kumaraswamy distribution. Garg (2008) derived the distribution of single order statistics and the distribution of the product and quotient of two order statistics when the random variables are independent and identically Kumaraswamy-distributed. Gholizadeh et al. (2011) discussed inference for the Kumaraswamy distribution by using both grouped and ungrouped data. Feroze and El-Batal (2013) considered maximum likelihood estimation of Kumaraswamy parameters based on progressive type II censored data with random removals.

Censoring schemes have attracted great attention due to their applicability in diverse areas such as medicine and engineering among others. In many life testing and reliability analysis, the experiment may be terminated before the failure of all items. Hence, the available observations are called censored samples. By the censoring, the test time can be reduced and further some experimental components are kept for future use. A hybrid type II censoring introduced by Childs et al. (2003), is a mixture of the conventional type I and type II censoring that can be described as follows. Put n identical items on test, and then stop the experiment at the random time $T^{*}=\max\{X_{l:n},T\}$, where $X_{l:n}$ denotes the time of lth failure.

There is a large amount of literature about the estimation of lifetime model parameters using hybrid type II censoring scheme. However, as we observed, the Kumaraswamy model is not considered. In this paper, we first obtain likelihood function based on a type II hybrid censored sample from Kumaraswamy distribution, and then discuss the maximum likelihood estimation of the parameters α and λ. We also construct approximate confidence interval of the unknown parameters by using the asymptotic distribution of the MLEs. In addition, a bootstrap confidence interval are also proposed. We further consider the Bayesian estimation of the parameters under the squared error loss function. Based on type II hybrid censored data, the Bayes estimate of the parameters cannot be obtained analytically; therefore, we adapt Tierney and Kadane's approximation to compute the Bayes estimates.

The rest of this paper is organized as follows. In Section 2, we describe the model and the available data, and obtain the maximum likelihood estimate of the parameters. The asymptotic confidence intervals of the parameters α and λ are also derived. In Section 3, by using the parametric bootstrap method, construction of the confidence intervals for the parameters is discussed. The Bayesian analyses are presented in Section 4. In Section 5, the estimation procedures are compared via Monte Carlo simulations in terms of their average values and mean squared errors and a numerical example is given to illustrate the proposed approaches.

2 Data descriptions and maximum likelihood estimation

Suppose that n identical units are placed on a life test with the corresponding lifetimes $X_1, ..., X_n$. It is assumed that these variables are independent and identically distributed as $KW(\alpha, \lambda)$. For known r and T, and under the type II hybrid censoring scheme, we can observe the following three types of observations:

Case I: $\{x_{(1)}, ..., x_{(r)}\}$ if $x_{(r)} > T$.

Case II: $\{x_{(1)}, ..., x_{(r)}, x_{(r+1)}, ..., x_{(m)}\}$ if $r \leq m < n$ and $x_{(m)} < T$.

Case III: $\{x_{(1)}, ..., x_{(n)}\}$ if $x_{(n)} < T$.

The likelihood functions for the above three different cases become, respectively, as

Case I:

$$L(\alpha, \lambda) = (\alpha\lambda)^r \left(1 - x_{(r)}^{\alpha}\right)^{\lambda(n-r)} \prod_{i=1}^{r} x_{(i)}^{\alpha-1}(1 - x_{(i)}^{\alpha})^{\lambda-1}. \tag{3}$$

Case II:

$$L(\alpha, \lambda) = (\alpha\lambda)^m \left(1 - T^{\alpha}\right)^{\lambda(n-m)} \prod_{i=1}^{m} x_{(i)}^{\alpha-1}(1 - x_{(i)}^{\alpha})^{\lambda-1}. \tag{4}$$

Case III:

$$L(\alpha, \lambda) = (\alpha\lambda)^n \prod_{i=1}^{n} x_{(i)}^{\alpha-1}(1 - x_{(i)}^{\alpha})^{\lambda-1}. \tag{5}$$

Combining the three cases, the likelihood function under type II hybrid censoring scheme can be written as

$$L(\alpha, \lambda) = (\alpha\lambda)^{\delta} \left(1 - \nu^{\alpha}\right)^{\lambda(n-\delta)} \prod_{i=1}^{\delta} x_{(i)}^{\alpha-1}(1 - x_{(i)}^{\alpha})^{\lambda-1}. \tag{6}$$

where δ denotes the number of failures and $\nu = x_{(r)}$ if $\delta = r$, and $\nu = T$ if $\delta > r$.

Now, the maximum likelihood estimate of the parameters α and λ, say $\hat{\alpha}$ and $\hat{\lambda}$, can be derived by maximizing the log-likelihood

$$\begin{aligned} L^*(\alpha, \lambda) &= \log L(\alpha, \lambda) = \delta(\log\alpha + \log\lambda) + \lambda(n - \delta)\log(1 - \nu^{\alpha}) \\ &\quad +(\alpha - 1)\sum_{i=1}^{\delta} \log x_{(i)} + (\lambda - 1)\sum_{i=1}^{\delta} \log(1 - x_{(i)}^{\alpha}). \end{aligned} \tag{7}$$

Equating the partial derivatives of the log-likelihood (7) with respect to α and λ to zero, the resulting two equations are:

$$\frac{\partial L^*(\alpha,\lambda)}{\partial\alpha} = \frac{\delta}{\alpha} + \sum_{i=1}^{\delta}\log x_{(i)} - (\lambda-1)\sum_{i=1}^{\delta}\frac{x_{(i)}^{\alpha}\log x_{(i)}}{1-x_{(i)}^{\alpha}} - \lambda(n-\delta)\frac{\nu^{\alpha}\log\nu}{1-\nu^{\alpha}} = 0, \tag{8}$$

$$\frac{\partial L^*(\alpha,\lambda)}{\partial\lambda} = \frac{\delta}{\lambda} + (n-\delta)\log(1-\nu^{\alpha}) + \sum_{i=1}^{\delta}\log(1-x_{(i)}^{\alpha}) = 0. \tag{9}$$

Thus, we obtain the MLE of the parameter λ as

$$\hat{\lambda} = \lambda(\hat{\alpha}) = -\frac{\delta}{(n-\delta)\log(1-\nu^{\hat{\alpha}}) + \sum_{i=1}^{\delta}\log(1-x_{(i)}^{\hat{\alpha}})} \tag{10}$$

and $\hat{\alpha}$ can be obtained as a solution of the non-linear equation $G(\alpha)=\alpha$ where

$$G(\alpha) = -\delta\left[\sum_{i=1}^{\delta}\log x_{(i)} - (\lambda(\alpha)-1)\sum_{i=1}^{\delta}\frac{x_{(i)}^{\alpha}\log x_{(i)}}{1-x_{(i)}^{\alpha}} - \lambda(\alpha)(n-\delta)\frac{\nu^{\alpha}\log\nu}{1-\nu^{\alpha}}\right]^{-1}. \tag{11}$$

Since $\hat{\alpha}$ is a fixed point solution of the non-linear Eq. 11, its value can be obtained using an iterative scheme as $G(\alpha_{(j)}) = \alpha_{(j+1)}$, where $\alpha_{(j)}$ is the jth iterate of $\hat{\alpha}$. The iteration procedure should be stopped when $\mid \alpha_{(j+1)} - \alpha_{(j)} \mid$ is sufficiently small.

Once the maximum likelihood estimates of α and λ are obtained, we can use the asymptotic normality of the MLEs to compute the approximate confidence intervals for the parameters. The asymptotic variances and covariances of the MLEs $\hat{\alpha}$ and $\hat{\lambda}$ are given by the elements of the inverse of Fisher information matrix

$$I(\alpha,\lambda) = -\begin{bmatrix} E\left(\frac{\partial^2 L^*(\alpha,\lambda)}{\partial\alpha^2}\right) & E\left(\frac{\partial^2 L^*(\alpha,\lambda)}{\partial\alpha\partial\lambda}\right) \\ E\left(\frac{\partial^2 L^*(\alpha,\lambda)}{\partial\lambda\partial\alpha}\right) & E\left(\frac{\partial^2 L^*(\alpha,\lambda)}{\partial\lambda^2}\right) \end{bmatrix}. \tag{12}$$

But, the exact mathematical expressions for the above expectations in (12) are difficult to obtain. Therefore, we take the approximate asymptotic variance-covariance matrix for the MLE of the parameters as

$$\hat{\Sigma} = \begin{bmatrix} -\left(\frac{\partial^2 L^*(\alpha,\lambda)}{\partial\alpha^2}\right) & -\left(\frac{\partial^2 L^*(\alpha,\lambda)}{\partial\alpha\partial\lambda}\right) \\ -\left(\frac{\partial^2 L^*(\alpha,\lambda)}{\partial\lambda\partial\alpha}\right) & -\left(\frac{\partial^2 L^*(\alpha,\lambda)}{\partial\lambda^2}\right) \end{bmatrix}^{-1}_{(\alpha=\hat{\alpha},\lambda=\hat{\lambda})} = \begin{bmatrix} \sigma_{\alpha}^2 & \sigma_{\alpha,\lambda}^2 \\ \sigma_{\lambda,\alpha}^2 & \sigma_{\lambda}^2 \end{bmatrix}_{(\alpha=\hat{\alpha},\lambda=\hat{\lambda})} \tag{13}$$

with

$$\begin{aligned} \frac{\partial^2 L^*(\alpha,\lambda)}{\partial\alpha^2} &= -\frac{\delta}{\alpha^2} - \lambda(n-\delta)\frac{\nu^{\alpha}(\log\nu)^2(1-\nu^{\alpha}) + (\nu^{\alpha}(\log\nu))^2}{(1-\nu^{\alpha})^2} \\ &\quad -(\lambda-1)\sum_{i=1}^{\delta}\frac{x_{(i)}^{\alpha}(\log x_{(i)})^2(1-x_{(i)}^{\alpha}) + (x_{(i)}^{\alpha}(\log x_{(i)}))^2}{(1-x_{(i)}^{\alpha})^2}, \end{aligned}$$

$$\frac{\partial^2 L^*(\alpha,\lambda)}{\partial\alpha\partial\lambda} = -\sum_{i=1}^{\delta} \frac{x_{(i)}^{\alpha}\log x_{(i)}}{1-x_{(i)}^{\alpha}} - (n-\delta)\frac{\nu^{\alpha}\log\nu}{1-\nu^{\alpha}}.$$

$$\frac{\partial^2 L^*(\alpha,\lambda)}{\partial\lambda^2} = -\frac{\delta}{\lambda^2}.$$

Thus, $100(1-\gamma)\%$ confidence interval for the parameters α and λ can be easily obtained as

$$(\hat{\alpha} \pm Z_{\frac{\gamma}{2}}\sqrt{\hat{\sigma}_{\alpha}^2}) \qquad \text{and} \qquad (\hat{\lambda} \pm Z_{\frac{\gamma}{2}}\sqrt{\hat{\sigma}_{\lambda}^2}),$$

respectively. Here $\hat{\sigma}_{\alpha}^2$ and $\hat{\sigma}_{\lambda}^2$ are the elements on the main diagonal of the variance covariance matrix $\hat{\Sigma}$.

3 Bootstrap confidence intervals

In this section, we discuss the construction of confidence intervals for the unknown parameter α and λ using a bias-corrected and accelerated (BCa) percentile bootstrap method (see Efron , 1987 for pertinent details). The bootstrap confidence intervals can then be compared to the asymptotic confidence intervals in terms of coverage probabilities.

Suppose n identical items are put on a test and in the presence of type II hybrid censoring scheme, the observed lifetimes are reported as $\{x_{(1)},...,x_{(\delta)}\}$. Before we discuss the construction of the bootstrap confidence intervals for the parameters , the following algorithm is used to generate the bootstrap sample based on this original hybrid type II censored data.

Step 1: Given the original hybrid type II censored sample $\{x_{(1)},...,x_{(\delta)}\}$, compute the MLEs $\hat{\alpha}$ and $\hat{\lambda}$.

Step 2: Based on the pre-specied n, r, and T, generate a hybrid type II censored sample with the underlying distribution as $KW(\hat{\alpha},\hat{\lambda})$, using the algorithm described in Ganguly et al. (2012).

Step 3: Based on the simulated type II hybrid censored sample in Step 2, calculate the bootstrap MLEs of α and λ, denoted by $\hat{\alpha}^*$ and $\hat{\lambda}^*$.

Step 4: Repeat step 2 and 3, M times. Then, arrange all the bootstrapped values of $\hat{\alpha}^*$ and $\hat{\lambda}^*$ in ascending order to obtain the bootstrap samples $\left(\hat{\theta}_k^{*(1)},...,\hat{\theta}_k^{*(M)}\right)$, $k=1,2$, where $\hat{\theta}_1^* = \hat{\alpha}^*$ and $\hat{\theta}_2^* = \hat{\lambda}^*$.

With the bootstrap samples generated as above, a two sided $100(1-\gamma)\%$ BCa percentile bootstrap confidence interval for α and λ can be obtained as

$$\left(\hat{\theta}_k^{*[\eta_k M]}, \hat{\theta}_k^{*[\omega_k M]}\right),$$

where

$$\eta_k = \Phi\left(\hat{z}_{0k} + \frac{\hat{z}_{0k} - z_{\alpha/2}}{1-\hat{\rho}_k(\hat{z}_{0k} - z_{\alpha/2})}\right)$$

and

$$\omega_k = \Phi\left(\hat{z}_{0k} + \frac{\hat{z}_{0k} + z_{\alpha/2}}{1 - \hat{\rho}_k(\hat{z}_{0k} + z_{\alpha/2})}\right), \qquad k = 1, 2.$$

Here, $\Phi(.)$ denotes the CDF of the standard normal distribution, z_γ is the upper $\gamma-$point of the standard normal distribution and $[x]$ denotes the integer part of x. The value of bias-correction $\hat{z}_{0k}$ is given by

$$\hat{z}_{0k} = \Phi^{-1}\left(\frac{\sum_{j=1}^{M} I(\hat{\theta}_k^{*(j)} < \hat{\theta}_k)}{M}\right),$$

and a good estimate of the acceleration factor $\hat{\rho}_k$ is suggested to be

$$\hat{\rho}_k = \frac{\sum_{j=1}^{l}(\hat{\theta}_k^{(.)} - \hat{\theta}_k^{j})^3}{6\left\{\sum_{j=1}^{l}(\hat{\theta}_k^{(.)} - \hat{\theta}_k^{j})^2\right\}^{3/2}},$$

where $\hat{\theta}_1^j$ and $\hat{\theta}_2^j$ are the MLEs of α and λ based on the original hybrid type II censored sample with the jth observation deleted for $j = 1, ..., l$, and

$$\hat{\theta}_k^{(.)} = \frac{1}{l}\sum_{j=1}^{l}\hat{\theta}_k^{j}.$$

4 Bayes estimation

In this section we obtain the Bayesian estimates of the parameters assuming that α and λ are independent random variables and follow the gamma prior distributions

$$\begin{cases} \pi_1(\alpha) \propto \alpha^{a_1-1}\exp(-\alpha b_1) \\ \pi_2(\lambda) \propto \lambda^{a_2-1}\exp(-\lambda b_2) \end{cases}, \tag{14}$$

where the hyperparameters a_1, a_2, b_1, and b_2 are nonnegative. By combining (6) with (14), the joint density function of α, λ and the *data* can be written as

$$\pi(data, \alpha, \lambda) \propto \alpha^{\delta+a_1-1}\lambda^{\delta+a_2-1}\exp(-\alpha b_1)\exp(-\lambda b_2)\,(1-\nu^{\alpha})^{\lambda(n-\delta)}\prod_{i=1}^{\delta} x_{(i)}^{\alpha-1}(1-x_{(i)}^{\alpha})^{\lambda-1}. \tag{15}$$

Thus, the posterior density function of α and λ given the *data* can be obtained as

$$\pi(\alpha, \lambda \mid data) = \frac{\pi(data, \alpha, \lambda)}{\int_0^{\infty}\int_0^{\infty}\pi(data, \alpha, \lambda)d\alpha d\lambda}. \tag{16}$$

It is well known that the Bayes estimate of any function of α and λ, say $g(\alpha, \lambda)$, under squared error loss function is the posterior mean which is obtained by

$$\int_0^\infty \int_0^\infty \pi(\alpha, \lambda \mid data) g(\alpha, \lambda) d\alpha d\lambda. \tag{17}$$

The Eq. (17) do not simplified to a nice closed form due to the complex form of the posterior density function $\pi(\alpha, \lambda \mid data)$; therefore, in the following, we provide Tierney and Kadane's approximation for computing the Bayes estimates.

Setting $H(\alpha, \lambda) = Q(\alpha, \lambda)/n$ and $H^*(\alpha, \lambda) = [\ln g(\alpha, \lambda) + Q(\alpha, \lambda)]/n$, where $Q(\alpha, \lambda) = \ln \pi(data, \alpha, \lambda)$, the Bayes estimate of $g(\alpha, \lambda)$ can be expressed as

$$E(g(\alpha, \lambda) \mid data) = \frac{\int_0^\infty \int_0^\infty e^{nH^*(\alpha,\lambda)} d\alpha d\lambda}{\int_0^\infty \int_0^\infty e^{nH(\alpha,\lambda)} d\alpha d\lambda}. \tag{18}$$

Following Tierney and Kadane (1986), Eq. (18) can be approximated as the following form:

$$\hat{g}_{BT}(\alpha, \lambda) = \left[\frac{\det \Psi^*}{\det \Psi}\right]^{1/2} \exp\left\{n\left[H^*(\bar{\alpha}^*, \bar{\lambda}^*) - H(\bar{\alpha}, \bar{\lambda})\right]\right\}, \tag{19}$$

where $(\bar{\alpha}^*, \bar{\lambda}^*)$ and $(\bar{\alpha}, \bar{\lambda})$ maximize $H^*(\alpha, \lambda)$ and $H(\alpha, \lambda)$, respectively, and Ψ^* and Ψ are the negatives of the inverse Hessians of $H^*(\alpha, \lambda)$ and $H(\alpha, \lambda)$ at $(\bar{\alpha}^*, \bar{\lambda}^*)$ and $(\bar{\alpha}, \bar{\lambda})$, respectively.

In our case, we have

$$\begin{aligned} H(\alpha, \lambda) \quad = \quad & \frac{1}{n}\{k + (\delta + a_1 - 1)\log\alpha + (\delta + a_2 - 1)\log\lambda + \lambda(n - \delta)\log(1 - \nu^\alpha) \\ & -\alpha b_1 - \lambda b_2 + (\alpha - 1)\sum_{i=1}^{\delta}\log x_{(i)} + (\lambda - 1)\sum_{i=1}^{\delta}\log(1 - x_{(i)}^\alpha)\}. \end{aligned} \tag{20}$$

where k is a constant. Hence, $(\bar{\alpha}, \bar{\lambda})$ can be obtained by solving the following two equations

$$\begin{aligned} \frac{\partial}{\partial\alpha}H(\alpha, \lambda) \quad = \quad & \frac{1}{n}\{\frac{\delta + a_1 - 1}{\alpha} - b_1 + \sum_{i=1}^{\delta}\log x_{(i)} \\ & -(\lambda - 1)\sum_{i=1}^{\delta}\frac{x_{(i)}^\alpha \log x_{(i)}}{1 - x_{(i)}^\alpha} - \lambda(n - \delta)\frac{\nu^\alpha \log\nu}{1 - \nu^\alpha}\} = 0, \end{aligned} \tag{21}$$

$$\frac{\partial}{\partial\lambda}H(\alpha, \lambda) = \frac{1}{n}\left\{\frac{\delta + a_2 - 1}{\lambda} - b_2 + (n - \delta)\log(1 - \nu^\alpha) + \sum_{i=1}^{\delta}\log(1 - x_{(i)}^\alpha)\right\} = 0. \tag{22}$$

By using Eq. (22), $\bar{\lambda}$ is derived as

$$\bar{\lambda} = \lambda(\bar{\alpha}) = \frac{\delta + a_2 - 1}{b_2 - (n-\delta)\log(1-\nu^{\bar{\alpha}}) - \sum_{i=1}^{\delta}\log(1-x_{(i)}^{\bar{\alpha}})} \tag{23}$$

and $\bar{\alpha}$ is the solution of the non-linear equation

$$\frac{\delta + a_1 - 1}{\alpha} - b_1 + \sum_{i=1}^{\delta}\log x_{(i)} - (\lambda(\bar{\alpha})-1)\sum_{i=1}^{\delta}\frac{x_{(i)}^{\alpha}\log x_{(i)}}{1-x_{(i)}^{\alpha}} - \lambda(\bar{\alpha})(n-\delta)\frac{\nu^{\alpha}\log\nu}{1-\nu^{\alpha}} = 0. \tag{24}$$

The fixed point method can be applied as in the ML estimation to compute $\bar{\alpha}$. Then, from the second derivatives of $H(\alpha, \lambda)$, the determinant of the negative of the inverse Hessian of $H(\alpha, \lambda)$ at $(\bar{\alpha}, \bar{\lambda})$ is obtained by

$$\det \Psi = (H_{11}H_{22} - H_{12}^2)^{-1},$$

where

$$\begin{aligned}
H_{11} &= \frac{1}{n}\{-\frac{\delta + a_1 - 1}{\bar{\alpha}^2} - \bar{\lambda}(n-\delta)\frac{\nu^{\bar{\alpha}}(\log\nu)^2(1-\nu^{\bar{\alpha}}) + (\nu^{\bar{\alpha}}(\log\nu))^2}{(1-\nu^{\bar{\alpha}})^2} \\
&\quad -(\bar{\lambda}-1)\sum_{i=1}^{\delta}\frac{x_{(i)}^{\bar{\alpha}}(\log x_{(i)})^2(1-x_{(i)}^{\bar{\alpha}}) + (x_{(i)}^{\bar{\alpha}}(\log x_{(i)}))^2}{(1-x_{(i)}^{\bar{\alpha}})^2}\},
\end{aligned}$$

$$H_{12} = \frac{1}{n}\{-\sum_{i=1}^{\delta}\frac{x_{(i)}^{\bar{\alpha}}\log x_{(i)}}{1-x_{(i)}^{\bar{\alpha}}} - (n-\delta)\frac{\nu^{\bar{\alpha}}\log\nu}{1-\nu^{\bar{\alpha}}}\},$$

$$H_{22} = \frac{1}{n}\{-\frac{\delta + a_2 - 1}{\bar{\lambda}^2}\}.$$

Now, following the same arguments with $g(\alpha, \lambda) = \alpha$ and λ, respectively, in $H^*(\alpha, \lambda)$, $\hat{\alpha}_{BT}$ and $\hat{\lambda}_{BT}$ in Eq. (19) can then be obtained straightforwardly.

5 Numerical Experiments

5.1 Simulation study

This section consists the simulation study to compare the performances of different estimators and also different confidence intervals. The performance of the competitive estimates has been compared on the basis of their average values and mean squared errors. Also, the average lengths of the confidence intervals and their coverage percentages are compared. The computations are performed using R 2.14.0 (R Development Core

Team , 2011), which is a non-commercial, open source software package for statistical computing and graphics.
First, for different sets of parameter values, namely $(\alpha, \lambda) = (2, 2), (3, 5)$ and different choices of n, r and T, we have generated random samples from Kumaraswamy distribution. Then, the estimate of the parameters α and λ for the generated samples were computed using the maximum likelihood procedure. For computing the Bayes estimates, we have assumed the following priors:

Prior I (non-informative gamma prior): $(a_1, b_1) = (0, 0)$, $(a_2, b_2) = (0, 0)$,

Prior II (informative gamma prior): $(a_1, b_1) = (0.2, 2)$, $(a_2, b_2) = (5, 1)$.

We replicate the process 1000 times and report the average values (AV) and mean squared errors (MSE) of the estimates in Tables 1-4.

We have also computed approximate 95% confidence intervals and bootstrap confidence intervals of the unknown parameters α and λ. Criteria appropriate to the evaluation of the two methods under scrutiny include: closeness of the coverage probability to its nominal value and expected interval width. For each simulated sample, we have computed confidence intervals and checked whether the true value of the parameter lay within the intervals and recorded the length of the intervals. The estimated coverage probability was computed as the number of intervals that covered the true value divided by 1000 while the estimated expected width of the intervals was computed as the sum of the lengths for all intervals divided by 1000. The coverage probabilities and the expected widths for different sample sizes are presented in Tables 5-6.

From Tables 1-4, some of the points are quite clear. For both the estimators, it is observed that for fixed n as r increases or T increases, the biases and MSEs decrease. The performances of the Bayes estimates obtained from non-informative priors and the maximum likelihood estimates are very similar in all aspects. Moreover, using the informative gamma prior distributions, results in reasonable improvements in the performances of Bayes estimates. It is also seen that, for small sample sizes ($n = 15, 20$), the coverage percentages of the asymptotic confidence intervals are lower than the nominal level 95%, however for larger sample sizes ($n \geq 30$), the asymptotic results work quite well in most of the cases. The coverage percentage of the adjusted bootstrap method is somewhat close to its nominal level. It can be further observed that the widths of the confidence intervals narrow down with increases in r or T for fixed n.

5.2 Data Analysis

To demonstrate the application of the proposed methods to real data, let us consider a data set collected during the experiment reported in Eldin et al. (2014). It is the water capacity data from the Shasta reservoir in California, USA, $http : //cdec.water.ca.gov/reservoir_map.html$. The maximum capacity of the reservoir is 4552000 AF and the data were transformed to the interval $[0, 1]$. Table 7, gives the date, actual and transformed data. Eldin et al. (2014) observed that Kumaraswamy distribution works quite well for these capacity data.

We have considered hybrid type II censored samples from these data using the following two sampling schemes:

Scheme 1: $T = 0.75$, $r = 5, 10, 12$;
Scheme 2: $T = 0.8$, $r = 10, 12, 15$.
Based on the above type II hybrid censored samples, we obtained the estimate of the parameters using ML and Bayesian procedures. We have also computed approximate 95% confidence intervals of the unknown parameters . Furthermore, using the algorithm described in Section 3 of the BCa bootstrap method, we presented the 95% bootstrap confidence intervals. All the results are summarized in Table 8.

6 Conclusions

In this paper, we have considered the classical and Bayesian inference procedures for the parameters of Kumaraswamy distribution under type II hybrid censoring scheme. We have provided likelihood function based on type II hybrid censored sample and obtained maximum likelihood estimate of the parameters. In the Bayesian setting, we have computed the estimate of the unknown parameters by using Tierney and Kadane's approximation under the assumption of both non-informative and informative gamma priors. We have further constructed approximate confidence interval and bootstrap confidence interval of the parameters. The performances of the different methods have been compared by Monte Carlo simulations. Based on the results of the simulation study, we see clearly that, the Bayesian procedure based on non-informative prior and the ML procedure give similar estimation results. However, using the ML method, we can also obtain approximate confidence interval of the parameters. On the other hand, the Bayes estimates of the parameters based on informative priors give better performances than the MLEs in terms of minimum MSEs. As a result, when information about the hyperparameters are available, we suggest using Bayesian approach to estimate the parameters of Kumaraswamy distribution. Moreover, as we expected, the performances of all estimators are improved when the effective sample size increases. It can be further observed that, in most of the cases, the coverage probabilities of confidence intervals are close to the nominal level. Also, approximate method of construction of confidence interval is better with respect to coverage percentage and average length.

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Table 1: AVs and MSEs of the ML and Bayes estimates when $n = 15, 20$, $T = 0.4, 0.6$ and $(\alpha, \lambda) = (2, 2)$.

n	T	r	MLE		Non-informative Bayes		Informative Bayes	
			$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\alpha}_{BT}$	$\hat{\lambda}_{BT}$	$\hat{\alpha}_{BT}$	$\hat{\lambda}_{BT}$
15	0.4	7	2.2164	2.3894	2.2182	2.3922	2.2063	2.3817
			0.3147	0.6521	0.3159	0.6527	0.3094	0.6482
		9	2.1913	2.3816	2.1974	2.3803	2.1885	2.3790
			0.3091	0.4776	0.3116	0.6503	0.3028	0.6412
		12	2.1836	2.3629	2.1850	2.3685	2.1871	2.3645
			0.2782	0.6248	0.2746	0.6292	0.2715	0.6137
	0.6	7	2.1973	2.3706	2.1945	2.3741	2.1907	2.3723
			0.3065	0.6317	0.3077	0.6358	0.2973	0.6219
		9	2.1822	2.3652	2.1854	2.3628	2.1797	2.3611
			0.2917	0.6125	0.2958	0.6134	0.2832	0.6053
		12	2.1653	2.3544	2.1692	2.3572	2.1728	2.3508
			0.2648	0.6063	0.2673	0.6047	0.2604	0.5984
20	0.4	7	2.1962	2.4173	2.1986	2.4192	2.1845	2.4027
			0.2748	0.6395	0.2790	0.6423	0.2633	0.6352
		9	2.1643	2.3915	2.1675	2.3923	2.1727	2.3842
			0.2650	0.6109	0.2618	0.6147	0.2452	0.6073
		12	2.1827	2.3354	2.1884	2.3378	2.1914	2.3279
			0.2466	0.5813	0.2472	0.5819	0.2436	0.5745
	0.6	7	2.1879	2.4107	2.1935	2.4165	2.1828	2.4093
			0.2681	0.6168	0.2685	0.6180	0.2645	0.6132
		9	2.1804	2.3862	2.1841	2.3902	2.1764	2.3831
			0.2635	0.6091	0.2653	0.6117	0.2595	0.6077
		12	2.1748	2.3427	2.1792	2.3451	2.1803	2.3479
			0.2417	0.5735	0.2436	0.5752	0.2413	0.5716

Notes: The first and second rows show the AVs and MSEs of the parameters, respectively.

Table 2: AVs and MSEs of the ML and Bayes estimates when $n = 30, 40$, $T = 0.5, 0.75$ and $(\alpha, \lambda) = (2, 2)$.

n	T	r	MLE		Non-informative Bayes		Informative Bayes	
			$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\alpha}_{BT}$	$\hat{\lambda}_{BT}$	$\hat{\alpha}_{BT}$	$\hat{\lambda}_{BT}$
30	0.5	20	2.1751	2.4168	2.1773	2.4172	2.1638	2.3512
			0.2376	0.6134	0.2384	0.6152	0.2275	0.5804
		22	2.1683	2.3852	2.1689	2.3894	2.1579	2.3462
			0.2193	0.5765	0.2217	0.5771	0.2088	0.5529
		26	2.1473	2.3118	2.1492	2.3137	2.1533	2.3474
			0.1981	0.5048	0.1987	0.5063	0.1767	0.4811
	0.75	20	2.1736	2.3931	2.1758	2.3942	2.1641	2.2851
			0.2218	0.5473	0.2234	0.5478	0.2011	0.5265
		22	2.1628	2.3472	2.1638	2.3475	2.1331	2.2249
			0.2079	0.5192	0.2093	0.5215	0.1962	0.5042
		26	2.1259	2.2865	2.1265	2.2891	2.1221	2.1672
			0.1875	0.4851	0.1869	0.4855	0.1425	0.4739
40	0.5	20	2.1282	2.3227	2.1316	2.3241	2.1284	2.3176
			0.1748	0.4835	0.1757	0.4839	0.1675	0.4772
		22	2.1139	2.2713	2.1148	2.2736	2.0931	2.2845
			0.1693	0.4470	0.1706	0.4492	0.1608	0.4216
		26	2.1041	2.1758	2.1055	2.1762	2.0753	2.1894
			0.1384	0.2572	0.1395	0.2584	0.1371	0.2527
	0.75	20	2.1174	2.2941	2.1177	2.2947	2.1092	2.2831
			0.2218	0.4165	0.1529	0.4180	0.1429	0.3958
		22	2.1086	2.2574	2.1093	2.2588	2.0966	2.2679
			0.1337	0.3319	0.1351	0.3329	0.1314	0.3307
		26	2.0937	2.1626	2.0945	2.1637	2.0619	2.1842
			0.1058	0.2291	0.1067	0.2307	0.1021	0.2296

Notes: The first and second rows show the AVs and MSEs of the parameters, respectively.

Table 3: AVs and MSEs of the ML and Bayes estimates when $n = 15, 20$, $T = 0.4, 0.6$ and $(\alpha, \lambda) = (3, 5)$.

n	T	r	MLE		Non-informative Bayes		Informative Bayes	
			$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\alpha}_{BT}$	$\hat{\lambda}_{BT}$	$\hat{\alpha}_{BT}$	$\hat{\lambda}_{BT}$
15	0.4	7	3.2781	5.6845	3.2762	5.6839	3.2619	5.6785
			0.5433	1.0715	0.5476	1.0784	0.5281	0.9835
		9	3.2537	5.6719	3.2539	5.6773	3.2541	5.6746
			0.5307	1.0138	0.5322	1.0218	0.5239	0.9748
		12	3.2419	5.6473	3.2485	5.6518	3.2308	5.6417
			0.5126	0.9613	0.5117	0.9642	0.5094	0.9522
	0.6	7	3.2644	5.6724	3.2673	5.6739	3.2611	5.6678
			0.5318	1.0247	0.5369	1.0308	0.5178	0.9814
		9	3.2592	5.6541	3.2612	5.6532	3.2536	5.6394
			0.5149	0.9739	0.5214	0.9763	0.5026	0.9456
		12	3.2471	5.6393	3.2478	5.6355	3.2454	5.6217
			0.5064	0.9514	0.5136	0.9581	0.4873	0.9316
20	0.4	7	3.2875	5.6418	3.2913	5.6472	3.2821	5.6371
			0.5137	0.9865	0.5174	0.9738	0.5044	0.9717
		9	3.2403	5.6487	3.2461	5.6533	3.2391	5.6231
			0.4906	0.9127	0.4939	0.9166	0.4618	0.9043
		12	3.2447	5.6132	3.2345	5.6154	3.2358	5.6046
			0.4819	0.8933	0.4855	0.9047	0.4523	0.8629
	0.6	7	3.2512	5.6277	3.2530	5.6281	3.2470	5.6183
			0.4826	0.9751	0.4893	0.9619	0.4806	0.9445
		9	3.2384	5.6108	3.2488	5.6219	3.2342	5.5729
			0.4723	0.9048	0.4711	0.9176	0.4536	0.8731
		12	3.2253	5.5813	3.2279	5.5845	3.2217	5.5534
			0.4592	0.8771	0.4581	0.8734	0.4428	0.8562

Notes: The first and second rows show the AVs and MSEs of the parameters, respectively.

Table 4: AVs and MSEs of the ML and Bayes estimates when $n = 30, 40$, $T = 0.5, 0.75$ and $(\alpha, \lambda) = (3, 5)$.

n	T	r	MLE		Non-informative Bayes		Informative Bayes	
			$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\alpha}_{BT}$	$\hat{\lambda}_{BT}$	$\hat{\alpha}_{BT}$	$\hat{\lambda}_{BT}$
30	0.5	7	3.2531	5.6571	3.2617	5.6637	3.2509	5.6436
			0.4806	0.9415	0.4861	0.9483	0.4759	0.9225
		9	3.2397	5.6328	3.2435	5.6395	3.2336	5.6054
			0.4562	0.9237	0.4627	0.9316	0.4539	0.9071
		12	3.2341	5.6219	3.2318	5.6248	3.2274	5.5973
			0.4454	0.8846	0.4452	0.8972	0.4361	0.8439
	0.75	7	3.2219	5.6347	3.2276	5.6213	3.2158	5.5708
			0.4782	0.9134	0.4831	0.9227	0.4519	0.9112
		9	3.2308	5.6038	3.2314	5.6129	3.2243	5.5582
			0.4577	0.9021	0.4524	0.9134	0.4271	0.8540
		12	3.2159	5.5436	3.2162	5.5486	3.2092	5.5317
			0.4122	0.8578	0.4179	0.8549	0.4056	0.8363
40	0.5	7	3.2248	5.6112	3.2369	5.6134	3.2113	5.5972
			0.4513	0.8635	0.4527	0.8746	0.4352	0.8524
		9	3.1935	5.5645	3.1976	5.5702	3.1863	5.5637
			0.4407	0.8419	0.4461	0.8439	0.4319	0.8265
		12	3.2078	5.5506	3.2103	5.5528	3.2138	5.5417
			0.4029	0.8067	0.4036	0.8121	0.3965	0.7962
	0.75	7	3.1946	5.5839	3.1948	5.5873	3.1827	5.5792
			0.4155	0.8352	0.4204	0.8386	0.4126	0.8233
		9	3.1902	5.5574	3.1874	5.5519	3.1856	5.5125
			0.3728	0.8245	0.3793	0.8132	0.3697	0.7958
		12	3.1865	5.5123	3.1851	5.5248	3.1731	5.4976
			0.3518	0.7906	0.3548	0.7954	0.3451	0.7822

Notes: The first and second rows show the AVs and MSEs of the parameters, respectively.

Table 5: Average confidence lengths (A.L.) and the corresponding coverage probabilities (C.P.) of the approximate and BCa confidence intervals for $(\alpha, \lambda) = (2, 2)$.

n	T	r	α				λ			
			ACI		BCa		ACI		BCa	
			A.L.	C.P.	A.L.	C.P.	A.L.	C.P.	A.L.	C.P.
15	0.4	20	1.2871	0.871	1.2612	0.894	1.3627	0.892	1.3451	0.852
		22	1.2806	0.882	1.2568	0.903	1.3308	0.897	1.3276	0.883
		26	1.2542	0.895	1.2391	0.906	1.3265	0.916	1.3028	0.909
	0.6	20	1.2833	0.881	1.2574	0.896	1.3591	0.903	1.3369	0.861
		22	1.2761	0.903	1.2407	0.905	1.3227	0.914	1.3041	0.895
		26	1.2459	0.908	1.280	0.913	1.3054	0.921	1.2975	0.911
20	0.4	20	1.2238	0.887	1.2173	0.874	1.2667	0.885	1.2507	0.889
		22	1.2076	0.906	1.2048	0.877	1.2582	0.902	1.2318	0.906
		26	1.1963	0.908	1.1721	0.891	1.2441	0.917	1.2092	0.911
	0.6	20	1.2214	0.906	1.2106	0.889	1.2605	0.893	1.2466	0.903
		22	1.2137	0.911	1.1964	0.895	1.2571	0.908	1.2217	0.917
		26	1.1892	0.919	1.1875	0.908	1.2136	0.926	1.2048	0.920
30	0.5	20	1.1751	0.931	1.1684	0.925	1.1963	0.927	1.1908	0.892
		22	1.1709	0.933	1.1631	0.929	1.1874	0.928	1.1811	0.898
		26	1.1575	0.941	1.1452	0.935	1.1551	0.936	1.1422	0.916
	0.75	20	1.1663	0.943	1.1518	0.936	1.1838	0.940	1.1765	0.926
		22	1.1511	0.948	1.1475	0.938	1.1676	0.945	1.1632	0.927
		26	1.1286	0.949	1.1174	0.939	1.1344	0.946	1.1261	0.935
40	0.5	20	1.1237	0.949	1.1081	0.940	1.1317	0.948	1.1192	0.939
		22	1.1194	0.950	1.1016	0.940	1.1253	0.948	1.1087	0.942
		26	1.1028	0.951	1.0891	0.943	1.1095	0.952	1.0841	0.945
	0.75	20	1.0924	0.952	1.0813	0.946	1.1082	0.952	1.0935	0.946
		22	1.0789	0.952	1.0547	0.947	1.0971	0.953	1.0878	0.947
		26	1.0317	0.955	1.0176	0.951	1.0663	0.954	1.0511	0.950

Table 6: Average confidence lengths (A.L.) and the corresponding coverage probabilities (C.P.) of the approximate and BCa confidence intervals for $(\alpha, \lambda) = (3, 5)$.

n	T	r	α				λ			
			ACI		BCa		ACI		BCa	
			A.L.	C.P.	A.L.	C.P.	A.L.	C.P.	A.L.	C.P.
15	0.4	20	1.8431	0.854	1.7952	0.872	2.2149	0.891	2.1891	0.865
		22	1.8067	0.861	1.7741	0.878	2.1975	0.895	2.1743	0.873
		26	1.7942	0.893	1.7633	0.895	2.1806	0.912	2.1236	0.892
	0.6	20	1.8255	0.857	1.8097	0.863	2.2127	0.901	2.1591	0.874
		22	1.7739	0.893	1.7615	0.877	2.1794	0.903	2.1473	0.880
		26	1.7518	0.902	1.7469	0.890	2.1635	0.917	2.1148	0.883
20	0.4	20	1.8063	0.867	1.7645	0.874	2.1773	0.903	2.1518	0.872
		22	1.7955	0.886	1.7593	0.881	2.1342	0.912	2.1263	0.884
		26	1.7608	0.895	1.7056	0.894	2.1169	0.913	2.1108	0.897
	0.6	20	1.8192	0.873	1.7714	0.897	2.1457	0.906	2.1036	0.893
		22	1.7721	0.905	1.7082	0.908	2.1083	0.915	2.0851	0.906
		26	1.7344	0.908	1.6649	0.913	2.0952	0.928	2.0762	0.14
30	0.5	20	1.6905	0.913	1.6718	0.884	2.0814	0.913	2.0318	0.897
		22	1.6638	0.917	1.6275	0.892	2.0728	0.918	2.0160	0.905
		26	1.6195	0.926	1.5873	0.903	2.0419	0.927	1.9736	0.918
	0.75	20	1.6417	0.922	1.5692	0.898	2.0736	0.914	2.0245	0.909
		22	1.6251	0.927	1.5564	0.907	2.0592	0.930	1.9805	0.917
		26	1.5778	0.938	1.5312	0.911	2.0117	0.936	1.9576	0.925
40	0.5	20	1.6113	0.918	1.5078	0.897	1.9841	0.922	1.9754	0.911
		22	1.5849	0.925	1.4491	0.905	1.9633	0.929	1.9412	0.917
		26	1.4507	0.937	1.4335	0.912	1.9578	0.938	1.9225	0.926
	0.75	20	1.5782	0.922	1.5260	0.903	1.9512	0.931	1.9437	0.911
		22	1.5162	0.928	1.4541	0.926	1.9407	0.933	1.9315	0.919
		26	1.4237	0.943	1.4127	0.933	1.9342	0.947	1.9176	0.932

Table 7: Capacity for August and proportion of total capacity for Shasta reservoir

Capacity	Proportion of total capacity	Capacity	Proportion of total capacity
1542838	0.338936	3495969	0.768007
1966077	0.431915	3839544	0.843485
3459209	0.759932	3584238	0.787408
3298496	0.724626	3868600	0.849868
3448519	0.757583	3168056	0.695970
3694201	0.811556	3834224	0.842316
3574861	0.785339	3772193	0.828689
3567220	0.783660	2641041	0.580194
3712733	0.815627	1960458	0.430681
3857423	0.847413	3380147	0.742563

Table 8: MLE, Bayes, approximate and BCa confidence intervals of the parameters for hybrid type II censored samples from capacity data.

			MLE	Bayes	ACI	BCa
0.75	5	α	4.5134	4.5166	(2.2742,8.5845)	(2.2716,8.1435)
		λ	2.2048	2.2072	(0.2184,5.3366)	(0.2062,5.3818)
	10	α	4.7578	4.7592	(2.3328,7.6884)	(2.2264,7.5729)
		λ	2.5473	2.5496	(0.2206,5.3047)	(0.2165,5.3476)
	12	α	4.7742	4.7811	(2.3351,7.6522)	(2.3470,7.6832)
		λ	2.5528	2.5563	(0.3177,5.3582)	(0.2461,5.3735)
0.8	12	α	4.7795	4.7826	(2.4417,7.6791)	(2.3522,7.6514)
		λ	2.5567	2.5582	(0.2316,5.2541)	(0.22475.3312)
	14	α	4.7840	4.7851	(2.3463,6.3845)	(2.3609,7.5033)
		λ	2.5619	2.5632	(0.4481,5.3812)	(0.2737,5.3759)
	16	α	4.8147	4.8169	(2.9621,6.7316)	(2.6224,7.7245)
		λ	2.5677	2.5745	(0.4750,5.2897)	(0.2814,5.3662)

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