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UBLUE FOR THE REGULAR TWO-STAGE LINEAR MODEL FROM THE PERSPECTIVE OF PROJECTION OPERATORS

Surendra P. Sinha[(1)*], Arnaldo Goitía[(1)], Jorge L. Valera[(2)]

[(1)]*Instituto de Estadística Aplicada y Computación, Universidad de Los Andes, Mérida, Venezuela*
[(2)]*Departamento de Matemática y Física, Universidad Nacional Experimental del Táchira, Táchira, Venezuela*

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Abstract:*The linear models in multi stages and in particular the two-stage models, have great potential that may be used in many scientific research. Nevertheless more than 20 years have elapsed since the two-stage model was defined by [5] and still the use of these models is not yet popular among applied research workers, which we attribute to the complicated expressions of the estimators of the mean vector and other parameters of the model obtained till now in the different papers published on this subject. This article gives alternative expressions for UBLUE of $\boldsymbol{\beta}$ and $\boldsymbol{X\beta}$ in the regular two-stage linear model, using projection operators onto $\mathcal{M}(\boldsymbol{X})$ and a linear transformation $\boldsymbol{F}$ of the observable random vector $\boldsymbol{y}$ which is linearly sufficient in order to provide new insights and facilitate the use of these models in applied research.*

Keywords*:UBLUE, two-stage lineal model, Projector, transformed model.*

1. Introduction

The statistical study of linear models is an area of much interest because of its usefulness in the analysis of data in different fields of the human knowledge. According to Graybill [3], a general linear model is $\boldsymbol{y} = \boldsymbol{X\beta} + \boldsymbol{\varepsilon}$ where $\boldsymbol{y}$ is an observable random vector, $\boldsymbol{X}$ is an observable matrix, $\boldsymbol{\beta}$ is a vector of unknown parameters and $\boldsymbol{\varepsilon}$ is an unobservable random vector with $E(\boldsymbol{\varepsilon}) = \mathbf{0}$ and $Cov(\boldsymbol{\varepsilon}) = \boldsymbol{\Sigma}$. The different variants of a model are related with the specification of the distributional properties of $\boldsymbol{\varepsilon}$, or $\boldsymbol{y}$, or with the specification of the assumptions about the structure of the matrix $\boldsymbol{\Sigma}$, or of the matrix $\boldsymbol{X}$.

*Email: sinha32@yahoo.com

At present, the case when the matrix $\boldsymbol{\Sigma} = \boldsymbol{\sigma}^2\boldsymbol{I}$, is widely known and used by many researchers from various disciplines. For this case, the BLUE of $\boldsymbol{X\beta}$ is given by $\widehat{\boldsymbol{X\beta}} = \boldsymbol{X}(\boldsymbol{X'X})^{-}\boldsymbol{X}'\boldsymbol{y} = \boldsymbol{P_X y}$ where $\boldsymbol{P_X} = \boldsymbol{X}(\boldsymbol{X'X})^{-}\boldsymbol{X}'$ is the perpendicular projection operator onto the column space of $\mathcal{M}(\boldsymbol{X})$ parallel to (or along) $\mathcal{M}(\boldsymbol{Z})$, with $\boldsymbol{Z} = \boldsymbol{X}^{\perp}$ which is a matrix of maximum rank such that $\boldsymbol{X}'\boldsymbol{Z} = \boldsymbol{0}$. In the case $\boldsymbol{\Sigma} = \boldsymbol{\sigma}^2\boldsymbol{V}$ with $\boldsymbol{V}$ positive definite, the BLUE of $\boldsymbol{X\beta}$ is $\widehat{\boldsymbol{X\beta}} = \boldsymbol{X}(\boldsymbol{X'V}^{-1}\boldsymbol{X})^{-}\boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{y} = \boldsymbol{P_{XV}y}$ and again $\boldsymbol{X}(\boldsymbol{X'V}^{-1}\boldsymbol{X})^{-}\boldsymbol{X}'\boldsymbol{V}^{-1}$ is the projection operator on $\mathcal{M}(\boldsymbol{X})$ parallel to $\mathcal{M}(\boldsymbol{VZ})$. In both cases we see that the estimator $\widehat{\boldsymbol{X\beta}}$ is a projection of the response vector $\boldsymbol{y}$ onto $\mathcal{M}(\boldsymbol{X})$.

The two-stage and p-stage models were defined by Kubáček in [5] and [4] respectively. Such models have also been investigated by Volaufova in several of her articles, particularly [9], [10] and [11]. Volaufova [9] gives explicit but rather complicated expressions for the LBLUE (Locally best linear unbiased estimate) of $\boldsymbol{\beta}_1$, $\boldsymbol{\beta}_2$ and establishes that the UBLUE obtained from the transformed model exists if and only if $\mathcal{M}(\boldsymbol{D}) \subset \mathcal{M}(\boldsymbol{X}_2)$, and in this case the LBLUE's are UBLUE's.

Although it has been more than 20 years since the definition of two-stage linear model was given by Kubácěk [5], the use of this model by most researchers in different areas of human knowledge has been almost negligible which we attribute to the complicated expressions of the estimators of the mean vector and other parameters of the model obtained till now in the different papers published on this subject. We think that in order to facilitate the use of the great potential possessed by these models in the practical applications, it will be necessary to obtain equivalent expressions of the parameters vector of the model using the results published by Bhimasankaram and Sengupta [1] and Rao [7] so that the estimators obtained can be expressed as functions of the projection operators.

We will use the following notation: $\boldsymbol{A}'$ is the transpose of $\boldsymbol{A}$. $\boldsymbol{A}^{-}$ denotes any generalized inverse (g-inverse) of $\boldsymbol{A}$ as defined by Rao [6], i.e. $\boldsymbol{A}^{-}$ is such that $\boldsymbol{AA}^{-}\boldsymbol{A} = \boldsymbol{A}$. $\boldsymbol{A}^{\perp}$ is a matrix of maximum rank such $\boldsymbol{A}'\boldsymbol{A}^{\perp} = \boldsymbol{0}$. $\mathcal{M}(\boldsymbol{A})$ is the vector space generated by the columns of the matrix $\boldsymbol{A}$. $\boldsymbol{P_A} = \boldsymbol{A}(\boldsymbol{A}'\boldsymbol{A})^{-}\boldsymbol{A}'$ is the perpendicular projection operator onto $\mathcal{M}(\boldsymbol{A})$ and $\boldsymbol{P_{AM}} = \boldsymbol{A}(\boldsymbol{A}'\boldsymbol{MA})^{-}\boldsymbol{A}'\boldsymbol{M}$ where $\boldsymbol{M}$ is a positive definite matrix, is a projection operator onto $\mathcal{M}(\boldsymbol{A})$. $\boldsymbol{P_{A|B}}$ is a projection operator onto $\mathcal{M}(\boldsymbol{A})$ parallel to (or along) $\mathcal{M}(\boldsymbol{B})$ as defined by Rao [7].

In this paper we use the definition of UBLUE (Uniformly best linear unbiased estimator) given by Wulff and Birkes [12], which says that for the model (y, $\boldsymbol{X\beta}$, $\boldsymbol{V_\psi}$) where $\boldsymbol{\beta}$ is a vector of unknown parameters of fixed effects and $\boldsymbol{V_\psi}$ a positive definite matrix, which depends on a vector of parameters $\boldsymbol{\psi}$ that belongs to a known $\boldsymbol{\Psi}$, a linear estimator $\boldsymbol{L}'\boldsymbol{y}$ is called UBLUE if it is the BLUE of its expectation for all $\boldsymbol{\psi} \in \boldsymbol{\Psi}$.

2. The regular two-stage linear model

Let:

$$\boldsymbol{y}_1 = \boldsymbol{X}_1\boldsymbol{\beta}_1 + \boldsymbol{\varepsilon}_1$$
$$\boldsymbol{y}_2 = \boldsymbol{D}\boldsymbol{\beta}_1 + \boldsymbol{X}_2\boldsymbol{\beta}_2 + \boldsymbol{\varepsilon}_2$$

be a model where $\boldsymbol{y}_1$ and $\boldsymbol{y}_2$ are random vectors with dimensions $n_1 \times 1$, $n_2 \times 1$ respectively; $\boldsymbol{X}_1$, $\boldsymbol{X}_2$ and $\boldsymbol{D}$ are known matrices of real numbers; $\boldsymbol{\beta}_1$, $\boldsymbol{\beta}_2$ are vectors of unknown parameters of dimension $p_1 \times 1$, $p_2 \times 1$; and $\boldsymbol{\varepsilon}_1$, $\boldsymbol{\varepsilon}_2$are random error vectors with $E(\boldsymbol{\varepsilon}_1) = \boldsymbol{0}$, $Cov(\boldsymbol{\varepsilon}_1) = \sigma_1^2 \boldsymbol{V}_1$, $E(\boldsymbol{\varepsilon}_2) = \boldsymbol{0}$, $Cov(\boldsymbol{\varepsilon}_2) = \sigma_2^2 \boldsymbol{V}_2$, $\boldsymbol{\varepsilon}_1$ and $\boldsymbol{\varepsilon}_2$ uncorrelated, $\sigma_1^2 > 0$, $\sigma_2^2 > 0$ unknown parameters and $\boldsymbol{V}_1, \boldsymbol{V}_2$ are known positive definite matrices of proper dimension. This model is denominated as two-stage linear model and according to Kubáček [5] if the ranks of the matrices $\boldsymbol{X}_1$, $\boldsymbol{X}_2$, $\boldsymbol{V}_1, \boldsymbol{V}_2$ are $r(\boldsymbol{X}_1) = p_1 < n_1$, $r(\boldsymbol{X}_2) = p_2 < n_2$, $r(\boldsymbol{V}_1) = n_1$, $r(\boldsymbol{V}_2) = n_2$, then the model is denominated as a regular two-stage linear model.
The regular two-stage linear model can be written as:

$$\begin{bmatrix} \boldsymbol{y}_1 \\ \boldsymbol{y}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{X}_1 & \boldsymbol{0} \\ \boldsymbol{D} & \boldsymbol{X}_2 \end{bmatrix} \begin{bmatrix} \boldsymbol{\beta}_1 \\ \boldsymbol{\beta}_2 \end{bmatrix} + \begin{bmatrix} \boldsymbol{\varepsilon}_1 \\ \boldsymbol{\varepsilon}_2 \end{bmatrix} \qquad (1)$$

$$\boldsymbol{y} = \boldsymbol{X\beta} + \boldsymbol{\varepsilon}$$

with

$$Cov(\boldsymbol{\varepsilon}) = \begin{bmatrix} \sigma_1^2 \boldsymbol{V}_1 & \boldsymbol{0} \\ \boldsymbol{0} & \sigma_2^2 \boldsymbol{V}_2 \end{bmatrix} = \boldsymbol{V}_{\boldsymbol{\varepsilon}}$$

where $\boldsymbol{X}_1$, $\boldsymbol{X}_2$ are of full rank by columns, $\boldsymbol{V}_1$, $\boldsymbol{V}_2$ are positive definite matrices and $\sigma_1^2 > 0$, $\sigma_2^2 > 0$ unknown parameters. When $\boldsymbol{y}$ in the model (1) is pre multiplied by the $\boldsymbol{F}$ matrix where:

$$\boldsymbol{F} = \begin{bmatrix} \boldsymbol{I}_1 & \boldsymbol{0} \\ -\boldsymbol{DQ} & \boldsymbol{I}_2 \end{bmatrix}$$

then, we obtain the transformed model as:

$$\begin{bmatrix} \boldsymbol{y}_1 \\ \boldsymbol{y}_2^* \end{bmatrix} = \begin{bmatrix} \boldsymbol{X}_1 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{X}_2 \end{bmatrix} \begin{bmatrix} \boldsymbol{\beta}_1 \\ \boldsymbol{\beta}_2 \end{bmatrix} + \begin{bmatrix} \boldsymbol{I}_1 & \boldsymbol{0} \\ -\boldsymbol{DQ} & \boldsymbol{I}_2 \end{bmatrix} \begin{bmatrix} \boldsymbol{\varepsilon}_1 \\ \boldsymbol{\varepsilon}_2 \end{bmatrix} \qquad (2)$$

$$\boldsymbol{y}^* = \boldsymbol{X}^* \boldsymbol{\beta} + \boldsymbol{\varepsilon}^*$$

with

$$Cov(\boldsymbol{\varepsilon}^*) = \boldsymbol{V}_{\boldsymbol{\varepsilon}^*} = \begin{bmatrix} \sigma_1^2 \boldsymbol{V}_1 & -\sigma_1^2 \boldsymbol{C}' \\ -\sigma_1^2 \boldsymbol{C} & \sigma_1^2 \boldsymbol{C} \boldsymbol{V}_1^{-1} \boldsymbol{C}' + \sigma_2^2 \boldsymbol{V}_2 \end{bmatrix}$$

where $\boldsymbol{y}^* = \boldsymbol{Fy}$, $\boldsymbol{X}^* = \boldsymbol{FX}$, $\boldsymbol{\varepsilon}^* = \boldsymbol{F\varepsilon}$, $\boldsymbol{y}_2^* = -\boldsymbol{DQ}\boldsymbol{y}_1 + \boldsymbol{y}_2$, $\boldsymbol{\varepsilon}_2^* = -\boldsymbol{DQ}\boldsymbol{\varepsilon}_1 + \boldsymbol{\varepsilon}_2$, $\boldsymbol{Q}$ is a matrix such that $\boldsymbol{QX}_1 = \boldsymbol{I}_1$, $\boldsymbol{I}_1$ an identity matrix, and $\boldsymbol{C} = \boldsymbol{DQV}_1$.
Volaufova [9] gives explicit expressions although a little complicated, for the LBLUE (Locally best linear unbiased estimate) of $\boldsymbol{\beta}_1$, $\boldsymbol{\beta}_2$ obtained from the transformed model (2), and establishes that the UBLUE of $\boldsymbol{\beta}_1$, $\boldsymbol{\beta}_2$ obtained from the transformed model exists if and only if $\mathcal{M}(\boldsymbol{D}) \subset \mathcal{M}(\boldsymbol{X}_2)$, and in this case the LBLUE's are UBLUE's.

3. UBLUE for $X^*\beta$

In this paper we are interested in getting the UBLUE for $\boldsymbol{X\beta}$ from another perspective. For doing this, we will use the theory given by Bhimasankaram and Sengupta [1]. Since in the model (2) σ_1^2, σ_2^2 are unknown, we will assume that the ratio $\rho = \sigma_2^2/\sigma_1^2$ is known, and so we obtain the model:

$$\begin{bmatrix} \boldsymbol{y}_1 \\ \boldsymbol{y}_2^* \end{bmatrix} = \begin{bmatrix} \boldsymbol{X}_1 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{X}_2 \end{bmatrix} \begin{bmatrix} \boldsymbol{\beta}_1 \\ \boldsymbol{\beta}_2 \end{bmatrix} + \begin{bmatrix} \boldsymbol{\varepsilon}_1 \\ \boldsymbol{\varepsilon}_2^* \end{bmatrix} \tag{3}$$

$$\boldsymbol{y}^* = \boldsymbol{X}^*\boldsymbol{\beta} + \boldsymbol{\varepsilon}^*$$

with

$$Cov(\boldsymbol{\varepsilon}^*) = \boldsymbol{V}_{\boldsymbol{\varepsilon}^*} = \sigma_1^2 \boldsymbol{V}_\rho = \sigma_1^2 \begin{bmatrix} \boldsymbol{V}_1 & -\boldsymbol{C}' \\ -\boldsymbol{C} & \boldsymbol{C}\boldsymbol{V}_1^{-1}\boldsymbol{C}' + \rho \boldsymbol{V}_2 \end{bmatrix}$$

where $\rho \in \varrho$ and ϱ consists of all positive real numbers.
In this way, the model (3) represents a Gauss-Markov model and using the lemma (2.1) given by Bhimasankaram and Sengupta [1], the BLUE of $\boldsymbol{X}^*\boldsymbol{\beta}$ in the model $\left(\boldsymbol{y}^*, \boldsymbol{X}^*\boldsymbol{\beta}, \boldsymbol{V}_{\boldsymbol{\varepsilon}^*} = \sigma_1^2\boldsymbol{V}_\rho\right)$ is obtained as:

$$\widehat{\boldsymbol{X}^*\boldsymbol{\beta}} = \left[\boldsymbol{I} - \boldsymbol{V}_\rho(\boldsymbol{I} - \boldsymbol{P}_{\boldsymbol{X}^*})\left\{(\boldsymbol{I} - \boldsymbol{P}_{\boldsymbol{X}^*})\boldsymbol{V}_\rho(\boldsymbol{I} - \boldsymbol{P}_{\boldsymbol{X}^*})\right\}^-(\boldsymbol{I} - \boldsymbol{P}_{\boldsymbol{X}^*})\right]\boldsymbol{y}^* \tag{4}$$

where

$$\boldsymbol{P}_{\boldsymbol{X}^*} = \begin{bmatrix} \boldsymbol{P}_{\boldsymbol{X}_1} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{P}_{\boldsymbol{X}_2} \end{bmatrix}$$

and $\boldsymbol{P}_{\boldsymbol{X}_1}$, $\boldsymbol{P}_{\boldsymbol{X}_2}$ are the perpendicular projection operator matrices onto $\mathcal{M}(\boldsymbol{X}_1)$, $\mathcal{M}(\boldsymbol{X}_2)$ respectively. Let:

$$\boldsymbol{Z} = \boldsymbol{I} - \boldsymbol{P}_{\boldsymbol{X}^*} = \begin{bmatrix} \boldsymbol{I}_1 - \boldsymbol{P}_{\boldsymbol{X}_1} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I}_2 - \boldsymbol{P}_{\boldsymbol{X}_2} \end{bmatrix} = \begin{bmatrix} \boldsymbol{Z}_1 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{Z}_2 \end{bmatrix}$$

then

i) $\boldsymbol{V}_\rho\boldsymbol{Z} = \begin{bmatrix} \boldsymbol{V}_1\boldsymbol{Z}_1 & -\boldsymbol{C}'\boldsymbol{Z}_2 \\ -\boldsymbol{C}\boldsymbol{Z}_1 & \boldsymbol{C}\boldsymbol{V}_1^{-1}\boldsymbol{C}'\boldsymbol{Z}_2 + \rho\boldsymbol{V}_2\boldsymbol{Z}_2 \end{bmatrix}$

ii) $\boldsymbol{Z}\boldsymbol{V}_\rho\boldsymbol{Z} = \begin{bmatrix} \boldsymbol{Z}_1\boldsymbol{V}_1\boldsymbol{Z}_1 & -\boldsymbol{Z}_1\boldsymbol{C}'\boldsymbol{Z}_2 \\ -\boldsymbol{Z}_2\boldsymbol{C}\boldsymbol{Z}_1 & \boldsymbol{Z}_2\boldsymbol{C}\boldsymbol{V}_1^{-1}\boldsymbol{C}'\boldsymbol{Z}_2 + \rho\boldsymbol{Z}_2\boldsymbol{V}_2\boldsymbol{Z}_2 \end{bmatrix}$

The required necessary and sufficient condition for the existence of the UBLUE's given by $\mathcal{M}(\boldsymbol{D}) \subset \mathcal{M}(\boldsymbol{X}_2)$, implies that $\boldsymbol{Z}_2\boldsymbol{C} = \boldsymbol{Z}_2\boldsymbol{D}\boldsymbol{Q}\boldsymbol{V}_1 = \boldsymbol{0}$ and in this way both expressions are simplified to:

i) $\boldsymbol{V}_\rho \boldsymbol{Z} = \begin{bmatrix} \boldsymbol{V}_1\boldsymbol{Z}_1 & \boldsymbol{0} \\ -\boldsymbol{C}\boldsymbol{Z}_1 & \rho\boldsymbol{V}_2\boldsymbol{Z}_2 \end{bmatrix}$

ii) $\boldsymbol{Z}\boldsymbol{V}_\rho \boldsymbol{Z} = \begin{bmatrix} \boldsymbol{Z}_1\boldsymbol{V}_1\boldsymbol{Z}_1 & \boldsymbol{0} \\ \boldsymbol{0} & \rho\boldsymbol{Z}_2\boldsymbol{V}_2\boldsymbol{Z}_2 \end{bmatrix}$

In fact, from (4) we get:

$$\widehat{\boldsymbol{X}^*\boldsymbol{\beta}} = \begin{bmatrix} \boldsymbol{I}_1 - \boldsymbol{V}_1\boldsymbol{Z}_1\{\boldsymbol{Z}_1\boldsymbol{V}_1\boldsymbol{Z}_1\}^-\boldsymbol{Z}_1 & \boldsymbol{0} \\ \boldsymbol{C}\boldsymbol{Z}_1\{\boldsymbol{Z}_1\boldsymbol{V}_1\boldsymbol{Z}_1\}^-\boldsymbol{Z}_1 & \boldsymbol{I}_2 - \boldsymbol{V}_2\boldsymbol{Z}_2\{\boldsymbol{Z}_2\boldsymbol{V}_2\boldsymbol{Z}_2\}^-\boldsymbol{Z}_2 \end{bmatrix} \begin{bmatrix} \boldsymbol{y}_1 \\ \boldsymbol{y}_2^* \end{bmatrix}$$

which implies that

$$\widehat{\boldsymbol{X}^*\boldsymbol{\beta}} = \begin{bmatrix} \boldsymbol{I}_1 - \boldsymbol{P}'_{\boldsymbol{Z}_1\boldsymbol{V}_1} & \boldsymbol{0} \\ \boldsymbol{D}\boldsymbol{Q}\boldsymbol{P}'_{\boldsymbol{Z}_1\boldsymbol{V}_1} & \boldsymbol{I}_2 - \boldsymbol{P}'_{\boldsymbol{Z}_2\boldsymbol{V}_2} \end{bmatrix} \begin{bmatrix} \boldsymbol{y}_1 \\ \boldsymbol{y}_2^* \end{bmatrix} \tag{5}$$

which is the BLUE and also the UBLUE for $\boldsymbol{X}^*\boldsymbol{\beta}$ in the model (3) as a consequence of the necessary and sufficient condition established by Volaufova [9].
In this article we express $\widehat{\boldsymbol{X}^*\boldsymbol{\beta}}$ as a function of an oblique projector defined by Rao [7] which in this case is a projection operator onto $\mathcal{M}(\boldsymbol{X}^*)$ parallel to $\mathcal{M}(\boldsymbol{V}_\rho\boldsymbol{Z})$. In this way in a simplified form we can write the UBLUE of $\widehat{\boldsymbol{X}^*\boldsymbol{\beta}}$ as:

$$\widehat{\boldsymbol{X}^*\boldsymbol{\beta}} = \boldsymbol{P}_{\boldsymbol{X}^*|\boldsymbol{V}_\rho\boldsymbol{Z}}\boldsymbol{y}^* \tag{6}$$

This result is proved in the part 1 of the theorem 1.
It may be noted that the estimator $\widehat{\boldsymbol{X}^*\boldsymbol{\beta}}$ given in (6) exists for all possible values de ρ contained in the parametric space $\boldsymbol{\varrho}$, and therefore according to Wulff and Birkes [12] we can assert that it is the UBLUE of $\boldsymbol{X}^*\boldsymbol{\beta}$.

Remark 1. Observe that using the lemma (2.1) of Bhimasankaram and Sengupta [1], the expression for the BLUE of $\boldsymbol{X}_1\boldsymbol{\beta}_1$ obtained from the model $\boldsymbol{y}_1 = \boldsymbol{X}_1\boldsymbol{\beta}_1 + \boldsymbol{\varepsilon}_1$ where $\boldsymbol{X}_1$ is of full rank by columns, $Cov(\boldsymbol{\varepsilon}_1) = \sigma_1^2\boldsymbol{V}_1$ with $\boldsymbol{V}_1$ non-singular is:

$$\widehat{\boldsymbol{X}_1\boldsymbol{\beta}_1} = [\boldsymbol{I}_1 - \boldsymbol{P}'_{\boldsymbol{Z}_1\boldsymbol{V}_1}]\boldsymbol{y}_1.$$

Furthermore, the BLUE $\widehat{\boldsymbol{X}_2\boldsymbol{\beta}_2}$ obtained from the model $\boldsymbol{y}_2^* = \boldsymbol{X}_2\boldsymbol{\beta}_2 + \boldsymbol{\varepsilon}_2^*$, $\boldsymbol{V}_{\boldsymbol{\varepsilon}_2^*} = Cov(\boldsymbol{\varepsilon}_2^*) = Cov(\boldsymbol{\varepsilon}_2 - \boldsymbol{D}\boldsymbol{Q}\boldsymbol{\varepsilon}_1) = Cov(\boldsymbol{\varepsilon}_2) + Cov(\boldsymbol{D}\boldsymbol{Q}\boldsymbol{\varepsilon}_1) = \sigma_2^2\boldsymbol{V}_2 + \sigma_1^2\boldsymbol{D}\boldsymbol{Q}\boldsymbol{V}_1\boldsymbol{Q}'\boldsymbol{D}'$, $\boldsymbol{X}_2$ of complete rank by columns and $\boldsymbol{V}_2$ non singular is:

$$\begin{aligned}\widehat{\boldsymbol{X}_2\boldsymbol{\beta}_2} &= [\boldsymbol{I}_2 - \boldsymbol{V}_{\boldsymbol{\varepsilon}_2^*}(\boldsymbol{I} - \boldsymbol{P}_{\boldsymbol{X}_2})\{(\boldsymbol{I} - \boldsymbol{P}_{\boldsymbol{X}_2})\boldsymbol{V}_{\boldsymbol{\varepsilon}_2^*}(\boldsymbol{I} - \boldsymbol{P}_{\boldsymbol{X}_2})\}^-(\boldsymbol{I} - \boldsymbol{P}_{\boldsymbol{X}_2})]\boldsymbol{y}_2^* = [\boldsymbol{I}_2 - \boldsymbol{P}'_{\boldsymbol{Z}_2\boldsymbol{V}_2}]\boldsymbol{y}_2^* \\ &= [\boldsymbol{I}_2 - \boldsymbol{P}'_{\boldsymbol{Z}_2\boldsymbol{V}_2}]\boldsymbol{y}_2^*\end{aligned}$$

4. UBLUE of $\boldsymbol{\beta}$ and $X^*\boldsymbol{\beta}$ when is $\rho = \sigma_2^2/\sigma_1^2$ known

Since $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ are of complete rank by columns, from (6) we obtain the UBLUE for $\boldsymbol{\beta}$ and $\boldsymbol{X\beta}$ given by:

$$\widehat{\boldsymbol{\beta}} = \boldsymbol{X}^{*-}\widehat{\boldsymbol{X}^*\boldsymbol{\beta}} = \boldsymbol{X}^{*-}\boldsymbol{P}_{\boldsymbol{X}^*|\boldsymbol{V}_\rho \boldsymbol{Z}}\boldsymbol{y}^* = \boldsymbol{X}^{*-}\boldsymbol{P}_{\boldsymbol{X}^*|\boldsymbol{V}_\rho \boldsymbol{Z}}\boldsymbol{F}\boldsymbol{y} \quad (7)$$

$$\widehat{\boldsymbol{X\beta}} = \boldsymbol{X}\boldsymbol{X}^{*-}\widehat{\boldsymbol{X}^*\boldsymbol{\beta}} = \boldsymbol{X}\boldsymbol{X}^{*-}\boldsymbol{P}_{\boldsymbol{X}^*|\boldsymbol{V}_\rho \boldsymbol{Z}}\boldsymbol{y}^* = \boldsymbol{X}\boldsymbol{X}^{*-}\boldsymbol{P}_{\boldsymbol{X}^*|\boldsymbol{V}_\rho \boldsymbol{Z}}\boldsymbol{F}\boldsymbol{y} \quad (8)$$

where:

$$\boldsymbol{X}^{*-} = \begin{bmatrix} \boldsymbol{X}_1^- & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{X}_2^- \end{bmatrix} = \begin{bmatrix} (\boldsymbol{X}_1'\boldsymbol{X}_1)^{-1}\boldsymbol{X}_1' & \boldsymbol{0} \\ \boldsymbol{0} & (\boldsymbol{X}_2'\boldsymbol{X}_2)^{-1}\boldsymbol{X}_2' \end{bmatrix}$$

Also we get:

$$\widehat{\boldsymbol{\beta}} = \begin{bmatrix} \boldsymbol{X}_1^- & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{X}_2^- \end{bmatrix} \begin{bmatrix} \boldsymbol{I}_1 - \boldsymbol{P}'_{\boldsymbol{Z}_1\boldsymbol{V}_1} & \boldsymbol{0} \\ -\boldsymbol{D}\boldsymbol{Q}(\boldsymbol{I}_1 - \boldsymbol{P}'_{\boldsymbol{Z}_1\boldsymbol{V}_1}) & \boldsymbol{I}_2 - \boldsymbol{P}'_{\boldsymbol{Z}_2\boldsymbol{V}_2} \end{bmatrix} \begin{bmatrix} \boldsymbol{y}_1 \\ \boldsymbol{y}_2 \end{bmatrix}$$

$$\widehat{\boldsymbol{X\beta}} = \begin{bmatrix} \boldsymbol{P}_{\boldsymbol{X}_1} & \boldsymbol{0} \\ \boldsymbol{D}\boldsymbol{X}_1^- & \boldsymbol{P}_{\boldsymbol{X}_2} \end{bmatrix} \begin{bmatrix} \boldsymbol{I}_1 - \boldsymbol{P}'_{\boldsymbol{Z}_1\boldsymbol{V}_1} & \boldsymbol{0} \\ -\boldsymbol{D}\boldsymbol{Q}(\boldsymbol{I}_1 - \boldsymbol{P}'_{\boldsymbol{Z}_1\boldsymbol{V}_1}) & \boldsymbol{I}_2 - \boldsymbol{P}'_{\boldsymbol{Z}_2\boldsymbol{V}_2} \end{bmatrix} \begin{bmatrix} \boldsymbol{y}_1 \\ \boldsymbol{y}_2 \end{bmatrix}$$

which are the estimators of $\boldsymbol{\beta}$ and $\boldsymbol{X\beta}$ for the model (1) with the assumption that $\rho = \sigma_2^2/\sigma_1^2$ is known. Also observe that both estimators are functions of $\boldsymbol{F}\boldsymbol{y}$. An equivalent expression to (8) for the estimator $\widehat{\boldsymbol{X\beta}}$ is given by:

$$\widehat{\boldsymbol{X\beta}} = \boldsymbol{F}^{-1}\boldsymbol{P}_{\boldsymbol{X}^*|\boldsymbol{V}_\rho \boldsymbol{Z}}\boldsymbol{F}\boldsymbol{y}$$

since $\boldsymbol{X}\boldsymbol{X}^{*-} = \boldsymbol{F}^{-1}\boldsymbol{X}^*\boldsymbol{X}^{*-}$ and $\boldsymbol{X}^*\boldsymbol{X}^{*-}\boldsymbol{P}_{\boldsymbol{X}^*|\boldsymbol{V}_\rho \boldsymbol{Z}} = \boldsymbol{P}_{\boldsymbol{X}^*|\boldsymbol{V}_\rho \boldsymbol{Z}}$.

Also from $\boldsymbol{Z} = \boldsymbol{F}'\boldsymbol{Z}$ we obtain the following:

a) $\boldsymbol{X}\boldsymbol{X}^{*-}\boldsymbol{P}_{\boldsymbol{X}^*|\boldsymbol{V}_\rho \boldsymbol{Z}}\boldsymbol{F}\boldsymbol{X} = \boldsymbol{X}\boldsymbol{X}^{*-}\boldsymbol{P}_{\boldsymbol{X}^*|\boldsymbol{V}_\rho \boldsymbol{Z}}\boldsymbol{X}^* = \boldsymbol{X}\boldsymbol{X}^{*-}\boldsymbol{X}^* = \boldsymbol{X}$.

b) $\boldsymbol{X}\boldsymbol{X}^{*-}\boldsymbol{P}_{\boldsymbol{X}^*|\boldsymbol{V}_\rho \boldsymbol{Z}}\boldsymbol{F}\boldsymbol{V}_{\boldsymbol{\varepsilon}}\boldsymbol{Z} = \boldsymbol{X}\boldsymbol{X}^{*-}\boldsymbol{P}_{\boldsymbol{X}^*|\boldsymbol{V}_\rho \boldsymbol{Z}}\boldsymbol{F}\boldsymbol{V}_{\boldsymbol{\varepsilon}}\boldsymbol{F}'\boldsymbol{Z} = \boldsymbol{X}\boldsymbol{X}^{*-}\boldsymbol{P}_{\boldsymbol{X}^*|\boldsymbol{V}_\rho \boldsymbol{Z}}\boldsymbol{V}_{\boldsymbol{\varepsilon}^*}\boldsymbol{Z} = \sigma_1^2\boldsymbol{X}\boldsymbol{X}^{*-}\boldsymbol{P}_{\boldsymbol{X}^*|\boldsymbol{V}_\rho \boldsymbol{Z}}\boldsymbol{V}_\rho\boldsymbol{Z} = \sigma_1^2\boldsymbol{X}\boldsymbol{X}^{*-}\boldsymbol{0} = \boldsymbol{0}$.

Using a) and b) we observe that $\boldsymbol{X}\boldsymbol{X}^{*-}\boldsymbol{P}_{\boldsymbol{X}^*|\boldsymbol{V}_\rho \boldsymbol{Z}}\boldsymbol{F}$ (or equivalently $\boldsymbol{F}^{-1}\boldsymbol{P}_{\boldsymbol{X}^*|\boldsymbol{V}_\rho \boldsymbol{Z}}\boldsymbol{F}$) is a projection operator onto $\mathcal{M}(\boldsymbol{X})$ parallel to $\mathcal{M}(\boldsymbol{V}_{\boldsymbol{\varepsilon}}\boldsymbol{Z})$.
Also the estimators (7) and (8) are UBLUE's for $\boldsymbol{\beta}$ and $\boldsymbol{X\beta}$ respectively for the model (1) which are functions of the estimators in (6) when $\rho = \sigma_2^2/\sigma_1^2$ is known.

5. UBLUE of $\boldsymbol{\beta}$ and $\boldsymbol{X^*\beta}$ for all $\sigma_1^2 > 0, \sigma_2^2 > 0$

The UBLUE for $\boldsymbol{X\beta}$ in the model (1) for all $\sigma_1^2 > 0$, $\sigma_2^2 > 0$ can be obtained from the UBLUE of $\boldsymbol{X}^*\boldsymbol{\beta}$ in the model (2) using $\boldsymbol{V}_{\boldsymbol{\varepsilon}^*}$ in (4) and a similar procedure which was applied to obtain the UBLUE of $\boldsymbol{X\beta}$ from model (3). Using the results given in the previous sections we prove the following theorem:

Theorem 1. Consider the two-stage regular linear model given in (1). If the condition $\mathcal{M}(\boldsymbol{D}) \subset \mathcal{M}(\boldsymbol{X}_2)$ is satisfied then:

(i) The projector onto $\mathcal{M}(\boldsymbol{X}^*)$ parallel to (or along) $\mathcal{M}(\boldsymbol{V}_\rho \boldsymbol{Z})$ is:

$$\boldsymbol{P}_{\boldsymbol{X}^*|\boldsymbol{V}_\rho \boldsymbol{Z}} = \begin{bmatrix} \boldsymbol{I}_1 - \boldsymbol{P}'_{\boldsymbol{Z}_1\boldsymbol{V}_1} & \boldsymbol{0} \\ \boldsymbol{DQP}'_{\boldsymbol{Z}_1\boldsymbol{V}_1} & \boldsymbol{I}_2 - \boldsymbol{P}'_{\boldsymbol{Z}_2\boldsymbol{V}_2} \end{bmatrix}.$$

(ii) The BLUE of $\boldsymbol{X}^*\boldsymbol{\beta}$ obtained from model (3) is given by:

$$\widehat{\boldsymbol{X}^*\boldsymbol{\beta}} = \boldsymbol{P}_{\boldsymbol{X}^*|\boldsymbol{V}_\rho \boldsymbol{Z}}\boldsymbol{y}^*.$$

(iii) The BLUE $\widehat{\boldsymbol{X}^*\boldsymbol{\beta}} = \boldsymbol{P}_{\boldsymbol{X}^*|\boldsymbol{V}_\rho \boldsymbol{Z}}\boldsymbol{y}^*$ is also the UBLUE of $\boldsymbol{X}^*\boldsymbol{\beta}$.

(iv) The UBLUE's of $\boldsymbol{\beta}$ and $\boldsymbol{X\beta}$ for the model (1) when $\rho = \sigma_2^2/\sigma_1^2$ is known, are respectively.

$$\widehat{\boldsymbol{\beta}} = \mathbf{X}^{*-}\mathbf{P}_{\mathbf{X}^*|\mathbf{V}_\rho \mathbf{Z}}\mathbf{F}\mathbf{y}$$
$$\widehat{\boldsymbol{X\beta}} = \boldsymbol{XX}^{*-}\boldsymbol{P}_{\boldsymbol{X}^*|\boldsymbol{V}_\rho \boldsymbol{Z}}\boldsymbol{Fy}.$$

(v) The UBLUE's of $\boldsymbol{\beta}$ and $\boldsymbol{X\beta}$ for the model (1) for all $\sigma_1^2 > 0$, $\sigma_2^2 > 0$ are:

$$\widehat{\boldsymbol{\beta}} = \boldsymbol{X}^{*-}\boldsymbol{P}_{\boldsymbol{X}^*|\boldsymbol{V}_{\varepsilon^*}\boldsymbol{Z}}\boldsymbol{Fy}$$
$$\widehat{\boldsymbol{X\beta}} = \boldsymbol{XX}^{*-}\boldsymbol{P}_{\boldsymbol{X}^*|\boldsymbol{V}_{\varepsilon^*}\boldsymbol{Z}}\boldsymbol{Fy}$$

which are equal to the corresponding estimators in the case when $\rho = \sigma_2^2/\sigma_1^2$ is known.

(vi) The projector onto $\mathcal{M}(\boldsymbol{X})$ parallel to (or along) $\mathcal{M}(\boldsymbol{V}_{\boldsymbol{\varepsilon}}\boldsymbol{Z})$ is:

$$\boldsymbol{XX}^{*-}\boldsymbol{P}_{\boldsymbol{X}^*|\boldsymbol{V}_{\varepsilon^*}\boldsymbol{Z}}\boldsymbol{F}.$$

Proof.

(i) To show that:

$$\boldsymbol{P}_{\boldsymbol{X}^*|\boldsymbol{V}_\rho \boldsymbol{Z}} = \begin{bmatrix} \boldsymbol{I}_1 - \boldsymbol{P}'_{\boldsymbol{Z}_1\boldsymbol{V}_1} & \boldsymbol{0} \\ \boldsymbol{DQP}'_{\boldsymbol{Z}_1\boldsymbol{V}_1} & \boldsymbol{I}_2 - \boldsymbol{P}'_{\boldsymbol{Z}_2\boldsymbol{V}_2} \end{bmatrix}$$

is a projection operator onto $\mathcal{M}(\boldsymbol{X}^*)$ parallel to (or along) $\mathcal{M}(\boldsymbol{V}_\rho\boldsymbol{Z})$ is a consequence of the verification of the necessary and sufficient conditions given by Rao [7] in the lemma (2.4). According to these conditions, for a disjoint partition $(\boldsymbol{X}^*:\boldsymbol{V}_\rho\boldsymbol{Z})$, $\boldsymbol{P}_{\boldsymbol{X}^*|\boldsymbol{V}_\rho\boldsymbol{Z}}$ is a projector onto $\mathcal{M}(\boldsymbol{X}^*)$ parallel to (or along) $\mathcal{M}(\boldsymbol{V}_\rho\boldsymbol{Z})$ if and only if the followings are satisfied:

a) $\boldsymbol{P}_{\boldsymbol{X}^*|\boldsymbol{V}_\rho\boldsymbol{Z}}\boldsymbol{X}^* = \boldsymbol{X}^*$

b) $\boldsymbol{P}_{\boldsymbol{X}^*|\boldsymbol{V}_\rho\boldsymbol{Z}}\boldsymbol{V}_\rho\boldsymbol{Z} = \boldsymbol{0}$

which can be verified using the matrix products given below:

a) $$\begin{bmatrix}\boldsymbol{I}_1-\boldsymbol{P}'_{\boldsymbol{Z}_1\boldsymbol{V}_1} & \boldsymbol{0}\\ \boldsymbol{DQP}'_{\boldsymbol{Z}_1\boldsymbol{V}_1} & \boldsymbol{I}_2-\boldsymbol{P}'_{\boldsymbol{Z}_2\boldsymbol{V}_2}\end{bmatrix}\begin{bmatrix}\boldsymbol{X}_1 & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{X}_2\end{bmatrix}=\begin{bmatrix}\boldsymbol{X}_1 & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{X}_2\end{bmatrix}$$

b) $$\begin{bmatrix}\boldsymbol{I}_1-\boldsymbol{P}'_{\boldsymbol{Z}_1\boldsymbol{V}_1} & \boldsymbol{0}\\ \boldsymbol{DQP}'_{\boldsymbol{Z}_1\boldsymbol{V}_1} & \boldsymbol{I}_2-\boldsymbol{P}'_{\boldsymbol{Z}_2\boldsymbol{V}_2}\end{bmatrix}\begin{bmatrix}\boldsymbol{V}_1 & -\boldsymbol{C}'\\ -\boldsymbol{C} & \boldsymbol{C}\boldsymbol{V}_1^{-1}\boldsymbol{C}'+\rho\boldsymbol{V}_2\end{bmatrix}\begin{bmatrix}\boldsymbol{Z}_1 & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{Z}_2\end{bmatrix}=\begin{bmatrix}\boldsymbol{0} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{0}\end{bmatrix}$$

(ii) This result is obtained by applying the expression (4) of the BLUE of $\boldsymbol{X}^*\boldsymbol{\beta}$ (which is a special case of the part (b) of theorem 3.2 given by Rao [7]) to the model (3) under the condition $\mathcal{M}(\boldsymbol{D})\subset\mathcal{M}(\boldsymbol{X}_2)$, as indicated by Volaufova [9] in the theorem 1.2, in the same way as it was done in the pages 4 and 5 of this paper.

(iii) Since the BLUE $\widehat{\boldsymbol{X}^*\boldsymbol{\beta}} = \boldsymbol{P}_{\boldsymbol{X}^*|\boldsymbol{V}_\rho\boldsymbol{Z}}\boldsymbol{y}^*$ of $\boldsymbol{X}^*\boldsymbol{\beta}$ obtained in the part (ii) of this theorem exists for all $\rho\in\varrho$, therefore according to Wulff and Birkes [12] it is also the UBLUE of $\boldsymbol{X}^*\boldsymbol{\beta}$.
Another way to prove this result consists in considering that according to theorem 1.2 of Volaufova [9] the UBLUE of vector $\boldsymbol{\beta}=(\boldsymbol{\beta}_1',\boldsymbol{\beta}_2')'$ of the model (3) exists if and only if $\mathcal{M}(\boldsymbol{D})\subset\mathcal{M}(\boldsymbol{X}_2)$, which is the condition used together with the equation (4) in order to obtain the BLUE of $\boldsymbol{X}^*\boldsymbol{\beta}$ of the model (3) in the part (ii) of this theorem and since $E(\boldsymbol{X}^{*-}\widehat{\boldsymbol{X}^*\boldsymbol{\beta}}) = \boldsymbol{X}^{*-}E(\widehat{\boldsymbol{X}^*\boldsymbol{\beta}}) = \boldsymbol{X}^{*-}\boldsymbol{X}^*\boldsymbol{\beta}=\boldsymbol{\beta}$, it follows that $\widehat{\boldsymbol{\beta}}=\boldsymbol{X}^{*-}\widehat{\boldsymbol{X}^*\boldsymbol{\beta}}$ and $\widehat{\boldsymbol{X}^*\boldsymbol{\beta}}=\boldsymbol{X}^*\widehat{\boldsymbol{\beta}}$. Therefore $\widehat{\boldsymbol{\beta}}$ and $\widehat{\boldsymbol{X}^*\boldsymbol{\beta}}$ are related linearly and since $\widehat{\boldsymbol{\beta}}=\boldsymbol{X}^{*-}\widehat{\boldsymbol{X}^*\boldsymbol{\beta}}$ is the UBLUE of $\boldsymbol{\beta}$ for the model (3), it follows that $\widehat{\boldsymbol{X}^*\boldsymbol{\beta}}$ is also the UBLUE for $\boldsymbol{X}^*\boldsymbol{\beta}$.

(iv) To prove this result we observe that the model (3) is a linear transformation of the model (1) assuming that $\rho=\sigma_2^2/\sigma_1^2$is known. This transformation does not affect the vector of parameters $\boldsymbol{\beta}$ of the model (3), consequently $\widehat{\boldsymbol{\beta}}=\boldsymbol{X}^{*-}\widehat{\boldsymbol{X}^*\boldsymbol{\beta}}$ is the estimator UBLUE of $\boldsymbol{\beta}$ for both models. Therefore the estimator UBLUE of $\boldsymbol{\beta}$ for the model (1) can be written as $\widehat{\boldsymbol{\beta}}=\boldsymbol{X}^{*-}\widehat{\boldsymbol{X}^*\boldsymbol{\beta}}=\boldsymbol{X}^{*-}\boldsymbol{P}_{\boldsymbol{X}^*|\boldsymbol{V}_\rho\boldsymbol{Z}}\boldsymbol{y}^*=\boldsymbol{X}^{*-}\boldsymbol{P}_{\boldsymbol{X}^*|\boldsymbol{V}_\rho\boldsymbol{Z}}\boldsymbol{F}\boldsymbol{y}$. In order to obtain the UBLUE $\widehat{\boldsymbol{X}\boldsymbol{\beta}}$ for the model (1) assuming $\rho=\sigma_2^2/\sigma_1^2$ known the matrix $\boldsymbol{X}$ is premultiplied to the estimator $\widehat{\boldsymbol{\beta}}=\boldsymbol{X}^{*-}\widehat{\boldsymbol{X}^*\boldsymbol{\beta}}$ to obtain $\widehat{\boldsymbol{X}\boldsymbol{\beta}}=\boldsymbol{X}\widehat{\boldsymbol{\beta}}=\boldsymbol{X}\boldsymbol{X}^{*-}\widehat{\boldsymbol{X}^*\boldsymbol{\beta}}=\boldsymbol{X}\boldsymbol{X}^{*-}\boldsymbol{P}_{\boldsymbol{X}^*|\boldsymbol{V}_\rho\boldsymbol{Z}}\boldsymbol{F}\boldsymbol{y}$.

(v) Using a similar process as the one used to find the estimator $\widehat{\boldsymbol{X}^*\boldsymbol{\beta}}=\boldsymbol{P}_{\boldsymbol{X}^*|\boldsymbol{V}_\rho\boldsymbol{Z}}\boldsymbol{y}^*$ given in (6) but with $\boldsymbol{V}_{\boldsymbol{\varepsilon}^*}$ of the model (2) instead of $\boldsymbol{V}_\rho$ and considering again the condition $\mathcal{M}(\boldsymbol{D})\subset\mathcal{M}(\boldsymbol{X}_2)$, we obtain the expression:

$$\widehat{\boldsymbol{X}^*\boldsymbol{\beta}}=\begin{bmatrix}\boldsymbol{I}_1-\boldsymbol{P}'_{\boldsymbol{Z}_1\boldsymbol{V}_1} & \boldsymbol{0}\\ \boldsymbol{DQP}'_{\boldsymbol{Z}_1\boldsymbol{V}_1} & \boldsymbol{I}_2-\boldsymbol{P}'_{\boldsymbol{Z}_2\boldsymbol{V}_2}\end{bmatrix}\begin{bmatrix}\boldsymbol{y}_1\\ \boldsymbol{y}_2^*\end{bmatrix}$$

which is the same as given in (5). In consequence the BLUE of $\boldsymbol{X}^*\boldsymbol{\beta}$ for the model (2) for all $\sigma_1^2 > 0$, $\sigma_2^2 > 0$ is equal to the same which was obtained with $\rho = \sigma_2^2/\sigma_1^2$ known in the model (3). Since it estimator does not depend explicitly of the parameters σ_1^2, σ_2^2 of the model, $\widehat{\boldsymbol{X}^*\boldsymbol{\beta}}$ given en (6) is also the BLUE of $\boldsymbol{X}^*\boldsymbol{\beta}$ for the model (2) for all $\sigma_1^2 > 0, \sigma_2^2 > 0$ and according to Wulff and Birkes [12] it is also the UBLUE of $\boldsymbol{X}^*\boldsymbol{\beta}$. We will denote by $\boldsymbol{P}_{\boldsymbol{X}^*|\boldsymbol{V}_{\boldsymbol{\varepsilon}^*}\boldsymbol{Z}}$ the matrix:

$$\boldsymbol{P}_{\boldsymbol{X}^*|\boldsymbol{V}_{\boldsymbol{\varepsilon}^*}\boldsymbol{Z}} = \begin{bmatrix} \boldsymbol{I}_1 - \boldsymbol{P}'_{\boldsymbol{Z}_1\boldsymbol{V}_1} & \boldsymbol{0} \\ \boldsymbol{DQP}'_{\boldsymbol{Z}_1\boldsymbol{V}_1} & \boldsymbol{I}_2 - \boldsymbol{P}'_{\boldsymbol{Z}_2\boldsymbol{V}_2} \end{bmatrix}$$

for the case of the model (2) for all $\sigma_1^2 > 0, \sigma_2^2 > 0$. In consequence the UBLUE of $\boldsymbol{X}^*\boldsymbol{\beta}$ for this model is $\widehat{\boldsymbol{X}^*\boldsymbol{\beta}} = \boldsymbol{P}_{\boldsymbol{X}^*|\boldsymbol{V}_{\boldsymbol{\varepsilon}^*}\boldsymbol{Z}}\boldsymbol{y}^*$.

In this way the estimators of $\boldsymbol{\beta}$ and $\boldsymbol{X\beta}$ of the model (1) can be obtained as functions of $\widehat{\boldsymbol{\beta}}$ in the same way as it was done in the proof of the part (iv) of this theorem. In this case the estimators of $\boldsymbol{\beta}$ and $\boldsymbol{X\beta}$ for the model (1) for all $\sigma_1^2 > 0, \sigma_2^2 > 0$ are given by:

$$\widehat{\boldsymbol{\beta}} = \boldsymbol{X}^{*-}\widehat{\boldsymbol{X}^*\boldsymbol{\beta}} = \boldsymbol{X}^{*-}\boldsymbol{P}_{\boldsymbol{X}^*|\boldsymbol{V}_{\boldsymbol{\varepsilon}^*}\boldsymbol{Z}}\boldsymbol{y}^* = \boldsymbol{X}^{*-}\boldsymbol{P}_{\boldsymbol{X}^*|\boldsymbol{V}_{\boldsymbol{\varepsilon}^*}\boldsymbol{Z}}\boldsymbol{Fy}$$
$$\widehat{\boldsymbol{X\beta}} = \boldsymbol{X}\widehat{\boldsymbol{\beta}} = \boldsymbol{XX}^{*-}\widehat{\boldsymbol{X}^*\boldsymbol{\beta}} = \boldsymbol{XX}^{*-}\boldsymbol{P}_{\boldsymbol{X}^*|\boldsymbol{V}_{\boldsymbol{\varepsilon}^*}\boldsymbol{Z}}\boldsymbol{y}^* = \boldsymbol{XX}^{*-}\boldsymbol{P}_{\boldsymbol{X}^*|\boldsymbol{V}_{\boldsymbol{\varepsilon}^*}\boldsymbol{Z}}\boldsymbol{Fy}$$

(vi) In the parts a) and b) on the page 7 of this paper it was shown that:

$$\boldsymbol{XX}^{*-}\boldsymbol{P}_{\boldsymbol{X}^*|\boldsymbol{V}_{\rho}\boldsymbol{Z}}\boldsymbol{FX} = \boldsymbol{X}$$
$$\boldsymbol{XX}^{*-}\boldsymbol{P}_{\boldsymbol{X}^*|\boldsymbol{V}_{\rho}\boldsymbol{Z}}\boldsymbol{FV}_{\boldsymbol{\varepsilon}}\boldsymbol{Z} = \boldsymbol{0}$$

Now from the proof of the part 5 of this theorem we know that $\boldsymbol{P}_{\boldsymbol{X}^*|\boldsymbol{V}_{\rho}\boldsymbol{Z}} = \boldsymbol{P}_{\boldsymbol{X}^*|\boldsymbol{V}_{\boldsymbol{\varepsilon}^*}\boldsymbol{Z}}$, in consequence:

$$\boldsymbol{XX}^{*-}\boldsymbol{P}_{\boldsymbol{X}^*|\boldsymbol{V}_{\boldsymbol{\varepsilon}^*}\boldsymbol{Z}}\boldsymbol{FX} = \boldsymbol{X}$$
$$\boldsymbol{XX}^{*-}\boldsymbol{P}_{\boldsymbol{X}^*|\boldsymbol{V}_{\boldsymbol{\varepsilon}^*}\boldsymbol{Z}}\boldsymbol{FV}_{\boldsymbol{\varepsilon}}\boldsymbol{Z} = \boldsymbol{0}$$

and therefore $\boldsymbol{XX}^{*-}\boldsymbol{P}_{\boldsymbol{X}^*|\boldsymbol{V}_{\boldsymbol{\varepsilon}^*}\boldsymbol{Z}}\boldsymbol{F}$ is a projector onto $\mathcal{M}(\boldsymbol{X})$ parallel to (or along) $\mathcal{M}(\boldsymbol{V}_{\boldsymbol{\varepsilon}}\boldsymbol{Z})$.

It may be noted that the model (2) is a nonsingular transformation of the model (1) and that the expression for $\widehat{\boldsymbol{X\beta}}$ is a linear function of $\boldsymbol{Fy}$ and because it is a BLUE (and also UBLUE), and so it follows by the definition 3.1 given by Drygas [2] that $\boldsymbol{Fy}$ is a sufficient linear transformation.

Remark 2. The results obtained in this paper can also be derived using the expression (6) given in the Theorem 3.2 of Rao [7]. In this case the matrices $\boldsymbol{Z}_1$, $\boldsymbol{Z}_2$ are of maximum rank such that $\boldsymbol{Z}_1 = \boldsymbol{X}_1^{\perp}, \boldsymbol{Z}_2 = \boldsymbol{X}_2^{\perp}$.

6. Application

We will present an example of the application of the regular two stage model using the data obtained from the web page of U.N.E.T. (Universidad Nacional Experimental de Táchira, Táchira, Venezuela) [8]. For this purpose we will use only the grade data of the entrance exam and the preparatory courses of the high school graduates who wished to follow a career in Mechanical, Electronic, Informatics or Civil Engineering of the U.N.E.T. in the period 2009-1 and 2010-1. From each period a random sample of sample size 40 was taken. Since the data belongs to a group of students who followed a preparatory course and took admission exam in different and not consecutive periods, we will assume independence among the different groups of data. This assumption is required for the correct application of the two stage regular model.
In particular, $\boldsymbol{y}_1 = \text{CEA2009\_1}$ y $\boldsymbol{y}_2 = \text{CEA2010\_1}$ are the dependent variables in each of the two stages of the model corresponding to the grades obtained by the aspirants in the admission exam for the periods 2009-1 y 2010-1 respectively. The matrices $\boldsymbol{X}_1$ y $\boldsymbol{X}_2$ consists of two columns of data in each case where the first column consists of the one´s corresponding to the intercept and the second column is the explicatory variable. In this case $\boldsymbol{X}_1 = [\boldsymbol{J}, \boldsymbol{X}_{11}]$, $\boldsymbol{X}_2 = [\boldsymbol{J}, \boldsymbol{X}_{22}]$ where $\boldsymbol{X}_{11} = \text{CP2009\_1}$, $\boldsymbol{X}_{11} = \text{CP2010\_1}$ are the explicatory variables in the stage 1 and stage 2 respectively corresponding to the preparatory course grades and $\boldsymbol{J}$ is a matrix of the one´s of dimension 40×1. In this way $\boldsymbol{y}_1$, $\boldsymbol{X}_1$ correspond with the period 2009-1 y $\boldsymbol{y}_2$, $\boldsymbol{X}_2$ with the period 2010-1.
In relation to the covariance matrices we will assume that $\boldsymbol{V}_1$, $\boldsymbol{V}_2$ has the pattern given by:

$$\boldsymbol{V}_i = \begin{bmatrix} 1 & \rho_i & \cdots & \rho_i \\ \rho_i & 1 & \cdots & \rho_i \\ \vdots & \vdots & \ddots & \vdots \\ \rho_i & \rho_i & \cdots & 1 \end{bmatrix}$$

for $i = 1, 2$ y $\rho_1 = 0.3$, $\rho_2 = 0.3$. In other words we will assume that the elements within each sample are equally correlated. In order that the condition $\mathcal{M}(\boldsymbol{D}) \subset \mathcal{M}(\boldsymbol{X}_2)$ is satisfied, we will write $\boldsymbol{D} = \boldsymbol{X}_2\boldsymbol{A}$ where $\boldsymbol{A}$ has the dimension $p_2\times p_1$. For this example we will use:

$$\boldsymbol{A} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Other possibilities for $\boldsymbol{A}$ may be:

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \boldsymbol{A} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

In this way the two stage model can be written as:

$$\boldsymbol{y}_1 = \boldsymbol{X}_1\boldsymbol{\beta}_1 + \boldsymbol{\varepsilon}_1$$
$$\boldsymbol{y}_2 = \boldsymbol{X}_2\boldsymbol{A}\boldsymbol{\beta}_1 + \boldsymbol{X}_2\boldsymbol{\beta}_2 + \boldsymbol{\varepsilon}_2$$

or equivalently:

$$\boldsymbol{y}_1 = \beta_{11}\boldsymbol{J} + \beta_{12}\boldsymbol{X}_{11} + \boldsymbol{\varepsilon}_1$$
$$\boldsymbol{y}_2 = \beta_{21}\boldsymbol{J} + (\beta_{12} + \beta_{22})\boldsymbol{X_{22}} + \boldsymbol{\varepsilon}_2 = \beta_{21}^{*}\boldsymbol{J} + \beta_{22}^{*}\boldsymbol{X_{22}} + \boldsymbol{\varepsilon}_2$$

where, $\boldsymbol{\beta}_1 = (\beta_{11}, \beta_{12})'$, $\boldsymbol{\beta}_2 = (\beta_{21}, \beta_{22})'$. In this case the term β_{12} represents the influence of the stage 1 on the stage 2. The data corresponding to each period is tabulated in Table 1.

Table 1. Sample data used for the estimation of model parameters.

Period	2009-1		2010-1			2009-1		2010-1	
Obs	y_1	X_{11}	y_2	X_{22}	Obs	y_1	X_{11}	y_2	X_{22}
1	44,69	5,50	47,08	33,25	21	49,61	16,00	34,83	28,50
2	57,33	36,75	41,37	31,67	22	52,79	12,00	48,75	36,42
3	67,86	47,17	45,41	35,08	23	51,65	15,50	46,70	41,08
4	44,57	16,67	38,87	33,33	24	61,72	15,08	37,76	29,75
5	49,25	17,58	37,74	26,42	25	55,93	22,67	41,50	25,75
6	51,79	23,67	42,67	39,25	26	52,83	36,33	46,01	36,08
7	52,32	28,33	36,92	23,67	27	40,68	9,08	49,01	37,50
8	41,03	6,42	31,56	25,00	28	50,83	19,25	42,51	27,58
9	46,25	13,33	44,57	32,17	29	47,50	10,42	39,94	24,67
10	58,26	27,50	46,96	37,83	30	46,25	13,33	30,18	25,75
11	46,80	12,17	33,42	25,42	31	49,42	22,92	38,18	24,75
12	49,19	16,17	38,20	31,42	32	51,17	20,75	41,67	17,17
13	54,68	29,17	41,33	28,92	33	42,73	2,17	35,17	23,58
14	61,92	27,08	43,58	42,42	34	42,57	5,33	33,50	19,67
15	48,04	18,08	46,42	34,83	35	49,54	16,42	32,79	21,08
16	57,76	36,92	37,17	28,67	36	44,63	10,67	33,40	23,00
17	44,38	9,58	47,53	41,17	37	52,53	14,75	33,40	23,00
18	56,58	13,50	49,67	32,00	38	46,87	15,92	38,37	22,17
19	40,62	7,50	33,50	20,75	39	53,75	14,00	47,79	41,08
20	47,67	18,58	37,14	29,00	40	58,63	32,58	39,05	21,92

The results obtained by filling the two stage model are presented in Table 2 in which we observe that $\beta_{21}^{*} = \beta_{21}$ y $\beta_{22}^{*} = \beta_{12} + \beta_{22}$. A measure of the influence of the stage 1 on the stage 2 represented by $\hat{\beta}_{12} = 0.4934$, may be interpreted as a measure of the influence of the factors related to the preparatory course which may change as the correction process to improve this course is carried out. In this sense, the influence of the preparatory course of the period 2010_1 (CP2010_1) over the grade of the admission exam for the same period (CEA2010_1), can be expressed as the sum of two components. One component associated to factors inherent in the preparatory course represented by 0.4934*CP2010_1 and the component related to the factors related to the knowledge of the student 0,1257*CP2010_1. The total influence of the grade of the preparatory course of the period 2010_1 on the admission exam is given by the sum of these two components and is equal to 0.61907*CP2010_1.

Table 2. Regression coefficients obtained by fitting a two-stage model.

	Stage 1 coefficients		Stage 2 coefficients			
	$\hat{\beta}_{11}$	$\hat{\beta}_{12}$	$\hat{\beta}_{21}$	$\hat{\beta}_{22}$	$\hat{\beta}^*_{21}$	$\hat{\beta}^*_{22}$
$\boldsymbol{D} = \boldsymbol{X}_2\boldsymbol{A}$	41.4771	0.4934	21.9851	0.1257	21.9851	0,61907

This method of interpreting the regression coefficients reflects the reality as regards the deficiency of the necessary knowledge with which most of the students at present are admitted to the venezuelan universities. In this way the model corresponding to the second stage can be expressed as $\text{CEA2010\_1}_\text{i} = 21{,}9851 + 0.61907\ \text{CP2010\_1}_i,\ \ i = 1, 2, \dots, 40$.

7. Conclusion

The estimation theory of multi-stage linear models and particularly two stage linear model is known for more than twenty years, still the use of these models is very rare in applied research which we attribute to the lengthy and complicated expressions of the parameter estimators of the mean vector and other parameters of the model as published in different papers on this subject. In this article, equivalent alternative expressions for the UBLUE of $\boldsymbol{\beta}$, $\boldsymbol{X\beta}$ and $\boldsymbol{X}^*\boldsymbol{\beta}$ as functions of projector operators onto $\mathcal{M}(\boldsymbol{X})$ and $\mathcal{M}(\boldsymbol{X}^*)$ in the untransformed and transformed version of the regular two-stage linear model, are obtained which should facilitate and provide new insights for the use of these models in applied research.

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