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BOOTSTRAP GRAPHICAL TEST FOR EQUALITY OF VARIANCES

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***Abstract**: In this paper, a bootstrap graphical method is developed as an alternative to the Bartlett and Levene tests to test the hypothesis on equality of several variances. An example is given to demonstrate the advantage of bootstrap graphical procedure over the Bartlett and Levene tests from decision making point of view.*

***Keywords**: Bootstrap method, Bartlett test, Levene test, variance.*

1. Introduction

Testing of equality of variances is of great interest in a number of research areas. Increasing uniformity is an important objective in quality control of manufacturing processes, in agricultural production systems and in the development of educational methods. Procedures for comparing variances are also used as a preliminary to standard analysis of variance, dose- response modeling or discriminant analysis [2].

Let $\{X_{ij},\ i=1,2,\ldots,k,\quad j=1,2,\ldots,n\}$ represent k independent random samples of size n and we assume that $X_{ij} \sim N(\mu_i, \sigma_i^2)$ for $i=1,2,\ldots,k$. Since the k samples are drawn from k normal populations with different means and different variances, we wish to test the null hypothesis $H_0 : \sigma_1^2 = \sigma_2^2 = \cdots = \sigma_k^2 = \sigma^2$ against the alternative hypothesis that at least two variances are unequal. Bartlett and Levene tests are used for testing H_0 in the literature. These tests demonstrate only the statistical significance of the variances being compared. In Sections 2 and 3, we briefly review the Bartlett test and Levene test procedures [1] [2] [6] [8].

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Statistical inference based on data resampling has drawn a great deal of attention in recent years. The main idea about these resampling methods is not to assume much about the underlying population distribution and instead tries to get the information about the population from the data itself various types of resampling leads to various types of methods like the jackknife and the bootstrap. Bootstrap method [3] uses the relationship between the sample and resamples drawn from the sample, to approximate the relationship between the population and samples drawn from it. With the bootstrap method, the basic sample is treated as the population and a Monte Carlo style procedure is conducted on it. This is done by randomly drawing a large number of resamples of size n from this original sample with replacement. Both bootstrap and traditional parametric inference seek to achieve the same goal using limited information to estimate the sampling distribution of the chosen estimator $\hat{\theta}$. The estimate will be used to make inferences about a population parameter θ. The bootstrap method is distribution free which means that it is not dependent on a particular class of distributions. With the bootstrap method, the entire sampling distribution of $\hat{\theta}$ is estimated by relying on the fact that the sampling distribution is a good estimate of the population distribution [3] [4] [5]. In section 4, bootstrap method applied to testing of equality of several variances is explained.

2. Bartlett test for equality of variances

The Bartlett test statistic is designed to test for equality of variances across samples against the alternative that variances are unequal for at least two samples. Let $s_i^2 = \frac{1}{n-1}\sum_{j=1}^{n}\left(x_{ij} - \bar{x}_i\right)^2$ is the variance of i[th] sample and the pooled variance is the weighted average of the sample variances and is defined as $s_p^2 = \frac{n-1}{N-k}\sum_{i=1}^{k} s_i^2$ where N=nk. The Bartlett test statistic is given by:

$$T = \frac{(N-k)\ln s_p^2 - (n-1)\sum_{i=1}^{k}\ln s_i^2}{1 + \frac{1}{3(k-1)}\left(\frac{k}{n-1} - \frac{1}{N-k}\right)} \tag{1}$$

The variances are judged to be unequal if, $T \geq \chi^2_{(\alpha,k-1)}$ where $\chi^2_{(\alpha,k-1)}$ is the upper critical value of the chi-square distribution with k - 1 degrees of freedom and a significance level of α [1] [2] [8].

3. Levene test for equality of variances

Bartlett's test is sensitive to departures from normality. That is, if the samples come from non-normal distributions, then Bartlett's test may simply be testing for non-normality. The Levene test is an alternative to the Bartlett test that is less sensitive to departures from normality. Levene's test (1960) is used to test if k samples have equal variances. Let $z_{ij} = \left|x_{ij} - \tilde{x}_{i.}\right|$, $\tilde{x}_{i.}$ is the median of the i[th] sample, $\bar{z}_{i.} = \frac{1}{n}\sum_{j=1}^{n} z_{ij}$, $\bar{z}_{..} = \frac{1}{N}\sum_{i=1}^{k}\sum_{j=1}^{n} z_{ij}$ and N=nk then the Levene test statistic is given by:

$$W = \frac{(N-k)}{(k-1)} \frac{\sum_{i=1}^{k} n\left(\bar{z}_{i.} - \bar{z}_{..}\right)^2}{\sum_{i=1}^{k}\sum_{j=1}^{n}\left(z_{ij} - \bar{z}_{i.}\right)^2} \tag{2}$$

The Levene test rejects the hypothesis that the variances are equal if $W \geq F_\alpha(k-1,\ N-k)$, where $F_\alpha(k-1,\ N-k)$ is the upper critical value of the F distribution with (k-1, N-k) degrees of freedom at a significance level α [1][6].

4. Bootstrap graphical method for testing of equality of several variances

Let $\{X_{ij}, i = 1,2,\cdots,k;\ j = 1,2,\cdots,n\}$ represent k available independent random samples of size n and the sample variance of the i[th] sample is given by $s_i^2 = \frac{1}{n-1}\sum_{j=1}^{n}\left(X_{ij} - \bar{X}_i\right)^2$ for i=1, 2,...,k. Bootstrap graphical procedure for testing the equality of several variances is given in the following steps:

1. Let $\{Y_{ijb}, i = 1,2,\ \ldots, k;\ j = 1,2,\ldots,n; b = 1,2,\ldots, B = 3000\}$ represents the b-th bootstrap sample of size n, drawn from i[th] available sample.
2. Compute $\bar{y}_{ib}$ and s_{ib}^2, the mean and variance of b-th bootstrap sample form i[th] available sample and are given by $\bar{y}_{ib} = \frac{1}{n}\sum_{j=1}^{n} Y_{ijb}$ and $s_{ib}^2 = \frac{1}{n-1}\sum_{j=1}^{n}\left(Y_{ijb} - \bar{y}_{ib}\right)^2$.
3. Compute $s_b^2 = \frac{1}{k}\sum_{i=1}^{k} s_{ib}^2$, b=1,2, …, B.
4. Obtain the sampling distribution of sample variance using B-bootstrap estimates and compute the central decision line (CDL) as $s_*^2 = \frac{1}{B}\sum_{b=1}^{B} s_b^2$.
5. The lower decision line (LDL) and the upper decision line (UDL) for the comparison of each of the s_i^2 are given by:

$$LDL = s^2_{(m_1)} \tag{3}$$

$$UDL = s^2_{(m_2)} \tag{4}$$

where $m_1 = \max\left\{1, \left[\frac{\alpha}{2}(B+1)\right]\right\}$, $m_2 = \min\left\{\left[\left(1-\frac{\alpha}{2}\right)(B+1)\right], B\right\}$, $\alpha = 0.05$, $S^2_{(m_1)}$ and $S^2_{(m_2)}$ are the m_1^{th} and m_2^{th} order statistics respectively from the bootstrap estimates s_b^2, b=1, 2, …,B and [x] represents the integer part.

6. Plot s_i^2 against the decision lines. If any one of the points plotted lies outside the respective decision lines, H_0 is rejected at α level and conclude that the variances are not homogenous.

The proposed method is very useful in handling of small samples of size less than 30 and it depends only on the supplied information. This method not only tests the significant difference among the variances but also identify the source of heterogeneity of variances.

5. Numerical example

The data of the 20 observations on each of four process variables x_1, x_2, x_3 and x_4 under a chemical process are shown in upper panel of the Table 10-6 of Montgomery (2004, p538) [7]. The sample sizes are equal in this problem, and we have n=20, k=4 and N=80. We wish to test the equality of variances of the four variables by assuming that the observations have come from a normal population. For the given data $s_1^2 = 1.01, s_2^2 = 0.92, s_3^2 = 9.39,$ and $s_4^2 = 2.28$, we have calculated the test statistics of the Bartlett and Levene tests and compared them with the respective critical values at 5% level. The Bartlett test statistic value is 35.47 and the significant value at 5% level is $\chi^2_{3,0.05} = 7.81$. Since the Bartlett-test statistic value is greater than the critical value, therefore we reject H_0 at 5% level. The Levene test statistic value is 11.35 and the significant value at 5% level is $F_{0.05}(3,76) = 2.72$. Since the Levene-test statistic value is greater than the critical value, therefore we reject H_0 at 5% level.
By applying the bootstrap procedure explained in Section 4, the LDL, CDL and UDL are obtained as 2.04, 3.24 and 4.62 respectively. Prepare a chart as in Figure 1, with the above decision lines and plot the points s_i^2 $(i = 1,2,3,4)$. From the Figure 1, we observe that s_1^2, s_2^2 and s_3^2 lie outside the decision lines. Hence, H_0 may be rejected and it may be concluded that the variances of four chemical processes are not equal.

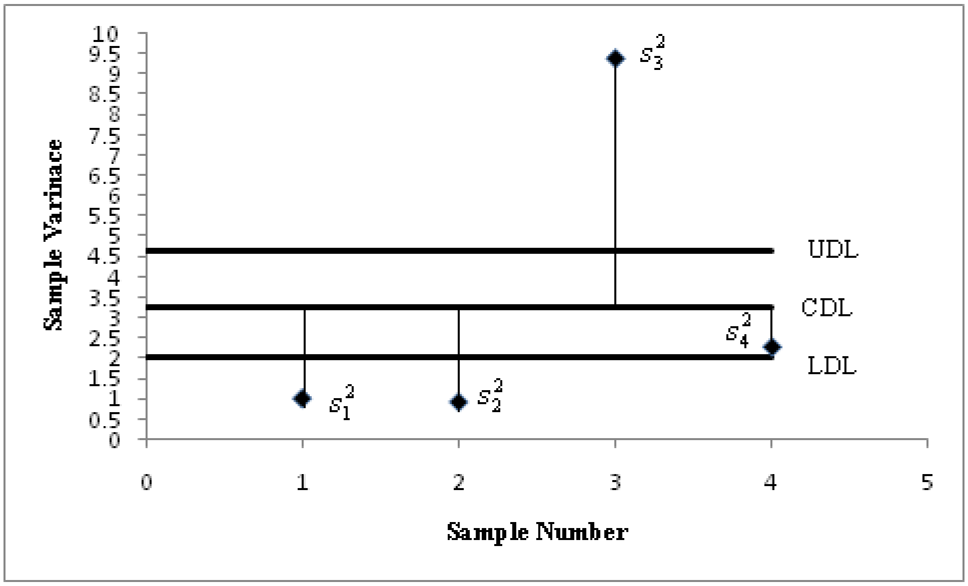

Figure 1. Chart for equality of variances

6. Conclusions

It is observed that the conclusion is the same (H_0 is rejected) in all the test procedures considered, but only the bootstrap graphical procedure gives additional information through Figure 1., that the process variables x_1, x_2 and x_3 are responsible for this heteroscedasticity. This information may be useful for the experimenter for taking further decisions. The proposed method being a graphical procedure simultaneously demonstrates the statistical significance and identifies the source of heterogeneity. This bootstrap graphical test is more useful in particular, for small samples and when the observations have come from a non-normal population.

References

[1]. Conover, W.J., Johnson, M.E. and Johnson, M.M. (1981). A comparative study of tests for homogeneity of variances, with applications to the outer continental shelf biding data. *Technometrics*, Vol. 23, 351-361.

[2]. Dennis D. Boss and Cavell Brownie (2004). Comparing variances and other measures of dispersion. *Statistical Science*, Vol. 19, No. 4, 571-578.

[3]. Efron, B, and Tibshirani, R.J. (1993). *An introduction to the bootstrap*, New York: Chapman & Hall.

[4]. Efron, B. (1979). Bootstrap methods: Another look at the Jackknife. *Annals of Statistics*, 7, 1-26.

[5]. Krishna Reddy, M., Naveen Kumar, B. and Ramu, Y., (2010). Bootstrap method for testing of equality of several means. *InterStat*.

[6]. Levene, H. (1960). In contributions to probability and statistics: *Essays in honor of Harold Hotelling*, I.Olkin *et.al.*, Eds. Stanford University Press, Stanford Calif. 278-292.

[7]. Montgomery, D.C. (2004). *Introduction to statistical quality control, 4^{th}*. New York: John Willey & Sons, Inc.

[8]. Rao, C. V. and Harikrishna, S. (1997). A graphical method for testing the equality of several variances. *Journal of Applied Statistics*, 24, 3, 279 — 288